

Control and dynamical system tools for optimization

Introduction to discrete-time system theory

Prof. Ivano Notarnicola

Dept. of Electrical, Electronic, Information Engineering
Alma Mater Studiorum Università di Bologna (Italy)
`ivano.notarnicola@unibo.it`

ENSTA • Institut Polytechnique de Paris (France)
26 January 2026

Example 1: Heron algorithm

Goal. Compute the square root of a positive number $a > 0$

Consider the sequence $\{x_k\}_{k \in \mathbb{N}}$ generated for all $k \in \mathbb{N}$ by the so-called Heron algorithm

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right), \quad x_0 > 0$$

Assume that the limit $x_{\text{eq}} \neq 0$ of $\{x_k\}_{k \in \mathbb{N}}$ exists, then it must satisfy the fixed-point condition

$$x_{\text{eq}} = \frac{1}{2} \left(x_{\text{eq}} + \frac{a}{x_{\text{eq}}} \right) \quad \implies \quad x_{\text{eq}}^2 = a$$

which means that one equilibrium is $x_{\text{eq}} = \sqrt{a}$

An additional step would be to show that the sequence actually converges to x_{eq} for any initial value $x_0 > 0$



Example 2: Fibonacci sequence

The Fibonacci sequence $\{y_k\}_{k \in \mathbb{N}}$ is generated for all $k \in \mathbb{N}$ by the recursion

$$y_k = y_{k-1} + y_{k-2}$$

with $y_0 = 1$ and $y_{-1} = 0$

We can express y_k as the **output** of the following (autonomous) linear discrete-time dynamical system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_k, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

with **state** $x_k \in \mathbb{R}^2$

Remark. The state matrix has eigenvalues $\frac{1}{2}(1 \pm \sqrt{5})$

Remark. The sequence $\{y_k\}_{k \in \mathbb{N}}$ does not converge, but it holds that

$$\lim_{k \rightarrow \infty} \frac{y_k}{y_{k-1}} = \frac{1 + \sqrt{5}}{2}$$

also known as the golden ratio



Example 3: gradient method



An unconstrained (local) minimizer of a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$x_{\star} \in \operatorname{argmin}_{x \in \mathbb{R}^n} f(x)$$

can be obtained using the so-called **gradient method**, described for all $k \in \mathbb{N}$ by

$$x_{k+1} = x_k - \alpha \nabla f(x_k), \quad x_0 \text{ given}$$

where $\alpha > 0$ is the stepsize and $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the gradient of f at x_k

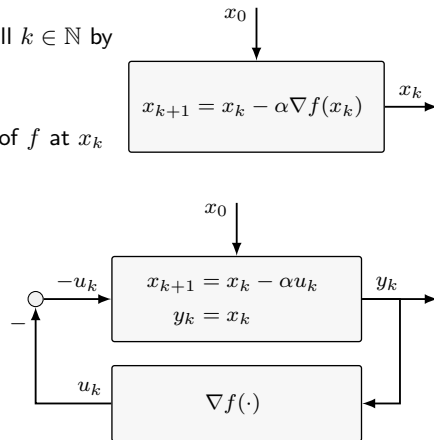
The algorithm update can be manipulated as

$$x_{k+1} = x_k - \alpha u_k, \quad x_0 \text{ given}$$

$$y_k = x_k$$

$$u_k = \nabla f(y_k)$$

which highlights a **feedback interconnection**



Discrete-time dynamical systems

A discrete-time dynamical system is generally defined as follows

$$\begin{aligned}x_{k+1} &= F(x_k, u_k), & x_0 \text{ given} \\ y_k &= G(x_k, u_k)\end{aligned}$$

with well-behaved $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$, where $x_k \in \mathbb{R}^n$ is the **state**, $u_k \in \mathbb{R}^m$ is the **input** and $y_k \in \mathbb{R}^p$ is the **output**

Let u_k be fixed to a constant value $\bar{u} \in \mathbb{R}^m$, then the state evolution depends on the initial condition x_0 only

Definition. A point $x_{\text{eq}} \in \mathbb{R}^n$ is an **equilibrium** of the system if it satisfies $x_{\text{eq}} = F(x_{\text{eq}}, \bar{u})$

Definition. An equilibrium x_{eq} is **stable** if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - x_{\text{eq}}\| < \delta \implies \|x_k - x_{\text{eq}}\| < \epsilon \quad \forall k \in \mathbb{N}$$

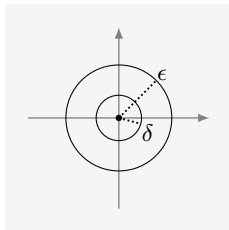
Definition. An equilibrium is **attractive** if there exists $\delta > 0$ such that

$$\|x_0 - x_{\text{eq}}\| < \delta \implies \lim_{k \rightarrow \infty} \|x_k - x_{\text{eq}}\| = 0$$

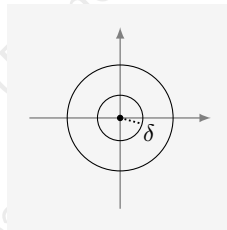
Definition. An equilibrium is **asymptotically stable** if it is both stable and attractive

Stability of equilibrium points

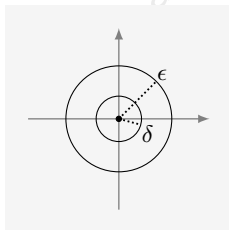
stable



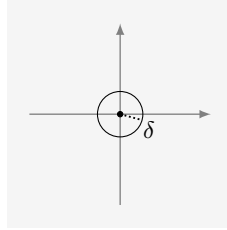
attractive



asymptotically stable



unstable



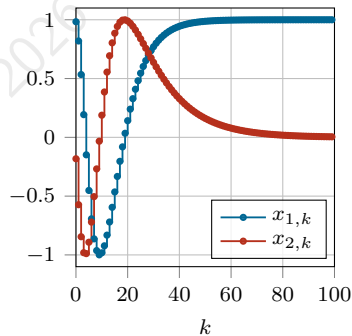
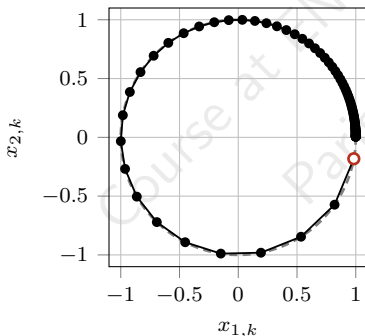
Example: attractive (unstable) equilibrium

Consider the nonlinear system

$$x_{k+1}^1 = \cos(\sigma \operatorname{atan2}(x_k^2, x_k^1))$$

$$x_{k+1}^2 = \sin(\sigma \operatorname{atan2}(x_k^2, x_k^1))$$

with $\sigma \in (0, 1)$

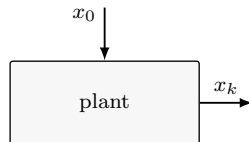


Lyapunov stability

Without loss of generality, let $x_{\text{eq}} = 0$ be the equilibrium of an unforced discrete-time dynamical system

$$x_{k+1} = F(x_k), \quad x_0 \text{ given}$$

with $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$



Direct Lyapunov Thm. If there exists a positive definite, radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that (strictly) decreases along the trajectories $\{x_k\}_{k \in \mathbb{N}}$ of the system $x_{k+1} = F(x_k)$, then the equilibrium (i.e., the origin) is (asymptotically) stable

Equivalently, check if

- the *increment* of V is non-positive across two consecutive times for all $k \in \mathbb{N}$

$$V(x_{k+1}) - V(x_k) \leq 0$$

- the value of V at any $T > 0$ is less than its initial value

$$V(x_T) - V(x_0) \leq 0$$

Lyapunov theory for linear systems

The Lyapunov theorem can be specialized for (unforced) linear time-invariant systems in the form

$$x_{k+1} = Ax_k, \quad x_0 \text{ given}$$

with $A \in \mathbb{R}^{n \times n}$

Lyapunov Thm. for LTI systems. The origin is asymptotically stable if and only if for any $Q = Q^\top > 0$ there exists a matrix $P = P^\top > 0$ satisfying the so-called (discrete-time) Lyapunov equation

$$A^\top P A - P = -Q$$

Remark. The matrix A must have all the eigenvalues strictly inside the unit circle (i.e., A is a *Schur* matrix)

Remark. The function $V(x) = \frac{1}{2}x^\top P x$ is a valid Lyapunov candidate for the linear system since its increment along trajectories satisfies

$$V(x_{k+1}) - V(x_k) = \frac{1}{2}x_k^\top \underbrace{(A^\top P A - P)}_{-Q} x_k$$

which implies that the scalar sequence $\{V(x_k)\}_{k \in \mathbb{N}}$ decreases along the trajectories of the system

Energy-based approach for input-output systems

The Lyapunov approach studies the convergence of the *state* trajectory $\{x_k\}_{k \in \mathbb{N}}$ by inspecting the evolution of a scalar (internal energy) function $\{V(x_k)\}_{k \in \mathbb{N}}$

Consider *input-output* ($u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$) systems in the form

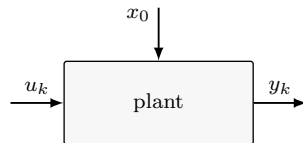
$$\begin{aligned}x_{k+1} &= F(x_k, u_k), & x_0 \text{ given} \\ y_k &= G(x_k, u_k)\end{aligned}$$

with well-behaved $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$

Some questions arise when considering the energy balance of the system

- Does the system (internally) *produce* energy?
- Does the system *dissipate* energy?
- Does the system *store* the externally supplied energy?

Next, focus on systems in which the increase in *internally stored* energy is less than the *externally supplied* energy provided through the input



Definition of dissipative system

Definition. A system is said to be *dissipative* from input u to output y with respect to the *supply rate* $\varphi : \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}$ if there exists a *storage function* $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$V(x_T) - V(x_0) \leq \sum_{k=0}^T \varphi(y_k, u_k)$$

for all admissible trajectories and all $T > 0$ (aka *dissipation inequality*)

Definition. A system is *strictly dissipative* from u to y wrt φ if V satisfies

$$V(x_T) - V(x_0) \leq \sum_{k=0}^T \varphi(y_k, u_k) - \sum_{k=0}^T \psi(x_k)$$

for all admissible trajectories and all $T > 0$, with $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ a positive definite function called *dissipation rate*

Remark. Dissipativity theory seamlessly applies to linear/nonlinear static/dynamical systems

Specialized terminology about passivity

A discrete-time dynamical system is said to be

- *passive* if the storage function satisfies

$$V(x_{k+1}) - V(x_k) \leq y_k^\top u_k$$

- *lossless* if the storage function satisfies

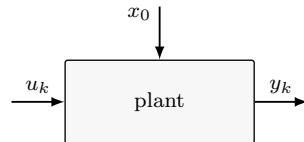
$$V(x_{k+1}) - V(x_k) = y_k^\top u_k$$

- *input strictly passive* (input-feedforward passive (IFP) for $u^\top \varphi_1(u)$) if there exists $\gamma_{\text{in}} > 0$ such that

$$V(x_{k+1}) - V(x_k) \leq y_k^\top u_k - \gamma_{\text{in}} \|u_k\|^2$$

- *output strictly passive* (output-feedback passive (OFP) for $y^\top \varphi_2(y)$) if there exists $\gamma_{\text{out}} > 0$ such that

$$V(x_{k+1}) - V(x_k) \leq y_k^\top u_k - \gamma_{\text{out}} \|y_k\|^2$$



Remark. The coefficients γ_{in} and γ_{out} are called passivity indices

Passivity for algebraic maps

An algebraic (aka static or memoryless) map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be

- *passive* (or monotone) if, for all $w \in \mathbb{R}^n$, it satisfies

$$\phi(w)^\top w \geq 0$$

- *input strictly passive* if there exists $\gamma_{\text{in}} > 0$ such that, for all $w \in \mathbb{R}^n$, it satisfies

$$\phi(w)^\top w - \gamma_{\text{in}} \|w\|^2 \geq 0$$

- *output strictly passive* if there exists $\gamma_{\text{out}} > 0$ such that, for all $w \in \mathbb{R}^n$, it satisfies

$$\phi(w)^\top w - \gamma_{\text{out}} \|\phi(w)\|^2 \geq 0$$

- *very strictly passive* if there exists $\gamma_{\text{in}}, \gamma_{\text{out}} > 0$ such that, for all $w \in \mathbb{R}^n$, it satisfies

$$\phi(w)^\top w - \gamma_{\text{in}} \|w\|^2 - \gamma_{\text{out}} \|\phi(w)\|^2 \geq 0$$



Example: discrete-time integrator (accumulator)

Consider the system

$$x_{k+1} = x_k + u_k$$

$$y_k = x_k$$

with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$ ($m = p = n$)

Consider the storage function $V(x) = \frac{1}{2}\|x\|^2$, then its increment along the system's trajectory satisfies

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \frac{1}{2}\|x_{k+1}\|^2 - \frac{1}{2}\|x_k\|^2 \\ &= \frac{1}{2}\|x_k + u_k\|^2 - \frac{1}{2}\|x_k\|^2 \\ &= x_k^\top u_k + \frac{1}{2}\|u_k\|^2 \\ &= y_k^\top u_k + \frac{1}{2}\|u_k\|^2 \end{aligned}$$

Remark. The system has a “shortage” of input passivity

Question. What if a gain $\alpha > 0$ on the input is considered, i.e., $x_{k+1} = x_k + \alpha u_k$ and $y_k = x_k$?

Example: modified discrete-time integrator

Consider the system

$$x_{k+1} = x_k + u_k$$

$$y_k = x_k + u_k$$

with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$

Consider the storage function $V(x) = \frac{1}{2}\|x\|^2$, then its increment along the system's trajectory satisfies

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \frac{1}{2}\|x_{k+1}\|^2 - \frac{1}{2}\|x_k\|^2 \\ &= \frac{1}{2}\|x_k + u_k\|^2 - \frac{1}{2}\|x_k\|^2 \\ &= x_k^\top u_k + \frac{1}{2}\|u_k\|^2 \\ &= y_k^\top u_k - \frac{1}{2}\|u_k\|^2 \end{aligned}$$

Remark. The system is input strictly passive, i.e., it has an “excess” of input passivity

Storage functions for general LTI systems

Consider a general linear time-invariant system

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k, & x_0 \text{ given} \\ y_k &= Cx_k + Du_k\end{aligned}$$

and a *quadratic storage function* $V(x) := \frac{1}{2}x^\top Px$, with $P \in \mathbb{R}^{n \times n}$ and $P = P^\top > 0$

Then, the increment of V along the system's trajectories satisfies

$$\begin{aligned}V(x_{k+1}) - V(x_k) &= \frac{1}{2}x_{k+1}^\top Px_{k+1} - \frac{1}{2}x_k^\top Px_k \\ &= \frac{1}{2}x_k^\top (A^\top PA - P)x_k + x_k^\top (A^\top PB)u_k + \frac{1}{2}u_k^\top B^\top PBu_k \\ &= \frac{1}{2} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} A^\top PA - P & A^\top PB \\ B^\top PA & B^\top PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}\end{aligned}$$

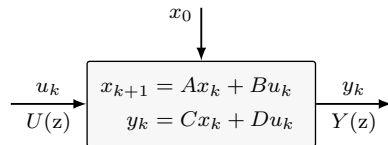
where

- the first Lyapunov-like term (quadratic in x_k) is possibly negative (if A is Schur)
- the last term (quadratic in u_k) is always positive, depending also on P
- the cross term must necessarily be exploited, e.g., recovering the output y_k

Frequency domain and transfer matrices

Another input-output description of linear time-invariant (LTI) systems is related to the **frequency domain** (Focus on square systems only, i.e., with $p = m$)

An LTI system can be represented by its **transfer matrix** $G(z)$ ($m \times m$ matrix of rational functions in the variable $z \in \mathbb{C}$) that relates the Z -transforms¹ of the input u_k and output y_k



The $m \times m$ transfer matrix $G(z)$ from input u_k and y_k can be computed from (A, B, C, D) as

$$G(z) = C(zI_n - A)^{-1}B + D, \quad z \in \mathbb{C}$$

Remark. Each element of $G(z)$ is a ratio of two polynomials in z , the roots of the denominator polynomials are called **poles** of the system

¹The Z -transform of a sequence $\{u_k\}_{k \in \mathbb{N}}$ is $U(z) := \sum_{k=0}^{\infty} u_k z^{-k}$ for all $z \in \mathbb{C}$ such that the series converges

Example: transfer matrix of the discrete-time integrator

Consider the system

$$x_{k+1} = x_k + \alpha u_k$$

$$y_k = x_k$$

with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$ ($m = n$) and $\alpha > 0$

Taking the Z -transform of the input-output relations we obtain

$$\begin{aligned} zX(z) - zx_0 &= X(z) + \alpha U(z) \\ Y(z) &= X(z) \end{aligned} \quad \Longrightarrow \quad Y(z) = \frac{z}{z-1}x_0 + \frac{\alpha}{z-1}U(z)$$

Setting $x_0 = 0$, we obtain the so-called *transfer matrix* given by

$$G(z) = \frac{\alpha}{z-1}I_n$$

Discrete positive realness

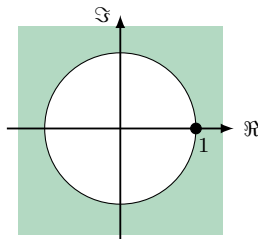
We introduce two useful properties for transfer matrices in discrete time

Definition. A transfer matrix $G(z)$ is *discrete positive real* (DPR) if

- $G(z)$ has analytic elements for all $z \in \mathbb{C}$ such that $|z| > 1$
- $G(z) + G(\bar{z})^\top \geq 0$ for all $z \in \mathbb{C}$ such that $|z| > 1$

Equivalently, one can show that $G(z)$ is DPR if and only if

- $G(z)$ has simple poles on the unit disk (if any)
- $G(e^{j\omega}) + G(e^{-j\omega})^\top \geq 0$ for all $\omega \in [-\pi, \pi)$ for which G exists
- If $p_0 = e^{j\omega_0}$ for a given $\omega_0 \in [-\pi, \pi)$ is a pole and K_0 is the residue matrix of $G(z)$ at $z = p_0$, then the matrix $e^{-j\omega_0} K_0$ is Hermitian and positive semi-definite



Definition. A transfer matrix $G(z)$ is *strictly discrete positive real* if there exists $\rho > 1$ such that $G(z/\rho)$ is DPR

Example: discrete-time integrator (revisited)

Consider the system

$$x_{k+1} = x_k + u_k$$

$$y_k = x_k$$

with state, input and output $x_k, u_k, y_k \in \mathbb{R}^n$ ($m = n$)

The transfer matrix is

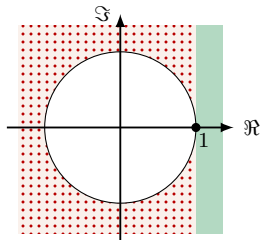
$$G(z) = \frac{1}{z-1} I_n$$

DPR needs $G(z) + G(\bar{z})^\top > 0$ outside the unit disk $|z| > 1$. It holds

$$G(z) + G(\bar{z})^\top = \frac{1}{z-1} I_n + \frac{1}{\bar{z}-1} I_n = \frac{z-1+\bar{z}-1}{|z-1|^2} I_n = 2 \frac{\Re(z)-1}{|z-1|^2} I_n$$

which is positive exclusively for $\Re(z) > 1$

Remark. The state-space description is $A = I$, $B = I$, $C = I$ and $D = 0$, thus it is strictly proper



Example: modified discrete-time integrator (revisited)

Consider the (scalar, $n = 1$) system

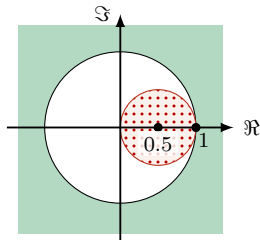
$$\begin{aligned} x_{k+1} &= x_k + u_k \\ y_k &= x_k + u_k \end{aligned} \quad \Longleftrightarrow \quad G(z) = \frac{Y(z)}{U(z)} = \frac{1}{z-1} + 1 = \frac{z}{z-1}$$

DPR needs $G(z) + G(\bar{z}) > 0$ outside the unit disk $|z| > 1$. It holds

$$\begin{aligned} G(z) + G(\bar{z}) &= \frac{z}{z-1} + \frac{\bar{z}}{\bar{z}-1} = \frac{2z\bar{z} - z - \bar{z}}{|z-1|^2} = 2 \frac{|z|^2 - \Re(z)}{|z-1|^2} \\ &= 2 \frac{(\Re(z) - \frac{1}{2})^2 + \Im(z)^2 - \frac{1}{4}}{|z-1|^2} \end{aligned}$$

which is positive outside the disk of radius $\sqrt{\frac{1}{4}} = \frac{1}{2}$ centered at $z = \frac{1}{2}$, hence also for all $|z| > 1$

Question. Is it also strictly DPR?



Positive real lemma

A connection between passivity and discrete positive realness does exist

Positive-real lemma. Let (A, B, C, D) be a minimal realization of a square transfer matrix $G(z)$, then the following statements are equivalent

- There exist a (describing) $n \times n$ matrix $P = P^\top > 0$, and matrices $M_1 \in \mathbb{R}^{\ell \times n}$ and $M_2 \in \mathbb{R}^{\ell \times m}$ solving the following matrix equations

$$A^\top P A - P = -M_1^\top M_1$$

$$A^\top P B = C^\top - M_1^\top M_2$$

$$B^\top P B = D + D^\top - M_2^\top M_2$$

- The transfer matrix $G(z)$ is discrete positive real

Remark. For general rectangular systems, the so-called *bounded-real lemma* must be taken into account

Implications of discrete positive realness

The increment along trajectories of the storage function $V(x) := \frac{1}{2}x^\top Px$ satisfies

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \frac{1}{2}x_k^\top (A^\top PA - P)x_k + x_k^\top (A^\top PB)u_k + \frac{1}{2}u_k^\top B^\top PBu_k \\ &= \frac{1}{2}x_k^\top (-M_1^\top M_1)x_k + x_k^\top (C^\top - M_1^\top M_2)u_k + \frac{1}{2}u_k^\top (D + D^\top - M_2^\top M_2)u_k \\ &= y_k^\top u_k - \frac{1}{2}\|M_1x_k + M_2u_k\|^2 + \frac{1}{2}\underbrace{u_k^\top (D - D^\top)u_k}_{\text{skew-symmetric term}} \end{aligned}$$

Remark. The matrix $D - D^\top$ is skew-symmetric

Remark. The vector $M_1x_k + M_2u_k$ represents a particular “output” of the system

Strict positive real lemma

Strict positive real lemma. Let (A, B, C, D) be a minimal realization of a square transfer matrix $G(z)$, then the following statements are equivalent

- There exist a $n \times n$ matrix $P = P^\top > 0$, matrices $M_1 \in \mathbb{R}^{\ell \times n}$ and $M_2 \in \mathbb{R}^{\ell \times m}$ and a scalar $\rho > 1$ solving the following matrix equations

$$A^\top P A - \frac{1}{\rho} P = -M_1^\top M_1$$

$$A^\top P B = C^\top - M_1^\top M_2$$

$$B^\top P B = D + D^\top - M_2^\top M_2$$

- The transfer matrix $G(z)$ is *strictly* discrete positive real

Implications of the strict positive real lemma

The increment along trajectories of the storage function $V(x) := \frac{1}{2}x^\top Px$ satisfies

$$V(x_{k+1}) - \frac{1}{\rho}V(x_k) = y_k^\top u_k - \frac{1}{2}\|M_1x_k + M_2u_k\|^2$$

where $\rho > 1$

Remark. If for all $k \in \mathbb{N}$ the input u_k is such that $y_k^\top u_k \leq 0$, then

$$V(x_k) \leq \left(\frac{1}{\rho}\right)^k V(x_0)$$

namely *exponential stability*² (or geometric/linear convergence) of the origin is certified

Remark. The negative term $-\frac{1}{2}\|M_1x_k + M_2u_k\|^2$ can be neglected

²An equilibrium x_{eq} is exponentially stable if $\|x_k - x_{\text{eq}}\| \leq M_0\sigma^k\|x_0 - x_{\text{eq}}\|$ for some $M_0 > 0$ and $\sigma \in (0, 1)$

Example: pure delay system

Consider the system

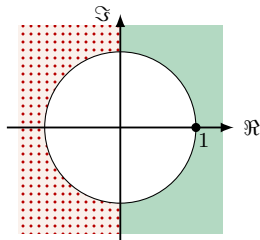
$$\begin{aligned} x_{k+1} &= u_k \\ y_k &= x_k \end{aligned} \iff G(z) = \frac{1}{z}$$

DPR needs $G(z) + G(\bar{z}) > 0$ outside the unit disk $|z| > 1$. It holds

$$G(z) + G(\bar{z}) = \frac{1}{z} + \frac{1}{\bar{z}} = 2 \frac{\Re(z)}{|z|^2}$$

which is positive exclusively when $\Re(z) > 0$

Remark. The state-space description is $A = 0$, $B = I$, $C = I$ and $D = 0$, thus it is strictly proper



Example: modified delay system

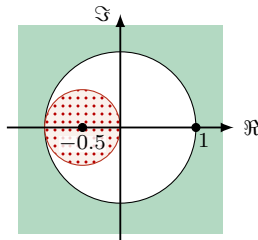
Consider the system

$$\begin{aligned} x_{k+1} &= u_k \\ y_k &= x_k + u_k \end{aligned} \iff G(z) = \frac{1}{z} + 1 = \frac{z+1}{z}$$

DPR needs $G(z) + G(\bar{z}) > 0$ outside the unit disk $|z| > 1$. It holds

$$\begin{aligned} G(z) + G(\bar{z}) &= \frac{z+1}{z} + \frac{\bar{z}+1}{\bar{z}} = \frac{(z+1)\bar{z} + (\bar{z}+1)z}{|z|^2} = 2 \frac{|z|^2 + \Re(z)}{|z|^2} \\ &= 2 \frac{(\Re(z) + \frac{1}{2})^2 + \Im(z)^2 - \frac{1}{4}}{|z|^2} \end{aligned}$$

which is positive outside the disk of radius $\frac{1}{2}$ centered at $z = -\frac{1}{2}$



Remark. For strictly DPR, scale the transfer function with $\rho > 1$, i.e., $G(z/\rho) = \frac{z/\rho+1}{z/\rho} = \frac{z+\rho}{z}$, but nm-phase

Example: general zero-pole system

A scalar LTI system with a (real) zero at $z = z_0$ and a (real) pole at $z = p_0$ and a static gain $\gamma > 0$ is

$$x_{k+1} = p_0 x_k + \gamma u_k$$

$$y_k = (p_0 - z_0)x_k + \gamma u_k$$

In fact, it holds (assuming initial condition $x_0 = 0$)

$$zX(z) = p_0 X(z) + \gamma U(z) \quad \Longrightarrow \quad X(z) = \frac{\gamma}{z - p_0} U(z)$$

$$Y(z) = (p_0 - z_0)X(z) + \gamma U(z)$$

then

$$G(z) = \frac{Y(z)}{U(z)} = \gamma \frac{p_0 - z_0}{z - p_0} + \gamma = \gamma \frac{z - z_0}{z - p_0}$$

Example: general zero-pole system

Consider the discrete-time system

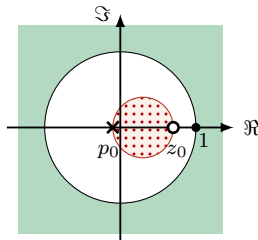
$$\begin{aligned}x_{k+1} &= p_0 x_k + \gamma u_k \\ y_k &= (p_0 - z_0)x_k + \gamma u_k\end{aligned} \iff G(z) = \gamma \frac{z - z_0}{z - p_0}$$

DPR needs $G(z) + G(\bar{z}) > 0$ outside the unit disk $|z| > 1$. It holds

$$\begin{aligned}G(z) + G(\bar{z}) &= \gamma \frac{z - z_0}{z - p_0} + \gamma \frac{\bar{z} - z_0}{\bar{z} - p_0} \\ &= \gamma \frac{(z - z_0)(\bar{z} - p_0) + (\bar{z} - z_0)(z - p_0)}{|z - p_0|^2} \\ &= 2\gamma \frac{|z|^2 - (z_0 + p_0)\Re(z) + z_0 p_0}{|z - p_0|^2} \\ &= 2\gamma \frac{\left(\Re(z) - \frac{z_0 + p_0}{2}\right)^2 + \Im(z)^2 - \left(\frac{z_0 - p_0}{2}\right)^2}{|z - p_0|^2}\end{aligned}$$

which is positive outside the disk of radius $\frac{|z_0 - p_0|}{2}$ centered at $z = \frac{z_0 + p_0}{2}$

Remark. For the (strictly) DPR condition to hold, z_0 and p_0 must lie in the (open) interval $[-1, 1]$



Stability and control of a passive (linear) system

Consider a passive LTI system (A, B, C, D)

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

with a storage function V satisfying the passivity inequality

$$V(x_{k+1}) - V(x_k) \leq y_k^\top u_k$$

- If $u_k \equiv 0$, then $V(x_{k+1}) - V(x_k) \leq 0$, i.e., marginal stability of the equilibrium is guaranteed
- If $y_k \equiv 0$, then $V(x_{k+1}) - V(x_k) \leq 0$, i.e., the so-called zero-dynamics of the system is stable
- A passive system can be easily stabilized using a static output negative feedback

$$u_k = -\alpha y_k$$

with an *arbitrary* gain $\alpha > 0$

The Passivity theorem

Theorem. Given two passive (nonlinear) systems Σ_1 and Σ_2 , their (negative) feedback interconnection (whenever well-posed) is also passive from (u_k^1, u_k^2) to (y_k^1, y_k^2)

Remark. Setting $(u_k^1, u_k^2) = 0$, the equilibrium (the origin) of the interconnected system is stable (possibly also asymptotically stable depending on Σ_1 and Σ_2)

Remark. An “*excess*” of passivity in one subsystem can compensate for a “*shortage*” in the other



The Lur'e problem (1944)

The Lur'e problem studies the (absolute) stability of the origin $x = 0_n$ for a dynamical system obtained as the *interconnection* of a LTI system

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k, & x_0 &= x^0 \\ y_k &= Cx_k + Du_k\end{aligned}$$

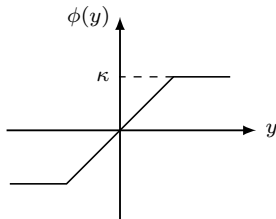
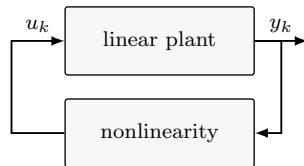
with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$, in feedback with a static nonlinearity

$$u_k = \phi(y_k)$$

with $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^m$ well behaved (namely, sector bounded)

Example. A saturation can be modeled as a sector bounded nonlinearity

$$u_k = \begin{cases} y_k, & \text{if } |y_k| < \kappa \\ \kappa \cdot \text{sign}(y_k), & \text{if } |y_k| \geq \kappa \end{cases}$$



Goal. Study the interconnection stability based on the properties of the *individual* components