SCATTERING FROM A THIN RANDOM COATING OF NANOPARTICLES: THE DIRICHLET CASE

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Abstract

We study the time-harmonic scattering by a heterogeneous object covered with a thin layer of randomly distributed sound-soft nanoparticles. The size of the particles, their distance between each other and the layer's thickness are all of the same order but small compared to the wavelength of the incident wave. Solving the Helmholtz equation in this context can be very costly and the simulation depends on the given distribution of particles. To circumvent this, we propose, via a multi-scale asymptotic expansion of the solution, an effective model where the layer of particles is replaced by an equivalent boundary condition. The coefficients that appear in this equivalent boundary condition depend on the solutions to corrector problems of Laplace type defined on unbounded random domains. Under the assumption that the particles are distributed given a stationary and mixing random point process, we prove that those problems admit a unique solution in the proper space. We then establish quantitative error estimates for the effective model and present numerical simulations that illustrate our theoretical results.

In this article we study the time-harmonic electromagnetic scattering by a heterogeneous object covered with a thin layer of randomly distributed sound-soft nanoparticles. The size of the particles, their spacing and the layer's thickness are of the order of a few nanometers and very small compared to the wavelength of the incident wave which is of the order of the centimeter. Two difficulties arise when trying to solve Maxwell's equation in this context : first, on account of the very different scales, the computational cost with a standard numerical method is prohibitive; second the distribution of particles for a given object is unknown. Our goal is to propose an effective model where the layer of particles is replaced by an equivalent boundary condition and whose solution approximates the original solution with an error that we can estimate.

Methods to build such effective models can be classified into two categories : methods based on optimization techniques and methods based on asymptotic techniques. The principle of the first class of methods is, given an effective boundary condition, to optimize the coefficients appearing in the condition so that the effective solution fits in a certain sense as much as possible a reference solution. Although the effective model is simple to implement, it is difficult to estimate its accuracy with respect to the physical parameters of the problem. See for instance [30, 43].

Our work lies in the second class of methods. It is based on a multi-scale asymptotic expansion which allows us to take into account the fast variations of the physical solution close to the layer and its slower variations far from the layer. These techniques in the context of scattering problem by thin layers are well known when the thin layer is penetrable and modeled by a constant coefficient (see for instance [6, 29, 44, 9]) or when the thin layer is periodic, *i.e.* the distribution of the particles is supposed to be periodic in the direction of the interface (see for instance [1, 19, 20, 15, 39]). Note that in this case, the technique works whichever the nature of the particles is: penetrable, sound-soft or sound-hard. It seems that our work is the first one dealing with scattering problem by a thin layer with random properties. Let us however mention [37] whose study is based on the "heuristic" quasi-crystalline approximation. Our work has of course lots of links with the literature on volume stochastic homogenenization (see for instance [28, 23, 21, 24]). We call it "volume" in contrast with our problem which could be called "surface" stochastic homogenization. Indeed, in volume stochastic homogenization, the question is to deal with (in general elliptic) PDEs with (stationary and ergodic) random coefficients and to look for an effective PDE in general of the same nature but with constant coefficients. Our problem is different since the random medium is constituted only by a thin layer and we wish to replace the thin layer by an effective boundary condition, while the PDE outside the layer stays the same. One will find of course similarities in the analysis (for instance in order to obtain the quantitative error estimates) but some difficulties are really specific to surface homogenization. Let us mention that similar problems were treated in the context of Laplace equation [14, 3] or Stokes equations [7, 26, 18, 33] in presence of random rough boundaries. The difficulty of dealing with a random distribution of particles instead of rough boundaries lies in, as we will see later, the derivation of Poincaré inequalities.

The problem that we consider in this paper is a simplification of the original application. First of all, we consider the time-harmonic acoustic scattering by an object. In other words, the unknown is

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a scalar function and the equation is Helmholtz's. When the dimension of the domain is $d = 2$, this problem can be derived from the study of electromagnetic scattering by a 3D object that is invariant in one direction, where the incident field is TE- or TM-polarized. More generally, this study is an important first step before addressing the more general case of electromagnetism, on which we are working. Also, for further simplification, we neglect the complexity of the geometry of the object and assume that the object is planar. Moreover, in the applications, the object is covered by multple layers of material whose goal is to attenuate the radar cross section of the object. This multilayer medium is modeled and replaced by an impedance boundary condition that is imposed on the boundary of the object. Finally, we impose Dirichlet boundary conditions on the particles. This corresponds, if we go back to the electromagnetic equivalent (see above) at $d = 2$, to perfectly conducting particles and a TE-polarization of the field. Imposing Neumann boundary conditions constitutes another challenge from a theoretical point of view and will be the subject of a forthcoming paper.

The paper is organized as follows: The model is presented in Section 1. We detail in particular the random setting with a classical stationary and ergodic assumption. Let us emphasize Hypothesis (H1) that ensures that arbitrarily large portions of the layer without Dirichlet particles can exist but only with a small probability. We also establish well-posedness of the scattering problem. The formal asymptotic expansion is described in Section 2. The proposed Ansatz involves so called far-field terms that approximate the solution far from the layer and the so called near-field terms, depending on microscopic variables and the longitudinal macroscopic variables, that capture specifically the scattering phenomena near the layer. The near-field terms satisfy Laplace-type PDEs parametrized by the macroscopic variables, set in a random halfspace. We study in Section 3 the well-posedness of the near-field problems for which Hypothesis (H1) is crucial. Imposing that the limit of the near-fields vanishes at infinity yields the boundary condition satisfied by each far-field term. The behavior at infinity of the near-fields is studied in Section 4. One can then derive an effective model (see Section 5) whose solution approaches the first two far-field terms. Quantitative error estimates are derived in Section 6. Those estimates are established under a new Hypothesis (L^∞), that supposes that the distance from any point in the layer to the nearest particule is bounded. If we assume additionally that the particles' distribution process satifies a quantitative mixing assumption (see Hypothesis (Mix)), we can impove the convergence rate of the effective model. A certain number of technical lemmas that are necessary to the proof of those improved estimates are proved in the appendices at the end of the paper. Finally, we present numerical simulations illustrating the theoretical results of the paper.

Notations used throughout the paper. The integer $d = 2$ or 3 denotes the dimension. Let $\boldsymbol{x} := (x_1, x_2)$ and $\boldsymbol{x} := (x_1, x_2, x_3)$ denote the Cartesian coordinate system in $\mathbb{R}^d$ respectively for $d = 2$ and $d = 3$. For all $\boldsymbol{x} \in \mathbb{R}^d$, $\boldsymbol{x}_{\shortparallel}$ denotes the tangential components of $\boldsymbol{x}$, i.e. $\boldsymbol{x}_{\shortparallel} := x_1$ when $d = 2$ and $\boldsymbol{x}_{\shortparallel} := (x_1, x_2)$ when $d = 3$.

The index ω indicates a dependency in the random distribution of the particles. We will specify the random setting that we consider in Section 1.5.

The positive real number $\varepsilon > 0$ denotes a small parameter, all the domains and functions that depend on ε are indexed by ε.

For any measurable bounded domain B, the spatial average over B is denoted by $⨍_B := |B|^{-1} \int_B$, where $|B|$ is the Lebesgue measure of B.

The centered cube of side R of dimension $d-1$ (we insist on the dimension $d-1$) is denoted $\square_R := (-R/2, R/2)^{d-1}$.

1 Presentation of the model

1.1 Domain of propagation

Let us consider an infinite plane denoted $\Sigma_0 := \{x_d = 0\}$ covered by a thin layer $\mathcal{L}_\varepsilon := \mathbb{R}^{d-1} \times (0, \varepsilon h)$ containing a set, denoted $\mathcal{P}_\varepsilon^\omega$, of randomly distributed particles of size ε. Let $D_\varepsilon^\omega := \mathbb{R}^{d-1} \times \mathbb{R}^+ \setminus \overline{\mathcal{P}_\varepsilon^\omega}$ be the domain above Σ_0 outside $\mathcal{P}_\varepsilon^\omega$.

For any $L > \varepsilon h$, let us introduce $\Sigma_L := \{x_d = L\}$, $\mathcal{B}_L^\omega := \mathbb{R}^{d-1} \times (0, L) \setminus \overline{\mathcal{P}_\varepsilon^\omega}$ the strip below Σ_L where lies the set of particles (and which then depends on the particles' distribution), $\mathcal{B}_L^\infty :=$

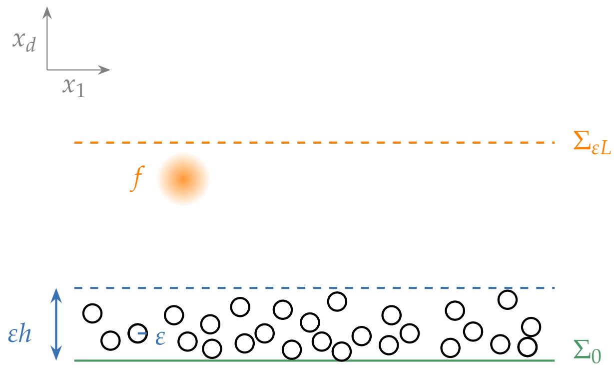


Figure 1: Illustration of the geometry of the model

$\mathbb{R}^{d-1} \times (L, +\infty)$ the half-space above Σ_L that does not depend on the distribution and finally for $\varepsilon h < L' < L$, $\mathcal{B}_{L',L} := \mathcal{B}^{\infty}_{L'} \setminus \mathcal{B}^{\infty}_{L}$ the strip between $\Sigma_{L'}$ and Σ_L.

1.2 Problem formulation

For a given source function $f \in L^2(D^{\omega}_{\varepsilon})$ whose support is compact and lies far away from the layer, *i.e.* there exists $L' < L \in (\varepsilon h, +\infty)$ such that

$$\mathrm{supp} f \subset \mathcal{B}_{L',L} \quad \text{with} \quad \mathrm{supp} f \text{ compact;} \tag{1}$$

we look for the solution u^{ω}_{ε} of the Helmholtz equation

$$-\Delta u^{\omega}_{\varepsilon} - k^2 u^{\omega}_{\varepsilon} = f \quad \text{in } D^{\omega}_{\varepsilon} \tag{2}$$

where $k > 0$ is the wavenumber. The infinite plane models a multi-layer object through a Robin boundary condition so that

$$-\partial_{x_d} u^{\omega}_{\varepsilon} + \mathrm{i}k\gamma\, u^{\omega}_{\varepsilon} = 0 \quad \text{on } \Sigma_0, \tag{3}$$

where the surface impedance coefficient $\gamma \in \mathbb{C}$ is such that

$$\mathrm{Re}\,[\gamma] > 0. \tag{4}$$

At the boundary of the particles $\partial\mathcal{P}^{\omega}_{\varepsilon}$, we impose to the field a homogeneous Dirichlet condition

$$u^{\omega}_{\varepsilon} = 0 \quad \text{on } \partial\mathcal{P}^{\omega}_{\varepsilon}. \tag{5}$$

Finally, the problem formulation has to be completed with a radiation condition. As in [12, 11] (where a Dirichlet boundary condition on the infinite hyper-plane is considered) and in [31, 4] (where an impedance boundary condition is considered), we make use of the so-called *upward propagating radiation condition* (UPRC). This outgoing wave condition is detailed in the following section.

1.3 Outgoing wave condition based on a Dirichlet-to-Neumann operator

To define the outgoing propagation radiation condition, we refer to [13, 12] which consider 2D scattering by a rough Dirichlet surface (defined as the graph of a function) and [4] which considers the case of 2D or 3D scattering by a rough impedance surface.

We introduce the fundamental solution of the Helmholtz equation in $\mathbb{R}^d$ given by

$$\left|\begin{array}{l} \text{for } d=2, \quad \Phi(\mathbf{x},\mathbf{y}) = \dfrac{\mathrm{i}}{4} H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{y}|) \\ \text{for } d=3, \quad \Phi(\mathbf{x},\mathbf{y}) = \dfrac{\exp\left(\mathrm{i}k|\boldsymbol{x}-\boldsymbol{y}|\right)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \end{array}\right. \qquad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d,\ \boldsymbol{x} \neq \boldsymbol{y},$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0. The UPRC then states that

$$u_\varepsilon^\omega(\boldsymbol{x}) = 2\int_{\Sigma_L} \partial_{x_d}\Phi(\boldsymbol{x},(\boldsymbol{y}_{\shortparallel},L))u_\varepsilon^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})d\boldsymbol{y}_{\shortparallel}, \quad \boldsymbol{x}\in\mathcal{B}_L^\infty. \tag{6}$$

For this integral representation to be well-defined, it is not necessary for $u_\varepsilon^\omega|_{\Sigma_L}$ to be in $L^2(\Sigma_L)$. For example for $d=2$, $u_\varepsilon^\omega|_{\Sigma_L}\in L^\infty(\Sigma_L)$ is sufficient since $\partial_{x_d}\Phi = O(\boldsymbol{y}_{\shortparallel}^{-3/2})$. Nevertheless, in the case where $u_\varepsilon^\omega|_{\Sigma_L}\in L^2(\Sigma_L)$, which is the case as we shall see in the next section, (6) can be rewritten in terms of the Fourier transform of $u_\varepsilon^\omega|_{\Sigma_L}$ as follows

$$u_\varepsilon^\omega(\boldsymbol{x}) \;=\; (2\pi)^{-(d-1)/2}\int_{\mathbb{R}^{d-1}} \widehat{\phi}_L(\boldsymbol{\zeta})\, e^{\mathrm{i}(x_d-L)\sqrt{k^2-|\boldsymbol{\zeta}|^2}}\, e^{i\boldsymbol{x}_{\shortparallel}\cdot\boldsymbol{\zeta}}\,\mathrm{d}\boldsymbol{\zeta}, \quad \boldsymbol{x}=(\boldsymbol{x}_{\shortparallel},x_d)\in\mathcal{B}_L^\infty. \tag{7}$$

where by convention $\sqrt{k^2-|\boldsymbol{\zeta}|^2} = \mathrm{i}\sqrt{|\boldsymbol{\zeta}|^2-k^2}$ when $k<|\boldsymbol{\zeta}|$ and

$$\widehat{\phi}_L(\boldsymbol{\zeta}) := (2\pi)^{-(d-1)/2}\int_{\mathbb{R}^{d-1}} \phi_L(\boldsymbol{y}_{\shortparallel})\, e^{-\mathrm{i}\boldsymbol{y}_{\shortparallel}\cdot\boldsymbol{\zeta}}\,\mathrm{d}\boldsymbol{y}_{\shortparallel}, \quad \boldsymbol{\zeta}\in\mathbb{R}^{d-1},$$

is the Fourier transform of $\phi_L := u_\varepsilon^\omega\big|_{\Sigma_L}$ (identifying Σ_L and $\mathbb{R}^{d-1}$).

Let us recall the characterization of Sobolev spaces in terms of Fourier transforms: for $s>0$

$$\begin{aligned} H^s(\Sigma_L) &:= \left\{\varphi\in L^2(\Sigma_L),\ \left(|\boldsymbol{\zeta}|^2+1\right)^{s/2}\widehat{\varphi}\in L^2(\mathbb{R}^{d-1})\right\}, \\ \text{and}\quad H^{-s}(\Sigma_L) &:= \left(H^s(\Sigma_L)\right)' = \left\{\varphi\in\mathcal{S}'(\mathbb{R}^{d-1}),\ \left(|\boldsymbol{\zeta}|^2+1\right)^{-s/2}\widehat{\varphi}\in L^2(\mathbb{R}^{d-1})\right\}. \end{aligned} \tag{8}$$

Definition. *The Dirichlet-to-Neumann (DtN) operator* $\Lambda^k : H^{\frac12}(\Sigma_L)\to H^{-\frac12}(\Sigma_L)$ *is defined as*

$$\forall\varphi,\psi\in H^{\frac12}(\Sigma_L), \quad \langle\Lambda^k\varphi,\psi\rangle_{\Sigma_L} = (2\pi)^{-(d-1)/2}\int_{\mathbb{R}^{d-1}} \mathrm{i}\sqrt{k^2-|\boldsymbol{\zeta}|^2}\,\widehat{\varphi}(\boldsymbol{\zeta})\,\overline{\widehat{\psi}(\boldsymbol{\zeta})}\,\mathrm{d}\boldsymbol{\zeta}, \tag{9}$$

where here and in the sequel $\langle\cdot,\cdot\rangle_{\Sigma_L}$ *denotes the sesquilinear duality product between* $H^{-\frac12}(\Sigma_L)$ *and* $H^{\frac12}(\Sigma_L)$.

The operator Λ^k satisfies then the following properties (see for instance [12]).

Proposition 1. *The operator* $\Lambda^k : H^{\frac12}(\Sigma_L)\to H^{-\frac12}(\Sigma_L)$ *is a continuous operator such that*

$$\forall\varphi\in H^{\frac12}(\Sigma_L), \quad \mathrm{Re}\left[\langle\Lambda^k\varphi,\varphi\rangle_{\Sigma_L}\right]\le 0 \quad \textit{and} \quad \mathrm{Im}\left[\langle\Lambda^k\varphi,\varphi\rangle_{\Sigma_L}\right]\ge 0. \tag{10}$$

1.4 Well-posedness of the scattering problem

The problem given in Section 1.2 and 1.3 can be rewritten as follows.

Find u_ε^ω in $H^1(\mathcal{B}_L^\omega)$ such that

$$\left|\begin{array}{lll} -\Delta u_\varepsilon^\omega - k^2 u_\varepsilon^\omega = f & \text{in} & \mathcal{B}_L^\omega\,, \\ -\partial_{x_d}u_\varepsilon^\omega + \mathrm{i}k\gamma u_\varepsilon^\omega = 0 & \text{on} & \Sigma_0\,, \\ u_\varepsilon^\omega = 0 & \text{on} & \partial\mathcal{P}_\varepsilon^\omega, \\ -\partial_{x_d}u_\varepsilon^\omega + \Lambda^k u_\varepsilon^\omega = 0 & \text{on} & \Sigma_L\,, \end{array}\right. \tag{11}$$

Let $H_0^1(\mathcal{B}_L^\omega) := \left\{u\in H^1(\mathcal{B}_L^\omega),\ u=0 \text{ on } \partial\mathcal{P}_\varepsilon^\omega\right\}$. The variational formulation associated with (11) is :

Find $u_\varepsilon^\omega\in H_0^1(\mathcal{B}_L^\omega)$ such that

$$\forall v\in H_0^1(\mathcal{B}_L^\omega), \quad a_\varepsilon^\omega\left(u_\varepsilon^\omega, v\right) \;=\; l_\varepsilon^\omega(v)\ , \tag{FV}$$

with

$$\forall u,v\in H_0^1(\mathcal{B}_L^\omega), \quad a_\varepsilon^\omega(u,v) \;:=\; \int_{\mathcal{B}_L^\omega}\nabla u\cdot\nabla\overline{v} \;-\; k^2\int_{\mathcal{B}_L^\omega} u\,\overline{v} - \;\langle\Lambda^k u,v\rangle_{\Sigma_L} \;+\; \mathrm{i}k\gamma\int_{\Sigma_0} u\,\overline{v}, \tag{12}$$

and

$$\forall v \in H^1_0(\mathcal{B}^\omega_L), \quad l^\omega_\varepsilon(v) \;:=\; \int_{\mathcal{B}^\omega_L} f\,\overline{v}.$$

The standard arguments to prove that the problem is of Fredholm type, *i.e.* that the operator associated with the sesquilinear form a^ω_ε is a compact perturbation of a coercive operator, do not apply here because, on contrary to a periodic distribution of particles, the problem cannot be reformulated in a bounded domain. Let us mention that in [10], the author proves that the scattering problem by a 2D plane with a Robin boundary condition (with $\gamma \in L^\infty$ such that $\mathcal{R}e(\gamma) \geq \eta > 0$) is well posed, using an integral equation reformulation of the problem . This result has been extended in [45] to the case of a rough $C^{1,1}$-surface. In the case of a more general rough Lipschitz surface, the well-posedness has been proved in [5]. The proof in this last paper can be directly extended to our setting.

Theorem 2. *[5, Lemma 15] Let ℓ^ω_ε be in $[H^1_0(\mathcal{B}^\omega_L)]'$,* i.e. *a continuous antilinear form on $H^1_0(\mathcal{B}^\omega_L)$. For all $\varepsilon > 0$ and ω, there exists a unique $u^\omega_\varepsilon \in H^1_0(\mathcal{B}^\omega_L)$ solution of*

$$\forall v \in H^1_0(\mathcal{B}^\omega_L), \quad a^\omega_\varepsilon\left(u^\omega_\varepsilon, v\right) \;=\; \ell^\omega_\varepsilon(v),$$

where a^ω_ε is defined in (12)*. Moreover, there exists $C > 0$ that depends only on k, L and γ such that*

$$\|u^\omega_\varepsilon\|_{H^1(\mathcal{B}^\omega_L)} \leq C\|\ell^\omega_\varepsilon\|_{[H^1_0(\mathcal{B}^\omega_L)]'}. \tag{13}$$

1.5 Random setting

We present in this section the different assumptions that we make on the random distribution of particles covering the object.

Let us consider the infinite normalized strip $\mathcal{L} := \mathbb{R}^{d-1} \times (0, h)$. Let $\{\boldsymbol{x}^\omega_n\}_{n\in\mathbb{N}}$ denote the point process corresponding to the centers of the particles. For $n \in \mathbb{N}$ we denote by $B(\boldsymbol{x}^\omega_n)$ the particle with radius 1 centered at $\boldsymbol{x}^\omega_n$. We suppose in the whole paper the following

- for all $n \in \mathbb{N}$, $B(\boldsymbol{x}^\omega_n) \in \mathcal{L}$,
- the particles lie at least at a distance of δ from one another and from Σ_0,
- $\{\boldsymbol{x}^\omega_n\}_{n\in\mathbb{N}}$ is stationary, *i.e.* the distribution law $\mathbb{P}$ of $\{\boldsymbol{x}^\omega_n\}_{n\in\mathbb{N}}$ is independent of the spatial variable.

Let Ω be the set of point processes in the strip such that the set of particles satisfy the previous assumptions. We equip Ω with its cylindrical σ-algebra $\mathcal{F}$.

We introduce the translation operator $\tau : \Omega \times \mathbb{R}^{d-1} \to \Omega$ defined as follows

$$\forall(\omega, \boldsymbol{y}_{\shortparallel}) \in \Omega \times \mathbb{R}^{d-1}, \quad \tau(\omega, \boldsymbol{y}_{\shortparallel}) = \omega', \qquad \text{where} \quad \boldsymbol{x}^{\omega'}_n = \boldsymbol{x}^\omega_n + \boldsymbol{y}_{\shortparallel}, \quad \text{for all } n \in \mathbb{N}.$$

Note that τ is measurable on $\Omega \times \mathbb{R}^{d-1}$. In what follows we denote $\tau_{\boldsymbol{y}_{\shortparallel}} := \tau(\cdot, \boldsymbol{y}_{\shortparallel})$ for $\boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}$.

The action $(\tau_{\boldsymbol{x}_{\shortparallel}})_{\boldsymbol{x}_{\shortparallel}\in\mathbb{R}^{d-1}}$ of the group $(\mathbb{R}^{d-1}, +)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ verifies then

$$\forall \boldsymbol{x}_{\shortparallel}, \boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}, \quad \tau_{\boldsymbol{x}_{\shortparallel}+\boldsymbol{y}_{\shortparallel}} = \tau_{\boldsymbol{x}_{\shortparallel}} \circ \tau_{\boldsymbol{y}_{\shortparallel}},$$

$$\forall \boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}, \forall A \in \mathcal{F}, \quad \mathbb{P}(\tau_{\boldsymbol{x}_{\shortparallel}} A) = \mathbb{P}(A)$$

where the last property means that τ preserves the law distribution and is due to the stationarity of the point process. Equipped with this action on $(\Omega, \mathcal{F}, \mathbb{P})$, we can now state the definition of a stationary process defined on $\Omega \times \mathbb{R}^{d-1}$.

Definition (Stationarity). *$f : \mathbb{R}^{d-1} \times \Omega \to \mathbb{R}^p$ is said to be stationary with respect to τ if*

$$\forall \boldsymbol{x}_{\shortparallel}, \boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}, \quad f^\omega(\boldsymbol{x}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}) = f^{\tau_{\boldsymbol{y}_{\shortparallel}}\omega}(\boldsymbol{x}_{\shortparallel}), \quad \omega \in \Omega.$$

It is easy to show that if f is stationary then

$$\mathbb{E}[f] = \mathbb{E}\left[\int_{\square_1} f(\boldsymbol{x}_{\shortparallel}) d\boldsymbol{x}_{\shortparallel}\right] = \left[\fint_{\square_R} f(\boldsymbol{x}_{\shortparallel}) d\boldsymbol{x}_{\shortparallel}\right],$$

for all $R > 0$. We suppose moreover that the action $(\tau_{\boldsymbol{x}_{\shortparallel}})_{\boldsymbol{x}_{\shortparallel}\in\mathbb{R}^{d-1}}$ is ergodic. Let us recall the definition of an ergodic action below.

Definition (Ergodicity). *$(\tau_{\boldsymbol{x}_{\shortparallel}})_{\boldsymbol{x}_{\shortparallel}\in\mathbb{R}^{d-1}}$ is said to be ergodic if any τ-invariant event has probability 0 or 1, i.e.*

$$\forall A \in \mathcal{F}, \quad (\forall \boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}, \tau_{\boldsymbol{x}_{\shortparallel}}^{-1} A = A) \implies (\mathbb{P}(A) \in \{0,1\}).$$

An easy way to grasp the concept of ergodicity is through Birkhoff ergodic Theorem.

Theorem 3 (Birkhoff ergodic Theorem). *Let f be a stationary process with respect to an ergodic action $(\tau_{\boldsymbol{x}_{\shortparallel}})_{\boldsymbol{x}_{\shortparallel}\in\mathbb{R}^{d-1}}$ such that $f \in L^p(\Omega, L^1_{loc}(\mathbb{R}^{d-1}))$. Then almost surely (a.s.) and in $L^p(\Omega)$ for any measurable bounded domain $K \subset \mathbb{R}^{d-1}$*

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \quad \fint_{K_t} f^\omega(\boldsymbol{y}_{\shortparallel} + \boldsymbol{x}_{\shortparallel}) \mathrm{d}\boldsymbol{y}_{\shortparallel} \xrightarrow[t\to\infty]{} \mathbb{E}[f],$$

where $K_t := \left\{\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1} | t^{-1}\boldsymbol{x}_{\shortparallel} \in K\right\}$ denotes the homothetic dilation with ratio $t > 0$ of K.

In other words, ergodicity implies that averaging with respect to the randomness corresponds to averaging spatially. We refer to [42, 36] for the proof of this theorem.

For $\boldsymbol{y} \in \mathbb{R}^d$ and $\omega \in \Omega$, let $R^\omega(\boldsymbol{y})$ be the distance from $\boldsymbol{y}$ to the nearest $\boldsymbol{x}_n^\omega$, *i.e.*

$$R^\omega : \begin{cases} \mathbb{R}^d \to \mathbb{R}_+ , \\ \boldsymbol{y} \mapsto \min_{n\in\mathbb{N}} |\boldsymbol{y} - \boldsymbol{x}_n^\omega|. \end{cases} \tag{14}$$

By stationarity of $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$, for all $y_d > 0$, $(\omega, \boldsymbol{y}_{\shortparallel}) \mapsto R^\omega(\boldsymbol{y}_{\shortparallel}, y_d)$ is stationary.

Hypothesis (H1). *There exists $m > 2d$ such that*

$$\sup_{y_d\in[0,h]} \mathbb{E}[R(\cdot, y_d)^m] < +\infty.$$

Intuitively Hypothesis (H1) implies that there exist arbitrarily large portions of $\mathcal{L}$ without particle but the probability to sample a large portion without particle is small. Classical hard-core point processes such as hard-core Poisson point process or random parking satisfy stationarity ergodicity and hypothesis (H1) (see [23]). Note that the necessity for m to be greater than $2d$ is important to obtain Lemma 7.

Let us define the weight

$$\mu^\omega(\boldsymbol{y}) := \left| \begin{array}{ll} R^\omega(\boldsymbol{y}_{\shortparallel}, y_d)^{-m} & \text{if } y_d \in [0,h], \\ (y_d^2 + R^\omega(\boldsymbol{y}_{\shortparallel}, h)^{2m})^{-1} & \text{if } y_d > h, \end{array} \right. \tag{15}$$

for $\boldsymbol{y} \in D^\omega$ and $\omega \in \Omega$. Assumption (H1) can then be rewritten as: there exists C independent of y_d such that

$$\forall y_d > 0, \qquad \mathbb{E}[\mu(\cdot, y_d)^{-1}] \le C y_d^2. \tag{16}$$

The set of normalized particles is denoted $\mathcal{P}^\omega \in \mathcal{L}$ and the set of scaled particles of size ε, $\mathcal{P}_\varepsilon^\omega = \varepsilon\mathcal{P}^\omega \subset \mathcal{L}^\varepsilon = \varepsilon\mathcal{L}$.

1.6 Well-posedness of the scattering problem in the random setting

For any Banach space V, we define $L^2(\Omega, V)$ the space of measurable applications $u : \Omega \to V$ such that $\mathbb{E}\left[\|u\|_V^2\right] < +\infty$, that we equip with the usual norm

$$\forall\, u \in L^2(\Omega, V), \quad \|u\|^2_{L^2(\Omega,V)} := \mathbb{E}\left[\|u\|_V^2\right].$$

Since the constant in Theorem 2 is independent of $\omega \in \Omega$, the following result can be easily deduced.

Corollary 4. *Let ℓ_ε be a continuous antilinear form on $L^2(\Omega, H^1_0(\mathcal{B}^\omega_L))$ then for all $\varepsilon > 0$, there exists a unique $u_\varepsilon \in L^2(\Omega, H^1_0(\mathcal{B}^\omega_L))$ such that for a.e. $\omega \in \Omega$, u^ω_ε is solution of* (FV). *Moreover, there exists $C > 0$ depending only on k, L and γ such that*

$$\mathbb{E}\left[\|u^\omega_\varepsilon\|^2_{H^1(\mathcal{B}^\omega_L)}\right] \;\le\; C\,\mathbb{E}\left[\|\ell_\varepsilon\|^2_{\left[H^1_0(\mathcal{B}^\omega_L)\right]'}\right]. \tag{17}$$

2 A formal asymptotic expansion

To build approximations of u^ω_ε at different orders of ε we use multiscale expansion techniques. The first step is to propose a multiscale aymptotic expansion as an Ansatz for u^ω_ε. We then inject this Ansatz in the equations verified by u^ω_ε and obtain formally the equations verified by the different terms of the expansion. This step is detailed in this section.

For a.e. $\omega \in \Omega$, we divide the upper half-space D^ω_ε in two regions :

- the domain $\mathcal{B}^\omega_{\varepsilon H} := \mathbb{R}^{d-1} \times (0, \varepsilon H) \setminus \overline{\mathcal{P}}^\omega_\varepsilon$ below the plane $\Sigma_{\varepsilon H} := \{x_d = \varepsilon H\}$, and
- the domain $\mathcal{B}^\infty_{\varepsilon H} := \mathbb{R}^{d-1} \times (\varepsilon H, +\infty)$ above $\Sigma_{\varepsilon H}$.

Here $H > h$ is a parameter, the only constraint being that the set of particles is always in the domain $\mathcal{B}^\infty_{\varepsilon H}$. We will explain in Section 5.3 how this parameter should be chosen.

We propose the following Ansatz for the solution u_ε. For all $\omega \in \Omega$,

$$\begin{aligned}
u^\omega_\varepsilon(\boldsymbol{x}) &= \sum_{n\in\mathbb{N}} \varepsilon^n\, U^{\omega,NF}_n\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}_{\shortparallel}}{\varepsilon}, \frac{x_d}{\varepsilon}\right) + \sum_{n\in\mathbb{N}} \varepsilon^n\, u^{\omega,FF}_n(\boldsymbol{x}), && \boldsymbol{x} \in \mathcal{B}^\infty_{\varepsilon H}\\
u^\omega_\varepsilon(\boldsymbol{x}) &= \sum_{n\in\mathbb{N}} \varepsilon^n\, U^{\omega,NF}_n\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}_{\shortparallel}}{\varepsilon}, \frac{x_d}{\varepsilon}\right), && \boldsymbol{x} \in \mathcal{B}^\omega_{\varepsilon H}.
\end{aligned} \tag{18}$$

where

- the so called far-field (FF) terms $u^{\omega,FF}_n$ defined only in $\mathcal{B}^\infty_{\varepsilon H}$ depend only on the macroscopic variable $\boldsymbol{x} := (\boldsymbol{x}_{\shortparallel}, x_d)$ and *a priori* on the random distribution;
- the so-called near-field (NF) terms $U^{\omega,NF}_n$ depend on $\boldsymbol{x}_{\shortparallel}$ and on a microscopic variable $\boldsymbol{y} := (\boldsymbol{y}_{\shortparallel}, y_d) = (\varepsilon^{-1}\boldsymbol{x}_{\shortparallel}, \varepsilon^{-1}x_d)$.

Given the stationarity of the particles distribution, we suppose that the near-field terms are stationary with respect to the tangential microscopic variables

$$\forall n \in \mathbb{N},\ \forall \boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1},\ \forall y_d \in \mathbb{R}^+, \quad (\omega, \boldsymbol{y}_{\shortparallel}) \mapsto U^{\omega,NF}_n(\boldsymbol{x}_{\shortparallel}; \boldsymbol{y}_{\shortparallel}, y_d) \quad \text{is stationary.} \tag{19}$$

We want for u^ω_ε to be well approximated by the far-field terms far away from the particles so we impose

$$\forall n \in \mathbb{N},\ \forall \boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1},\ \forall \boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}, \quad \lim_{y_d\to+\infty} U^{\omega,NF}_n(\boldsymbol{x}_{\shortparallel}; \boldsymbol{y}_{\shortparallel}, y_d) = 0 \text{ a.s.} \tag{20}$$

After injecting the Ansatz into (11), we obtain the following :

- For all $n \in \mathbb{N}$, the far-field term $u_n^{\omega,FF}$ should satisfy :

$$\left| \begin{array}{ll} -\Delta u_n^{\omega,FF} - k^2 u_n^{\omega,FF} = f\delta_{0,n} & \text{in} \quad \mathcal{B}_{\varepsilon H,L} \\ -\partial_{x_d} u_n^{\omega,FF} + \Lambda^k u_n^{\omega,FF} = 0 & \text{on} \quad \Sigma_L \end{array} \right. \tag{21}$$

  with $\delta_{0,n} = 1$ if $n = 0$, $\delta_{0,n} = 0$ otherwise and $L > \varepsilon H$ (that is always possible since the interface where the DtN condition is imposed is artificial). A boundary condition for $u_n^{\omega,FF}$ on $\Sigma_{\varepsilon H}$ is missing for now and will be derived by studying the near field terms.

- The near-field terms $U_n^{\omega,NF}$ are solutions of Laplace-type problems posed in an infinite half-space containing particles of size 1 following a stationary and ergodic distribution, $\boldsymbol{x}_{\shortparallel}$ playing the role of a parameter: for all $n \in \mathbb{N}$, for a.e. $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$, the near-field term $\boldsymbol{y} \mapsto U_n^{\omega,NF}(\boldsymbol{x}_{\shortparallel}; \boldsymbol{y})$ verifies

$$\left| \begin{array}{l} -\Delta_{\boldsymbol{y}} U_n^{\omega,NF} = 2\nabla_{\boldsymbol{x}_{\shortparallel}} \cdot \nabla_{\boldsymbol{y}_{\shortparallel}} U_{n-1}^{\omega,NF} + \Delta_{\boldsymbol{x}_{\shortparallel}} U_{n-2}^{\omega,NF} + k^2\, U_{n-2}^{\omega,NF} \quad \text{in} \quad \mathcal{B}_H^{\omega} \cup \mathcal{B}_H^{\infty}, \\ \partial_{y_d} U_n^{\omega,NF} = ik\gamma\, U_{n-1}^{\omega,NF} \text{ on } \Sigma_0, \quad \text{and} \quad U_n^{\omega,NF} = 0 \text{ on } \partial\mathcal{P}^{\omega}, \\ \Big[U_n^{\omega,NF}\Big]_{\Sigma_H} = -u_n^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) \text{ and } \Big[-\partial_{y_d} U_n^{\omega,NF}\Big]_{\Sigma_H} = \partial_{x_d} u_{n-1}^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}), \end{array} \right. \tag{22}$$

  where $u_{-1}^{\omega,FF} = 0$ and $U_{-1}^{\omega,NF} = U_{-2}^{\omega,NF} = 0$ by convention and

$$[v]_{\Sigma_H} := \Big(v|_{\mathcal{B}_H^{\infty}}\Big)\Big|_{\Sigma_H} - \Big(v|_{\mathcal{B}_H^{\omega}}\Big)\Big|_{\Sigma_H}.$$

The functional framework and the assumptions on the source terms for which these problems (22) set in an unbounded domain are well posed are not straight-forward. Neither is the condition for which the solution tends to 0 at infinity. The next two sections are devoted to the analysis of this problem. First in section 3 we study the well-posedness of the near field problems. Next in section 4, we focus on the behavior at infinity of the near-fields.

3 Well-posedness of the near-field problems

The near-field problems (22) can be rewritten as follows: for $n \geq 0$, we look for a stationary $U_n^{\omega,NF}$ solution for a.e. $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$ and $\omega \in \Omega$ of

$$\left| \begin{array}{l} -\Delta_{\boldsymbol{y}} U_n^{\omega,NF} = F_{n-2}^{\omega} + \nabla \cdot \boldsymbol{G}_{n-1}^{\omega} \quad \text{in} \quad \mathcal{B}_H^{\omega} \cup \mathcal{B}_H^{\infty}, \\ \partial_{y_d} U_n^{\omega,NF} = \Psi_{n-1}^{\omega} \text{ on } \Sigma_0, \quad \text{and} \quad U_n^{\omega,NF} = 0 \text{ on } \partial\mathcal{P}^{\omega}, \\ \Big[U_n^{\omega,NF}\Big]_{\Sigma_H} = \alpha_n^{\omega,D} \text{ and } \Big[-\partial_{y_d} U_n^{\omega,NF}\Big]_{\Sigma_H} = \alpha_{n-1}^{\omega,N}, \end{array} \right. \tag{23}$$

where $F_{n-2}(\boldsymbol{x}_{\shortparallel};\ \cdot,\ y_d)$, $\boldsymbol{G}_{n-1}(\boldsymbol{x}_{\shortparallel};\ \cdot,\ y_d)$ and $H_{n-1}(\boldsymbol{x}_{\shortparallel};\ \cdot)$ are stationary for any $y_d > 0$. $\mathbf{F}_{n-2}^{\omega}(\boldsymbol{x}_{\shortparallel};\ \cdot\ )$ depends on the previous term $U_{n-2}^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\ \cdot\ )$ and $\boldsymbol{G}_{n-1}^{\omega}(\boldsymbol{x}_{\shortparallel};\ \cdot,\ y_d)$ and $\Psi_{n-1}^{\omega}(\boldsymbol{x}_{\shortparallel};\ \cdot)$ on $U_{n-1}^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\ \cdot\ )$. $\alpha_n^{\omega,D}(\boldsymbol{x}_{\shortparallel})$ and $\alpha_{n-1}^{\omega,N}(\boldsymbol{x}_{\shortparallel})$ depend respectively on the far field terms $u_n^{\omega,NF}(\boldsymbol{x}_{\shortparallel})$ and $\partial_{x_d} u_{n-1}^{\omega,FF}(\boldsymbol{x}_{\shortparallel})$. Furthermore, $\alpha_n^{\omega,D}(\boldsymbol{x}_{\shortparallel})$ and $\alpha_{n-1}^{\omega,FF}(\boldsymbol{x}_{\shortparallel})$ are functions independent of $\boldsymbol{y}_{\shortparallel}$.

In (23) $\boldsymbol{x}_{\shortparallel}$ plays the role of a parameter. For convenience we will omit the dependency on $\boldsymbol{x}_{\shortparallel}$ in what follows. After lifting the jump of the solution across Σ_H we are interested in the well-posedness of the problems verified by $\widetilde{U}_n^{\omega,NF} := U_n^{\omega,NF} - \alpha_n^{\omega,D}\, \mathbb{1}_{\mathcal{B}_H^{\infty}}$, *i.e.* problems of the form

$$\left| \begin{array}{l} -\Delta_{\boldsymbol{y}} \widetilde{U}^{\omega} = F^{\omega} + \nabla \cdot \boldsymbol{G}^{\omega} \quad \text{in} \quad \mathcal{B}_H^{\omega} \cup \mathcal{B}_H^{\infty}, \\ -\partial_{y_d} \widetilde{U}^{\omega} = \Psi^{\omega} \text{ on } \Sigma_0, \quad \text{and} \quad \widetilde{U}^{\omega} = 0 \text{ on } \partial\mathcal{P}^{\omega}, \\ \Big[\widetilde{U}^{\omega}\Big]_H = 0 \quad \text{and} \quad \Big[-\partial_{y_d} \widetilde{U}^{\omega}\Big]_H = \alpha^{\omega,N}, \end{array} \right. \tag{24}$$

for given sources F, G, Ψ and α^N. Since one can find arbitrary large boxes $\square_R \times [0, H]$ without Dirichlet particles and since $\mathcal{B}_H^{\infty}$ is also unbounded im the y_d-direction, we cannot find a solution in the classical L^2-framework. Instead we consider weighted Sobolev spaces with weight μ previously introduced in (15).

Let us introduce the following weighted functional space

$$\begin{aligned}\mathcal{W}_0(D) := \Big\{&\text{a.s. } v^\omega \in H^1_{loc}(D^\omega),\ \mathbb{1}_D v(\cdot, y_d) \text{ stationary for any } y_d > 0,\\ &\text{a.s. } v^\omega = 0 \text{ on } \partial\mathcal{P}^\omega,\ \mathbb{E}\left[\int_{\square_1}\int_0^{+\infty} \mathbb{1}_D\ (\mu(\boldsymbol{y})|v(\boldsymbol{y})|^2 + |\nabla v(\boldsymbol{y})|^2)\mathrm{d}\boldsymbol{y}\right] < +\infty\Big\}, \quad (25)\end{aligned}$$

equipped with the norm

$$\forall v \in \mathcal{W}_0(D), \quad \|v\|^2_{\mathcal{W}_0(D)} := \mathbb{E}\left[\int_{\square_1}\int_0^{+\infty} \mathbb{1}_D\ (\mu(\boldsymbol{y})|v(\boldsymbol{y})|^2 + |\nabla v(\boldsymbol{y})|^2)\mathrm{d}\boldsymbol{y}\right].$$

We introduce the following functional spaces adapted to our source terms

$$\begin{aligned}\mathcal{L}^2(D) := \Big\{&\text{a.s. } v^\omega \in L^2_{loc}(D^\omega),\ \mathbb{1}_D v(\cdot, y_d) \text{ stationary for any } y_d > 0,\\ &\mathbb{E}\left[\int_{\square_1}\int_0^{+\infty} \mathbb{1}_D\ |v|^2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right] < +\infty\Big\},\end{aligned}$$

and for $t \geq 0$, $\mathcal{L}^2(\Sigma_t) := \Big\{\varphi \in L^2_{loc}(\Sigma_t, L^2(\Omega)), \varphi \text{ stationary},\ \mathbb{E}\left[|\varphi|^2\right] < +\infty\Big\}$. For all $v \in \mathcal{L}^2(D)$, let us denote $\|v\|^2_{\mathcal{L}^2(D)} := \mathbb{E}\left[\int_{\square_1}\int_0^{+\infty} \mathbb{1}_D\ |v|^2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right]$.

Remark 1. *Note that $\mathcal{W}_0(D) \not\subset \mathcal{L}^2(D)$.*

Let us write the variational formulation associated to (24) in $\mathcal{W}_0(D)$. We look for $\widetilde{U}$ in $\mathcal{W}_0(D)$ such that for any $V \in \mathcal{W}_0(D)$,

$$a(\widetilde{U}, V) = l(V) \tag{26}$$

where $a : \mathcal{W}_0(D) \times \mathcal{W}_0(D) \to \mathbb{C}$ is the sesquilinear form defined for all $U, V \in \mathcal{W}_0(D)$ by

$$a(U, V) := \mathbb{E}\left[\int_{\square_1}\int_0^{L} \mathbb{1}_D \Big[\nabla U(\boldsymbol{y}) \cdot \nabla \overline{V}(\boldsymbol{y})\Big]\, \mathrm{d}\boldsymbol{y}\right],$$

and $l : \mathcal{W}_0(D) \to \mathbb{C}$ the antilinear form defined for all $V \in \mathcal{W}_0(D)$ by

$$l(V) := \mathbb{E}\left[\int_{\square_1}\int_0^{L} \mathbb{1}_D(F^\omega \overline{V}^\omega + \boldsymbol{G}^\omega \cdot \nabla \overline{V}^\omega)(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} + \int_{\square_1} (\Psi^\omega\ \overline{V^\omega}|_{\Sigma_0} + \alpha^N\ \overline{V}^\omega|_{\Sigma_H})(\boldsymbol{y}_{\shortparallel})\, \mathrm{d}\boldsymbol{y}_{\shortparallel}\right].$$

The objective of the next two subsections is to show the following proposition.

Proposition 5. *Suppose that $\mathcal{P}$ verifies hypothesis (H1) then for all $\mu^{-\frac{1}{2}}F \in \mathcal{L}^2(D)$, $\boldsymbol{G} \in \mathcal{L}^2(D)^n$, $\mu^{-\frac{1}{2}}\Psi \in \mathcal{L}^2(\Sigma_0)$ and $\mu^{-\frac{1}{2}}\alpha^N \in \mathcal{L}^2(\Sigma_H)$, Problem* (26) *is well posed in $\mathcal{W}_0(D)$. Moreover the unique solution $\widetilde{U} \in \mathcal{W}_0(D)$ verifies*

$$\|\widetilde{U}\|_{\mathcal{W}_0(D)} \lesssim \|\mu^{-\frac{1}{2}}\Psi\|_{\mathcal{L}^2(\Sigma_0)} + \|\mu^{-\frac{1}{2}}\alpha^N\|_{\mathcal{L}^2(\Sigma_H)} + \|\mu^{-\frac{1}{2}}F\|_{\mathcal{L}^2(D)} + \|G\|_{\mathcal{L}^2(D)}. \tag{27}$$

Remark 2. *One can show that the unique solution in $\mathcal{W}_0(D)$ of Problem* (26) *satisfies almost surely Problem* (24) *in the sense of distributions. That is why, in what follows, we designate the unique solution of* (24) *in $\mathcal{W}_0(D)$ the variational solution.*

3.1 Functional inequalities in $\mathcal{W}_0(D)$

We start by proving that $(\mathcal{W}_0(D), \|\cdot\|_{\mathcal{W}_0(D)})$ is a Hilbert space.

Lemma 6. *Assume that $\mathcal{P}$ verifies hypothesis (H1) then $(\mathcal{W}_0(D), \|\cdot\|_{\mathcal{W}_0(D)})$ is a Hilbert space.*

Proof. It is easy to see that $\mathcal{W}_0(D)$ is a Pre-Hilbert space. Let us prove completeness. For all $R > 0$ and $\omega \in \Omega$, let $D_R^\omega := D^\omega \cap (\square_R \times (0, +\infty))$. Since a.s. $\mu^{-1} \in L^1_{loc}(D_R^\omega)$, the weighted Sobolev space

$$W_0(D_R^\omega) := \left\{ v \in H^1_{loc}(D_R^\omega), v = 0 \text{ on } \partial\mathcal{P}^\omega, \int_{\square_R} \int_0^{+\infty} \mathbb{1}_D \left(\mu(\boldsymbol{y})|v(\boldsymbol{y})|^2 + |\nabla v(\boldsymbol{y})|^2\right) \mathrm{d}\boldsymbol{y} < +\infty \right\}$$

equipped with the norm

$$\forall v \in W_0(D_R^\omega), \quad \|v\|_{W_0(D_R^\omega)} := \int_{\square_R} \int_0^{+\infty} \mathbb{1}_D \left(\mu(\boldsymbol{y})|v(\boldsymbol{y})|^2 + |\nabla v(\boldsymbol{y})|^2\right) \mathrm{d}\boldsymbol{y}$$

is a Hilbert space [38, Theorem 1.11]. Therefore $(L^2(\Omega, W_0(D_R^\omega)), \|\cdot\|_{L^2(\Omega, W_0(D_R^\omega))})$ is also a Hilbert space.

Let $(v_n)_{n\in n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{W}_0(D)$. By stationarity $(v_n)_{n\in n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, W_0(D_R^\omega))$ for any $R > 0$. As $\lim\limits_{R\to\infty} \uparrow \square_R = \mathbb{R}^{d-1}$, there exists $v \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega))$ such that for all $R > 0$, $v_n \xrightarrow[n\to\infty]{} v$ in $L^2(\Omega, W_0(D_R^\omega))$.

It remains to show that $\mathbb{1}_D v(\cdot, x_d)$ is stationary for all $x_d > 0$. Let $n \in \mathbb{N}$. By stationarity of $\mathbb{1}_D v_n(\cdot, x_d)$, we have for a.e. $\omega \in \Omega$, $\boldsymbol{x} \in \mathbb{R}^{d-1} \times (0, +\infty)$ and $\boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}$

$$\mathbb{E}\left[\left|\mathbb{1}_D v_n^\omega((\boldsymbol{x}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}, x_d)) - \mathbb{1}_D v_n^{\tau_{\boldsymbol{y}_{\shortparallel}}\omega}(\boldsymbol{x})\right|\right] = 0.$$

We would like to pass to the limit in this equality but we only have that $v_n \to v$ in $L^2(\Omega, W_0(D_R^\omega))$. Let R such that $\boldsymbol{x}_{\shortparallel}, \boldsymbol{x}_{\shortparallel} + \boldsymbol{y}_{\shortparallel} \in \square_R$. Using (16) and the stationarity of μ we obtain by Cauchy-Schwarz inequality

$$\begin{aligned}
\mathbb{E}\left[\int_{\square_R}\int_0^{x_d} |\mathbb{1}_D v^\omega((\tilde{\boldsymbol{x}}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}, \tilde{x}_d)) - \mathbb{1}_D v^{\tau_{\boldsymbol{y}_{\shortparallel}}\omega}(\tilde{\boldsymbol{x}})|\, d\tilde{\boldsymbol{x}}\right] &\leq \mathbb{E}\left[\int_{\square_R}\int_0^{x_d} \mu^{-1}(\tilde{\boldsymbol{x}}) d\tilde{\boldsymbol{x}}\right] \\
\Bigg(\mathbb{E}\left[\int_{\square_R}\int_0^{x_d} \mu^\omega((\tilde{\boldsymbol{x}}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}, \tilde{x}_d))\, |\mathbb{1}_D v_n^\omega((\tilde{\boldsymbol{x}}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}, \tilde{x}_d)) - \mathbb{1}_D v^\omega((\tilde{\boldsymbol{x}}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}, \tilde{x}_d))|^2\, d\tilde{\boldsymbol{x}}\right] \\
+\mathbb{E}\left[\int_{\square_R}\int_0^{x_d} \mu^\omega((\tilde{\boldsymbol{x}}_{\shortparallel} + \boldsymbol{y}_{\shortparallel}, \tilde{x}_d)) \left|\mathbb{1}_D v_n^{\tau_{\boldsymbol{y}_{\shortparallel}}\omega}(\tilde{\boldsymbol{x}}) - \mathbb{1}_D v^{\tau_{\boldsymbol{y}_{\shortparallel}}\omega}(\tilde{\boldsymbol{x}})\right|^2 d\tilde{\boldsymbol{x}}\right]\Bigg). \quad (28)
\end{aligned}$$

Since $v_n \to v$ in $L^2(\Omega, W_0(D_R^\omega))$, we deduce by passing to the limit in this last inequality that $\mathbb{1}_D v(\cdot, x_d)$ is stationary for all $x_d > 0$. □

In order to show well-posedness of (24) we now derive a trace inequality for functions in $\mathcal{W}_0(D)$.

Lemma 7 (Weighted trace Theorem). *Let $\mathcal{P}$ verify Hypothesis (H1). Let $\alpha \in \{0\} \cup [h, +\infty)$. There exists C_α such that for any $v \in \mathcal{W}_0(D)$*

$$\mathbb{E}\left[\int_{\square_1} \mu^\omega(\boldsymbol{x}_{\shortparallel}, \alpha) \left|v^\omega\big|_{\Sigma_\alpha}\right|^2 (\boldsymbol{y}_{\shortparallel}) \mathrm{d}\boldsymbol{y}_{\shortparallel}\right] \leq C_\alpha \|v\|^2_{\mathcal{W}_0(D)}.$$

Proof. We first prove the result for $\alpha \in \{0, h\}$. Let $v \in \mathcal{W}_0(D)$. Let us consider the Voronoï diagram $(V_n^\omega)_{n\in\mathbb{N}}$ in $\mathbb{R}^{d-1} \times [0, h]$ associated with the point process $\{\boldsymbol{x}_n^\omega\}_{n\in\mathbb{N}}$ defined by the sets

$$V_n^\omega := \left\{ \boldsymbol{y} \in \mathbb{R}^{d-1} \times [0, h] \text{ such that } |\boldsymbol{y} - \boldsymbol{x}_n^\omega| = \min_{j\in\mathbb{N}} \left|\boldsymbol{y} - \boldsymbol{x}_j^\omega\right| \right\}, \ n \in \mathbb{N}.$$

For $\omega \in \Omega, n \in \mathbb{N}$ let $\Sigma_n^\omega(\alpha)$ denote the intersection of the hyperplane $\{y_d = \alpha\}$ with the Voronoï cell V_n^ω . Let $(\rho, \boldsymbol{\theta})$ be the spherical coordinate system centered at $\boldsymbol{x}_n^\omega$. Since V_n^ω is convex, $\Sigma_n^\omega(\alpha)$ can be parametrized as $\Sigma_n^\omega(\alpha) := \left\{(r(\boldsymbol{\theta}),\boldsymbol{\theta}), r(\boldsymbol{\theta}) \in [1,+\infty), \boldsymbol{\theta} \in [\boldsymbol{\theta}_0,\boldsymbol{\theta}_1] \subset \mathbb{S}^{d-1}\right\}$. Since $v^\omega \in H^1(V_n^\omega)$ with $v^\omega = 0$ on $\partial B(\boldsymbol{x}_n^\omega)$, there exists $(v_p^\omega)_{p\in\mathbb{N}}$ in $\mathcal{C}^\infty(V_n^\omega)^{\mathbb{N}}$ with $v_p^\omega = 0$ on $\partial B(\boldsymbol{x}_n^\omega)$ such that $v_p^\omega \to v^\omega$ in $H^1(V_n^\omega)$. We have for all p,

$$\forall \boldsymbol{\theta} \in [\boldsymbol{\theta}_0,\boldsymbol{\theta}_1], \quad \mathbb{1}_{D^\omega} v_p^\omega(r(\boldsymbol{\theta}),\boldsymbol{\theta}) = \int_1^{r(\boldsymbol{\theta})} \mathbb{1}_{D^\omega} \partial_r v_p^\omega(s,\boldsymbol{\theta}) ds.$$

We deduce for any non-negative function $\tilde{\mu}$ on $(1,+\infty)$, for all $\boldsymbol{\theta} \in [\boldsymbol{\theta}_0,\boldsymbol{\theta}_1]$

$$\mathbb{1}_{D^\omega} |v_p^\omega(r(\boldsymbol{\theta}),\boldsymbol{\theta})|^2 \tilde{\mu}(r(\boldsymbol{\theta})) r(\boldsymbol{\theta})^{d-1} \leq \int_1^{r(\boldsymbol{\theta})} \mathbb{1}_{D^\omega} |\partial_r v_p^\omega(s,\boldsymbol{\theta})|^2 s^{d-1} ds \left(\int_1^{r(\boldsymbol{\theta})} s^{1-d} ds\right) \tilde{\mu}(r(\boldsymbol{\theta})) r(\boldsymbol{\theta})^{d-1}.$$

For $\tilde{\mu}(\rho) := \left(\int_1^\rho s^{1-d} ds\right)^{-1} \rho^{1-d}$, we get after integrating with respect to $\boldsymbol{\theta}$

$$\int_{\Sigma_n^\omega(\alpha)} \mathbb{1}_{D^\omega} \tilde{\mu}^\omega(\boldsymbol{y}_{\shortparallel},\alpha) |v_p^\omega(\boldsymbol{y}_{\shortparallel},\alpha)|^2 \,\mathrm{d}\boldsymbol{y}_{\shortparallel} \leq \int_{\boldsymbol{\theta}_0}^{\boldsymbol{\theta}_1} \int_1^{r(\boldsymbol{\theta})} \mathbb{1}_{D^\omega} |\partial_r v_p^\omega(s,\boldsymbol{\theta})|^2 \, s^{d-1} \mathrm{d}s\mathrm{d}\boldsymbol{\theta}.$$

We can now pass to the limit when $p \to +\infty$ to obtain the first inequality of

$$\begin{aligned} \int_{\Sigma_n^\omega(\alpha)} \mathbb{1}_{D^\omega} \tilde{\mu}^\omega(\boldsymbol{y}_{\shortparallel},\alpha) |v^\omega|_{\Sigma_\alpha}|^2(\boldsymbol{y}_{\shortparallel}) \,\mathrm{d}\boldsymbol{y}_{\shortparallel} &\leq \int_{\boldsymbol{\theta}_0}^{\boldsymbol{\theta}_1} \int_1^{r(\boldsymbol{\theta})} \mathbb{1}_{D^\omega} |\partial_r v^\omega(s,\boldsymbol{\theta})|^2 \, s^{d-1} \mathrm{d}s\mathrm{d}\boldsymbol{\theta} \\ &\lesssim \int_{\Sigma_n(\alpha)\times(0,h)} \mathbb{1}_{D^\omega} |\nabla v^\omega|^2(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}. \end{aligned} \tag{29}$$

By definition (15), for $\boldsymbol{y} \in V_n^\omega$, $\mu^\omega(\boldsymbol{y}) = r(\boldsymbol{\theta})^{-m}$ with $m > 2d$. Therefore $\boldsymbol{y} \mapsto \mu^\omega/\tilde{\mu}^\omega(\boldsymbol{y})$ is bounded by monotonicity and there exists a constant independent of $\boldsymbol{y} \in V_n^\omega$ such that $\mu^\omega(\boldsymbol{y}) \lesssim \tilde{\mu}^\omega(\boldsymbol{y})$. Consequently (29) holds replacing $\tilde{\mu}$ by μ. For any $L > 0$, we sum over all $\Sigma_n^\omega(\alpha)$ such that $\boldsymbol{x}_n^\omega \in \square_L \times (0,h)$ to obtain

$$\int_{\bigcup_{n\in N_L} \Sigma_n^\omega(\alpha)} \mathbb{1}_{D^\omega} \; \mu^\omega(\boldsymbol{y}_{\shortparallel},\alpha) |v^\omega|_{\Sigma_\alpha}|^2(\boldsymbol{y}_{\shortparallel}) \,\mathrm{d}\boldsymbol{y}_{\shortparallel} \lesssim \int_{\bigcup_{n\in N_L} \Sigma_n^\omega(\alpha)\times(0,h)} \mathbb{1}_{D^\omega} \; |\nabla v^\omega|^2(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y},$$

where $N_L := \left\{n \in \mathbb{N} \text{ such that } \boldsymbol{y}_n^\omega \in \square_L \times (0,h)\right\}$. Finally we apply Birkhoff Theorem (Theorem 3) to conclude.

Let us consider now $\alpha > h$. We write for $v \in \mathcal{C}^\infty(\overline{\square_1 \times (h,\alpha)})$

$$\forall \boldsymbol{y}_{\shortparallel} \in \square_1, \quad v(\boldsymbol{y}_{\shortparallel},\alpha) = v(\boldsymbol{y}_{\shortparallel},h) + \int_h^\alpha \partial_{y_d} v(\boldsymbol{y}) \mathrm{d}y_d.$$

By Cauchy-Schwarz inequality we get

$$\forall \boldsymbol{y}_{\shortparallel} \in \square_1, \quad |v(\boldsymbol{y}_{\shortparallel},\alpha)|^2 \leq 2\left(|v(\boldsymbol{y}_{\shortparallel},h)|^2 + (\alpha-h)\int_h^\alpha |\partial_{y_d} v(\boldsymbol{y})|^2 \mathrm{d}y_d\right).$$

We integrate over $\square_1$ to obtain

$$\begin{aligned} \int_{\square_1} \mu^\omega(\boldsymbol{y}_{\shortparallel},\alpha) \left| v|_{\Sigma_\alpha} \right|^2 (\boldsymbol{y}_{\shortparallel}) d\boldsymbol{y}_{\shortparallel} \leq 2 \Bigg((\alpha-h) \int_h^\alpha \int_{\square_1} \mu^\omega(\boldsymbol{y}_{\shortparallel},\alpha) |\partial_{y_d} v^\omega|^2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ + \int_{\square_1} \mu^\omega(\boldsymbol{y}_{\shortparallel},\alpha)| \left| v|_{\Sigma_h} \right|^2 (\boldsymbol{y}_{\shortparallel}) \mathrm{d}\boldsymbol{y}_{\shortparallel} \Bigg). \end{aligned}$$

By density, this equality also holds for $v \in \mathcal{W}_0(D)$. On one hand, by definition (15) and since $\alpha > h$, we have $\mu^\omega(\boldsymbol{y}_{\shortparallel},\alpha) \leq 1$ for all $\boldsymbol{y}_{\shortparallel} \in \square_1$ so we deduce for the first term of the r.h.s.

$$\mathbb{E}\left[\int_h^\alpha \int_{\square_1} \mu^\omega(\boldsymbol{y}_{\shortparallel},\alpha) |\partial_{y_d} v|^2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right] \leq \mathbb{E}\left[\int_h^\alpha \int_{\square_1} \mathbb{1}_{D^\omega} |\nabla v^\omega|^2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right]. \tag{30}$$

On the other hand using the fact that $\mu^\omega(\boldsymbol{y}_{\shortparallel}, \alpha) \leq \mu^\omega(\boldsymbol{y}_{\shortparallel}, h)$ for all $\boldsymbol{y}_{\shortparallel} \in \square_1$ and the first part of the proof, we get for the second term of the r.h.s.

$$\mathbb{E}\left[\int_{\square_1} \mu^\omega(\boldsymbol{y}_{\shortparallel}, \alpha) | \left| v^\omega\right|_{\Sigma_h}\Big|^2 (\boldsymbol{y}_{\shortparallel}) \mathrm{d}\boldsymbol{y}_{\shortparallel}\right] \lesssim \mathbb{E}\left[\int_0^h \int_{\square_1} \mathbb{1}_{D^\omega} |\nabla v^\omega|^2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right]. \tag{31}$$

Combining (30) and (31) we obtain the desired result.

□

Let us now derive a random Hardy-type inequality in $\mathcal{W}_0(D)$.

Proposition 8 (Random Hardy's inequality)**.** *Suppose that $\mathcal{P}$ verifies hypothesis (H1). Then*

$$\forall v \in \mathcal{W}_0(D), \quad \mathbb{E}\left[\int_{\square_1} \int_0^{+\infty} \mathbb{1}_D\, \mu(\boldsymbol{y}) |v|^2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\right] \lesssim \mathbb{E}\left[\int_{\square_1} \int_0^{+\infty} \mathbb{1}_D\, |\nabla v|^2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\right]. \tag{32}$$

Proof. We first prove that for all $v \in \mathcal{W}_0(D)$

$$\mathbb{E}\left[\int_{\square_1} \int_0^h \mathbb{1}_D\, \mu(\boldsymbol{y}) |v|^2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right] \lesssim \mathbb{E}\left[\int_{\square_1} \int_0^h \mathbb{1}_D\, |\nabla v|^2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right]. \tag{33}$$

Let us consider the Voronoï diagram $(V_n^\omega)_{n\in\mathbb{N}}$ in $\mathbb{R}^{d-1} \times [0, h]$ associated with the point process $\{\boldsymbol{x}_n^\omega\}_{n\in\mathbb{N}}$. Let $n \in \mathbb{N}$ and $\omega \in \Omega$. We first prove that the following inequality holds a.s. in V_n^ω

$$\int_{V_n^\omega} \mathbb{1}_{D^\omega} \mu^\omega(\boldsymbol{y}) |v^\omega|^2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \lesssim \int_{V_n^\omega} \mathbb{1}_{D^\omega} |\nabla v^\omega|^2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}. \tag{34}$$

The proof of (34) relies on similar arguments as in the proof of Lemma 1.3 of [8]. We reproduce it here for completeness. Let $(\rho, \boldsymbol{\theta})$ be the spherical coordinate system centered at $\boldsymbol{x}_n^\omega$. Since V_n^ω is convex, it can be parametrized as $V_n^\omega := \left\{(\rho, \boldsymbol{\theta}), \rho \in (1, R(\boldsymbol{\theta})], \boldsymbol{\theta} \in \mathbb{S}^{d-1}\right\}$. Let $v^\omega \in \mathcal{C}^\infty(\overline{V_n^\omega})$ such that $v^\omega = 0$ on $\partial B(\boldsymbol{x}_n^\omega)$. We have for all $(\rho, \boldsymbol{\theta}) \in V_n^\omega$

$$v^\omega(\rho, \boldsymbol{\theta}) = \int_1^\rho \partial_r v^\omega(s, \boldsymbol{\theta}) \mathrm{d}s.$$

Therefore we derive by Cauchy-Schwarz inequality

$$|v^\omega(\rho, \boldsymbol{\theta})|^2 \leq \int_1^\rho |\partial_r v^\omega(s, \boldsymbol{\theta})|^2 s^{d-1} \mathrm{d}s \int_1^\rho s^{1-d} \mathrm{d}s, \tag{35}$$

and for $\tilde{\mu}$ a non-negative function on $(1, +\infty)$

$$\int_1^{R(\boldsymbol{\theta})} \tilde{\mu}(\rho) |v^\omega(\rho, \boldsymbol{\theta})|^2 \rho^{d-1} \mathrm{d}\rho \leq \int_1^{R(\boldsymbol{\theta})} \int_1^\rho |\partial_r v^\omega(s, \boldsymbol{\theta})|^2 s^{d-1} \mathrm{d}s \left(\int_1^\rho s^{1-d} \mathrm{d}s\right) \tilde{\mu}(\rho) \rho^{d-1} \mathrm{d}\rho.$$

By Fubini-Tonelli's Theorem we get

$$\int_1^{R(\boldsymbol{\theta})} \tilde{\mu}(\rho) |v^\omega(\rho, \boldsymbol{\theta})|^2 \rho^{d-1} \mathrm{d}\rho \leq \int_1^{R(\boldsymbol{\theta})} |\partial_r v^\omega(s, \boldsymbol{\theta})|^2 s^{d-1} \left(\int_s^{R(\boldsymbol{\theta})} \left(\int_1^\rho s^{1-d} \mathrm{d}s\right) \tilde{\mu}(\rho) \rho^{d-1} \mathrm{d}\rho\right) \mathrm{d}s.$$

For $\tilde{\mu}(\rho) := \rho^{-m}$ with $m > d$, $\displaystyle\int_1^{+\infty} \left(\int_1^\rho s^{1-d} ds\right) \tilde{\mu}(\rho) \rho^{d-1} d\rho < +\infty$. We integrate with respect to $\boldsymbol{\theta}$ to get

$$\int_{V_n^\omega} \mathbb{1}_{D^\omega} \tilde{\mu}^\omega(\boldsymbol{y}) |v^\omega|^2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \lesssim \int_{V_n^\omega} \mathbb{1}_{D^\omega} |\nabla v^\omega|^2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}.$$

To obtain (34) for $v \in \mathcal{W}_0(D)$, we use a density argument and that for $\boldsymbol{y} \in V_n^\omega$, $\mu^\omega(\boldsymbol{y}) \leq \tilde{\mu}^\omega(\boldsymbol{y})$. We conclude using Birkhoff Theorem (Theorem 3).

Let us now prove (32). By integrating by parts, we obtain

$$\begin{aligned}\int_h^{+\infty}\int_{\square_1}\mu^\omega(\boldsymbol{y})|v^\omega|^2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} &\leq \frac{1}{2}\int_h^{+\infty}\int_{\square_1}\frac{|v^\omega|^2(\boldsymbol{y})}{(y_d+(R^\omega(\boldsymbol{y}_{\shortparallel},h))^m)^2}\,\mathrm{d}\boldsymbol{y},\\ &\leq \int_h^{+\infty}\int_{\square_1}\frac{\overline{v^\omega}\partial_{y_d}v^\omega(\boldsymbol{y})}{y_d+(R^\omega(\boldsymbol{y}_{\shortparallel},h))^m}\mathrm{d}\boldsymbol{y}+\int_{\square_1}\frac{\left|v^\omega|_{\Sigma_h}\right|^2(\boldsymbol{y}_{\shortparallel})}{h+(R^\omega(\boldsymbol{y}_{\shortparallel},h))^m}\mathrm{d}\boldsymbol{y}_{\shortparallel}.\end{aligned}$$

We deduce the following by applying Cauchy-Schwarz inequality and using the weighted trace theorem (Lemma 7)

$$\begin{aligned}\mathbb{E}\left[\int_h^{+\infty}\int_{\square_1}\mu^\omega(\boldsymbol{y})|v^\omega|^2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right] \leq \mathbb{E}\left[\int_h^{+\infty}\int_{\square_1}\frac{|v^\omega(\boldsymbol{y})|^2}{(y_d+(R^\omega)^m)^2}\mathrm{d}\boldsymbol{y}\right]^{\frac{1}{2}}\mathbb{E}\left[\int_h^{+\infty}\int_{\square_1}|\partial_{y_d}v^\omega|^2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]^{\frac{1}{2}}\\ +\mathbb{E}\left[\int_{\square_1}\int_0^{+\infty}\mathbb{1}_D\,|\nabla v|^2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right].\end{aligned}$$

(32) is then a consequence of Young's inequality. ☐

Remark 3. *Note that using similar arguments we can prove that*

$$\int_{D^\omega}\mu^\omega(\boldsymbol{y})|v^\omega|^2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\lesssim\int_{D^\omega}|\nabla v^\omega|^2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{36}$$

where for $v^\omega\in W_0(D^\omega)$ for $\omega\in\Omega$, where

$$W_0(D^\omega):=\left\{v\in H^1_{loc}(D^\omega), v=0\,on\,\partial\mathcal{P}^\omega, \int_{\mathbb{R}^{d-1}}\int_0^{+\infty}\mathbb{1}_D\left(\mu(\boldsymbol{y})|v|^2(\boldsymbol{y})+|\nabla v|^2(\boldsymbol{y})\right)\,\mathrm{d}\boldsymbol{y}<+\infty\right\}.$$

3.2 Proof of proposition 5

The continuity of the sesquilinear form a given by (3) is straightforward. The coercivity of a is guaranteed by the weighted Hardy's inequality (32). The continuity of the anti-linear form l given by (3) can be deduced from Cauchy-Schwarz inequality and the weighted trace lemma (Lemma 7). Indeed for all $V\in\mathcal{W}_0(D)$

$$\begin{aligned}\mathbb{E}\left[\int_{\square_1}\int_0^L\mathbb{1}_D(F^\omega\overline{V}^\omega+\boldsymbol{G}^\omega\cdot\nabla\overline{V}^\omega)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right] &\leq \|\mu^{-\frac{1}{2}}F\|_{\mathcal{L}^2(D)}\|\mu^{\frac{1}{2}}V\|_{\mathcal{L}^2(D)}+\|G\|_{\mathcal{L}^2(D)}\|\nabla V\|_{\mathcal{L}^2(D)},\\ &\leq\left(\|\mu^{-\frac{1}{2}}F\|_{\mathcal{L}^2(D)}+\|G\|_{\mathcal{L}^2(D)}\right)\|V\|_{\mathcal{W}_0(D)},\end{aligned}$$

and

$$\mathbb{E}\left[\int_{\square_1}(\Psi^\omega\,\overline{V}^\omega|_{\Sigma_0}+\alpha^{\omega,N}\,\overline{V}^\omega(\cdot,\,H))\right]\leq(\|\mu^{-\frac{1}{2}}\Psi\|_{\mathcal{L}^2(\Sigma_0)}+\|\mu^{-\frac{1}{2}}\alpha^N\|_{\mathcal{L}^2(\Sigma_H)}\|V\|_{\mathcal{W}_0(D)}. \tag{37}$$

By Lax-Milgram Theorem, problem (24) is well posed in $\mathcal{W}_0(D)$. The estimate (27) is a direct consequence of the coercivity of a and estimate (37).

Finally, using similar arguments than in , onw shows that the unique solution satisfies almost surely Problem (24) in the sense of distributions.

3.3 Application to the first near-field terms

Let us now apply this uniqueness and existence result to our first near-field terms, and more specifically to $\widetilde{U}_n^{\omega,NF}:=U_n^{\omega,NF}+u_n^{\omega,FF}|_{\Sigma_{\varepsilon H}}\mathbb{1}_{\mathcal{B}_H^\infty}$ which satisfy (24) with

$$F:=\Delta_{\boldsymbol{x}_{\shortparallel}}U_{n-2}^{\omega,NF}\;+\;k^2\,U_{n-2}^{\omega,NF},\quad G:=2\nabla_{\boldsymbol{x}_{\shortparallel}}U_{n-1}^{\omega,NF},\quad\Psi:=ik\gamma\,U_{n-1}^{\omega,NF},\quad\text{and}\quad\alpha^N:=\partial_{x_d}u_{n-1}^{\omega,FF}|_{\Sigma_{\varepsilon H}}.$$

By Proposition 5, for almost every $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$, there exists a unique $\widetilde{U}_0^{NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) \in \mathcal{W}_0(D)$ solution of

$$\left|\begin{array}{l} -\Delta_{\boldsymbol{y}}\widetilde{U}_0^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) = 0 \quad \text{in} \quad \mathcal{B}_H^{\omega}\cup\mathcal{B}_H^{\infty}, \\ -\partial_{y_d}\widetilde{U}_0^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) = 0 \text{ on } \Sigma_0, \quad \text{and} \quad \widetilde{U}_0^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) = 0 \text{ on } \partial\mathcal{P}^{\omega}, \\ \Big[\widetilde{U}_0^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,)\Big]_H = 0 \quad \text{and} \quad \Big[-\partial_{y_d}\widetilde{U}_0^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,)\Big]_H = 0. \end{array}\right. \tag{38}$$

We deduce that $\widetilde{U}_0^{\omega,NF} = 0$ and then $U_0^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\boldsymbol{y}) = -u_0^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel})\mathbb{1}_{\mathcal{B}_H^{\infty}}(\boldsymbol{y})$. Since we look for a solution that tends to 0 when y_d tends to $+\infty$, we obtain the following boundary condition for u_0^{FF} on $\Sigma_{\varepsilon H}$

$$\text{a.s. } u_0^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) = 0. \tag{39}$$

Note that u_0^{FF} that satisfies (21) is then independent of ω.

By proposition 5, for almost any $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$, there exists a unique $\widetilde{U}_1^{NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) \in \mathcal{W}_0(D)$ solution of

$$\left|\begin{array}{l} -\Delta_{\boldsymbol{y}}\widetilde{U}_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) = 0 \quad \text{in} \quad \mathcal{B}_H^{\omega}\cup\mathcal{B}_H^{\infty}, \\ -\partial_{y_d}\widetilde{U}_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) = 0 \text{ on } \Sigma_0, \quad \text{and} \quad \widetilde{U}_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,) = 0 \text{ on } \partial\mathcal{P}^{\omega}, \\ \Big[\widetilde{U}_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,)\Big]_H = 0 \quad \text{and} \quad \Big[-\partial_{y_d}\widetilde{U}_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\,\cdot\,)\Big]_H = \partial_{x_d}u_0^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}). \end{array}\right. \tag{40}$$

where we used the fact that $U_0^{\omega,NF} = 0$.

By linearity, $\widetilde{U}_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\boldsymbol{y}) = \partial_{x_d}u_0^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel})W_1^{\omega}(\boldsymbol{y})$, where W_1^{ω} is the unique solution in $\mathcal{W}_0(D)$ of

$$\left|\begin{array}{l} -\Delta_{\boldsymbol{y}}W_1^{\omega} = 0 \quad \text{in} \quad \mathcal{B}_H^{\omega}\cup\mathcal{B}_H^{\infty}, \\ -\partial_{y_d}W_1^{\omega} = 0 \text{ on } \Sigma_0, \quad \text{and} \quad W_1^{\omega} = 0 \text{ on } \partial\mathcal{P}^{\omega}, \\ \Big[W_1^{\omega}\Big]_H = 0 \quad \text{and} \quad \Big[-\partial_{y_d}W_1^{\omega}\Big]_H = 1. \end{array}\right. \tag{41}$$

If we want to impose that $U_1^{\omega,NF}$ that is given by

$$U_1^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\boldsymbol{y}) = -u_1^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel})\mathbb{1}_{\mathcal{B}_H^{\infty}}(\boldsymbol{y}) + \partial_{x_d}u_0^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel})W_1^{\omega}(\boldsymbol{y}), \tag{42}$$

tends to 0 when y_d tends to $+\infty$, we need to understand the behavior at infinity of W_1^{ω}, which is done in the following section.

4 Integral representation and behavior at infinity of the near-fields

In order to study the behavior of the near-fields when y_d tends to $+\infty$, we first derive an integral representation for $\widehat{U}^{\omega}$ the unique solution of (24) in $\mathcal{W}_0(D)$ for $F = 0$, $G = 0$,$\mu^{-\frac{1}{2}}\Psi \in \mathcal{L}^2(\Sigma_0)$ and $\mu^{-\frac{1}{2}}\alpha^N \in \mathcal{L}^2(\Sigma_H)$. Let us introduce the Green's function associated to the Laplace operator in $\mathbb{R}^d$

$$\Gamma(\boldsymbol{z}) := -\frac{1}{2\pi}\ln|\boldsymbol{z}| \quad \text{for} \quad d=2\,, \quad \frac{1}{4\pi|\boldsymbol{z}|} \quad \text{for} \quad d=3\,, \quad \boldsymbol{z}\in\mathbb{R}^d. \tag{43}$$

Proposition 9 (Integral representation for $\widehat{U}^{\omega}$)**.** *Suppose that $\mathcal{P}$ verifies Hypothesis (H1). A.s. for $\boldsymbol{y} \in \mathbb{R}^{d-1}\times(L,+\infty)$, $\widehat{U}^{\omega}(\boldsymbol{y})$ has the following integral representation*

$$\widehat{U}^{\omega}(\boldsymbol{y}) = -2\int_{\mathbb{R}^{d-1}} \partial_{z_d}\Gamma\big(\boldsymbol{y},(\boldsymbol{z}_{\shortparallel},L)\big)\ \varphi^{\omega}(\boldsymbol{z}_{\shortparallel})\mathrm{d}\boldsymbol{z}_{\shortparallel} \tag{44}$$

where φ denotes the trace of $\widehat{U}^{\omega}$ on Σ_L.

We postpone the proof of Proposition 9 to appendix A. With this integral representation we prove that a.s. and in $L^2(\Omega)$ $\widehat{U}$ tends to a deterministic constant as $y_d \to +\infty$. This constant is the ensemble average of its trace on an hyperplane above the layer of particles. In the periodic case one can establish a similar result with the constant being the spatial mean on the hyperplane over a period. Before proving this result, let us start with a useful technical lemma.

Lemma 10. *Let Γ be the Green's function defined in* (43). *Let us introduce*

$$\partial_d\Gamma := \begin{cases} \partial_{z_1}\partial_{z_2}\Gamma & \text{for } d=2 \\ \partial_{z_1}\partial_{z_2}\partial_{z_3}\Gamma & \text{for } d=3, \end{cases} \quad \text{and} \quad \pi(\boldsymbol{z}_{\shortparallel}) := \begin{cases} z_1 & \text{for } d=2 \\ z_1 z_2 & \text{for } d=3. \end{cases} \tag{45}$$

We have for a fixed R and for y_d large enough that

$$\int_{\square_R} |\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)|\, d\boldsymbol{z}_{\shortparallel} \lesssim \frac{1}{y_d^{3(d-1)}} \quad \text{and} \quad \int_{\mathbb{R}^{d-1}\setminus\square_R} \left|\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\left(y_d^{d-1} + \pi(\boldsymbol{z}_{\shortparallel})\right)\right| d\boldsymbol{z}_{\shortparallel} \lesssim 1 \tag{46}$$

$$\int_{\square_R} |\partial_{z_i}\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)|\, d\boldsymbol{z}_{\shortparallel} \lesssim \frac{1}{y_d^{3(d-1)}} \quad \text{and} \quad \int_{\mathbb{R}^{d-1}\setminus\square_R} |\partial_{z_i}\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\pi(\boldsymbol{z}_{\shortparallel})|\, d\boldsymbol{z}_{\shortparallel} \lesssim \frac{1}{y_d},\ i \in [\![1,d]\!], \tag{47}$$

where the constants depend only on R and L.

Proof. A straighforward computation yields

$$\partial_d\Gamma(\boldsymbol{z}) = \frac{1}{\pi}\frac{z_1 z_2}{(z_1^2+z_2^2)^2} \ \text{ for } d=2 \quad \text{and} \quad \frac{1}{4\pi}\frac{-15 z_1 z_2 z_3}{(z_1^2+z_2^2+z_3^2)^{7/2}} \ \text{ for } d=3,$$

it is then easy to show the first inequality of (46). Moreover, by a change of variable $\boldsymbol{z}_{\shortparallel} \mapsto \boldsymbol{z}_{\shortparallel}/y_d$, we obtain the second inequality of (46)

$$\int_{\mathbb{R}^{d-1}\setminus\square_R} \left|\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\left(y_d^{d-1} + \pi(\boldsymbol{z}_{\shortparallel})\right)\right| d\boldsymbol{z}_{\shortparallel} \lesssim \left| \begin{array}{ll} \displaystyle\int_{\mathbb{R}} \frac{|u|(1+|u|)\,du}{(u^2+1)^2} & \text{for } d=2 \\ \displaystyle\int_{\mathbb{R}^2} \frac{|u_1u_2|(1+|u_1u_2|)\,du_1du_2}{(u_1^2+u_2^2+1)^{7/2}} & \text{for } d=3. \end{array}\right.$$

The proof of (47) can be done using similar arguments. □

Proposition 11. *Let $\mathcal{P}$ verify Hypothesis (H1). $\widehat{U}^\omega$ the unique solution of* (24) *in $\mathcal{W}_0(D)$ with $F=0$, $G=0$,$\mu^{-\frac{1}{2}}\Psi \in \mathcal{L}^2(\Sigma_0)$ and $\mu^{-\frac{1}{2}}\alpha^N \in \mathcal{L}^2(\Sigma_H)$ verifies*

$$a.s. \quad \lim_{y_d\to+\infty} \widehat{U}^\omega(\boldsymbol{y}_{\shortparallel}, y_d) = \mathbb{E}[\varphi]; \quad \boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}, \tag{48}$$

this limit being locally uniform in $\boldsymbol{y}_{\shortparallel}$ and

$$\mathbb{E}\left[\left|\widehat{U}^\omega(\cdot, y_d) - \mathbb{E}[\varphi]\right|^2\right] \underset{y_d\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad y_d^2\,\mathbb{E}\left[\left|\nabla\widehat{U}^\omega(\cdot, y_d)\right|^2\right] \underset{y_d\to+\infty}{\longrightarrow} 0\,. \tag{49}$$

Recall that by Lemma 7, $\mu_{|\Sigma_L}^{\frac{1}{2}}\varphi \in \mathcal{L}^2(\Sigma_L)$, and thus since $\mu_{|\Sigma_L}^{-1} \in L^1(\Omega)$ by Hypothesis (H1), $\varphi \in L^1(\Omega)$ by Cauchy-Schwarz inequality.

Proof. The proof is based as in section 4.6 of [7] on the integral representation established in Proposition 9 and Birkhoff ergodic Theorem.

Let us first rewrite $\widehat{U}^\omega(\cdot, y_d) - \mathbb{E}[\varphi]$ using the integral representation (44). We then have

$$\begin{aligned} \widehat{U}^\omega(\cdot, y_d) - \mathbb{E}[\varphi] &:= -2\int_{\mathbb{R}^{d-1}} \partial_{z_d}\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\left(\varphi^\omega(\boldsymbol{y}_{\shortparallel} - \boldsymbol{z}_{\shortparallel}) - \mathbb{E}[\varphi]\right) \mathrm{d}\boldsymbol{z}_{\shortparallel}, \\ &= -2\int_{\mathbf{R}^{d-1}} \partial_{z_d}\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\partial_{\boldsymbol{z}_{\shortparallel}} \int_0^{\boldsymbol{z}_{\shortparallel}} \left(\varphi^\omega(\boldsymbol{y}_{\shortparallel} - \mathbf{w}) - \mathbb{E}[\varphi]\right) \mathrm{d}\mathbf{w}\,\mathrm{d}\boldsymbol{z}_{\shortparallel}, \\ &= (-1)^d 2\int_{\mathbf{R}^{d-1}} \partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\pi(\boldsymbol{z}_{\shortparallel}) \left[\fint_0^{\boldsymbol{z}_{\shortparallel}} \varphi^\omega(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi]\right] \mathrm{d}\boldsymbol{z}_{\shortparallel}. \end{aligned} \tag{50}$$

where we used the notations introduced in (45) and also that $\partial_{\boldsymbol{z}_{\shortparallel}} := \partial_{z_1}$ for $d = 2$ and $:= \partial_{z_1}\partial_{z_2}$ for $d = 3$ and $\int_0^{\boldsymbol{z}_{\shortparallel}} := \int_0^{z_1}$ for $d = 2$ and $:= \int_0^{z_1}\int_0^{z_2}$ for $d = 3$.

We first prove (48). Let $\varepsilon > 0$. By Birkhoff ergodic Theorem (Theorem 3), we know that there exists $R > 0$ such that for all $R' > R$

$$\text{a.s.} \quad \left| \fint_{\square_{R'}} \varphi^{\omega}(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\,\mathrm{d}\mathbf{w} \;-\; \mathbb{E}[\varphi] \right| \;\leq\; \varepsilon, \quad \boldsymbol{y}_{\shortparallel} \in \square_{R'}\,.$$

On one hand we obtain then using the second inequality of (46)

$$\left| \int_{\mathbf{R}^{d-1}\setminus\square_R} \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\pi(\boldsymbol{z}_{\shortparallel}) \left[\fint_0^{\boldsymbol{z}_{\shortparallel}} \varphi^{\omega}(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi] \right] \mathrm{d}\boldsymbol{z}_{\shortparallel} \right| \leq \varepsilon.$$

On the other hand we have

$$\begin{aligned} &\left| \int_{\square_R} \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\pi(\boldsymbol{z}_{\shortparallel}) \left[\fint_0^{\boldsymbol{z}_{\shortparallel}} \varphi^{\omega}\Big(\boldsymbol{y}_{\shortparallel} - \mathbf{w}\Big)\,\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi^{\omega}] \right] \mathrm{d}\boldsymbol{z}_{\shortparallel} \right| \\ &\qquad \lesssim \left(\int_{\square_R} \left| \varphi^{\omega}(\boldsymbol{y}_{\shortparallel} - \mathbf{w}) \right| \mathrm{d}\mathbf{w} + R^2 \mathbb{E}[\varphi^{\omega}] \right) \int_{\square_R} \left| \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \right| \mathrm{d}\boldsymbol{z}_{\shortparallel}\,, \end{aligned}$$

and it suffices to use the first inequality of (46) to obtain the wanted result.

Let us now prove the first limit in (49). Let $\varepsilon > 0$. By Birkhoff's Theorem, since $\int_{\square_1} \varphi \in L^2(\Omega)$ there exists $R > 0$ such that for all $R' > R$

$$\mathbb{E}\left[\left| \fint_{\square_{R'}} \varphi(\boldsymbol{z}_{\shortparallel})\mathrm{d}\boldsymbol{z}_{\shortparallel} - \mathbb{E}[\varphi] \right|^2 \right] \;\leq\; \varepsilon\,. \tag{51}$$

By applying Cauchy-Schwarz inequality and taking the expectation, we obtain for $\boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}$, $y_d > L$,

$$\begin{aligned} \mathbb{E}\left[\left| \widehat{U}^{\omega}(\cdot, y_d) - \mathbb{E}[\varphi] \right|^2 \right] &\lesssim \int_{\mathbb{R}^{d-1}\setminus\square_R} \left| \pi(\boldsymbol{z}_{\shortparallel}) \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \right| \mathrm{d}\boldsymbol{z}_{\shortparallel} \\ &\int_{\mathbf{R}^{d-1}\setminus\square_R} \left| \pi(\boldsymbol{z}_{\shortparallel}) \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \right| \;\mathbf{E}\left[\left| \fint_0^{\boldsymbol{z}_{\shortparallel}} \varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\,\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi] \right|^2 \right] \mathrm{d}\boldsymbol{z}_{\shortparallel} \\ + \int_{\square_R} \left| \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \right| \mathrm{d}\boldsymbol{z}_{\shortparallel} &\int_{\square_R} \left| \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \right| \;\mathbf{E}\left[\left| \int_0^{\boldsymbol{z}_{\shortparallel}} (\varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w}) \;- \mathbf{E}[\varphi])\mathrm{d}\mathbf{w} \right|^2 \right] \mathrm{d}\boldsymbol{z}_{\shortparallel}. \end{aligned} \tag{52}$$

To bound the second term of the right-hand side of (52), note that since $\varphi \in L^2_{loc}(\mathbb{R}^{d-1}, L^2(\Omega))$, we have

$$\begin{aligned} &\left| \int_{\square_R} \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \mathbf{E}\left[\left| \int_0^{\boldsymbol{z}_{\shortparallel}} (\varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w}) \;- \mathbf{E}[\varphi])\mathrm{d}\mathbf{w} \right|^2 \right] \mathrm{d}\boldsymbol{z}_{\shortparallel} \right| \\ &\qquad \lesssim R^{d-1} \mathbf{E}\left[\int_{\square_R} \left| \varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w}) \;- \mathbf{E}[\varphi] \right|^2 \mathrm{d}\mathbf{w} \right] \int_{\square_R} \left| \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L) \right| \mathrm{d}\boldsymbol{z}_{\shortparallel}. \end{aligned} \tag{53}$$

We then use the first inequality of (46) to conclude. Let us now bound the first term of the right-hand side of (52). From (51) we obtain

$$\begin{aligned} &\left| \int_{\mathbf{R}^{d-1}\setminus\square_R} \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\pi(\boldsymbol{z}_{\shortparallel}) \mathbf{E}\left[\left| \fint_0^{\boldsymbol{z}_{\shortparallel}} \varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\,\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi] \right|^2 \right] \mathrm{d}\boldsymbol{z}_{\shortparallel} \right| \\ &\qquad \leq \varepsilon \int_{\mathbb{R}^{d-1}\setminus\square_R} \left| \partial_d \Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\,\pi(\boldsymbol{z}_{\shortparallel}) \right| \mathrm{d}\boldsymbol{z}_{\shortparallel}. \end{aligned}$$

By the second inequality of (46) the integral in the right-hand side is bounded. Finally we proved that for y_d large enough

$$\mathbb{E}\left[\left|\widehat{U}^\omega(\cdot, y_d) - \mathbb{E}[\varphi]\right|^2\right] \lesssim \frac{1}{y_d^{6(d-1)}} + C\varepsilon.$$

To prove the second limit in (49), we replace $\widehat{U}^\omega - \mathbb{E}[\varphi]$ by $\nabla\widehat{U}^\omega$ and $\partial_d\Gamma$ by $\nabla\partial_d\Gamma$ in (52) and use the two inequalities of (47). □

In the periodic case, the convergence of $\widehat{U}$ to its limit as $y_d \to +\infty$ is exponential with a rate inversely proportional to the period. In the random stationary ergodic setting, the speed of convergence can be estimated provided that its trace φ satisfies some additional mixing assumption.

Proposition 12. *If*

$$R^{d-1}\mathbb{E}\left[\left|\fint_{\square_R} \varphi(\boldsymbol{z}_{\shortparallel})\mathrm{d}\boldsymbol{z}_{\shortparallel} - \mathbb{E}[\varphi]\right|^2\right] \underset{R\to+\infty}{=} \mathcal{O}(1), \tag{54}$$

then

$$y_d^{d-1}\mathbb{E}\left[\left|\widehat{U}^\omega(\cdot, y_d) - \mathbb{E}[\varphi]\right|^2\right] + y_d^{d+1}\,\mathbb{E}\left[\left|\nabla\widehat{U}^\omega(\cdot, y_d)\right|^2\right] \underset{y_d\to+\infty}{=} \mathcal{O}(1). \tag{55}$$

Proof. Let us fix $\varepsilon > 0$. We suppose now that there exist $C \geq 0$ and $R > 0$ such that for all $R' > R$

$$(R')^{d-1}\mathbb{E}\left[\left|\fint_{\square_{R'}} \varphi^\omega(\boldsymbol{z}_{\shortparallel})\mathrm{d}\boldsymbol{z}_{\shortparallel} \;-\; \mathbb{E}[\varphi]\right|\right] \leq C.$$

Using similar arguments as in the proof of Proposition 11, we have for $\boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}$, $y_d > L$,

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{U}^\omega(\cdot, y_d) - \mathbb{E}[\varphi]\right|^2\right] &\lesssim \int_{\mathbb{R}^{d-1}\setminus\square_R} |\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)|\,\mathrm{d}\boldsymbol{z}_{\shortparallel} \\ &\int_{\mathbf{R}^{d-1}\setminus\square_R} |\pi(\boldsymbol{z}_{\shortparallel})\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)|\;\pi(\boldsymbol{z}_{\shortparallel})\mathbf{E}\left[\left|\fint_0^{\boldsymbol{z}_{\shortparallel}} \varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\,\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi]\right|^2\right]\mathrm{d}\boldsymbol{z}_{\shortparallel} \\ &+ \int_{\square_R} \left|\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)\right|\mathrm{d}\boldsymbol{z}_{\shortparallel}\int_{\square_R} |\partial_d\Gamma(\boldsymbol{z}_{\shortparallel}, y_d - L)|\;\mathbf{E}\left[\left|\int_0^{\boldsymbol{z}_{\shortparallel}} \varphi(\boldsymbol{y}_{\shortparallel} - \mathbf{w})\,\mathrm{d}\mathbf{w} - \mathbb{E}[\varphi]\right|^2\right]\mathrm{d}\boldsymbol{z}_{\shortparallel}.\end{aligned}$$

We can bound the second term of the right-hand side as in the proof of Proposition 11 (see (53)). To bound the first term, we use the second inequality of (46).

To prove the second part of (55), we decompose similarly $\mathbb{E}\left[\left|\nabla\widehat{U}^\omega\right|^2\right]$ and use the two inequalities of (47). □

5 Effective model

5.1 Construction of U_1^{NF}

Recall that U_1^{NF} is given by (42) where W_1 is the unique solution in $\mathcal{W}_0(D)$ of (41). In particular W_1 satisfies proposition 11. As we look for a solution $U_1^{\omega,NF}$ which tends to 0 when y_d tends to $+\infty$, we obtain the following boundary condition for u_1^{FF} on $\Sigma_{\varepsilon H}$

$$\text{a.s.} \quad -u_1^{\omega,FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) + \partial_{x_d}u_0^{FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel})\,c_1 = 0, \tag{56}$$

where c_1 is given by

$$c_1 := \lim_{y_d\to+\infty} W_1^\omega(\cdot, y_d) = \mathbb{E}\left[W_1|_{\Sigma_L}\right]. \tag{57}$$

Notice that u_1^{FF} that satisfies (21) is deterministic.

5.2 Construction of the first far-field terms

The boundary conditions (39) and (56) which guarantee the existence of two near-field terms going to 0 at infinity, allow us to complete the equations satisfied by the far field terms. We can now write the problems verified by u_0^{FF} and u_1^{FF}.

Construction of u_0^{FF}

Definition. *By using* (21) *and the boundary condition* (39) *on* $\Sigma_{\varepsilon H}$*, we define* u_0^{FF} *as the unique deterministic solution in* $H^1(\mathcal{B}_{\varepsilon H,L})$ *of*

$$\begin{cases} -\Delta u_0^{FF} - k^2 u_0^{FF} = f & in \quad \mathcal{B}_{\varepsilon H,L}, \\ u_0^{FF} = 0 & on \quad \Sigma_{\varepsilon H}, \\ -\partial_{x_d} u_0^{FF} + \Lambda^k u_0^{FF} = 0 & on \quad \Sigma_L. \end{cases} \tag{58}$$

Since u_0^{FF} is solution of the homogeneous Helmholtz equation in a half-space with a homogeneous Dirichlet condition on the hyperplane and a source term in $L^2(\mathcal{B}_{\varepsilon H,L})$ with compact support, completed by the outgoing wave condition (6), we can show that the problem (58) is well posed in $H^1(\mathcal{B}_{\varepsilon H,L})$ [12]. Actually, u_0^{FF} can be computed explicitly using a Fourier transform in the $\boldsymbol{y}_{\shortparallel}$-direction and one can deduce classical elliptic regularity results such as $u_0^{FF} \in H^s(\mathcal{B}_{\varepsilon H,L'})$ for any $s > 0$ with $L' < L$ since the support of f lies far away from $\Sigma_{\varepsilon H}$ by hypothesis (see (1)).

Construction of u_1^{FF}

Definition. *By using* (21) *and the boundary condition* (56) *on* $\Sigma_{\varepsilon H}$*, we define* u_1^{FF} *as the unique deterministic solution in* $H^1(\mathcal{B}_{\varepsilon H,L})$ *of*

$$\begin{cases} -\Delta u_1^{FF} - k^2 u_1^{FF} = 0 & in \quad \mathcal{B}_{\varepsilon H,L}, \\ u_1^{FF} = c_1 \partial_{x_d} u_0^{FF} & on \quad \Sigma_{\varepsilon H}, \\ -\partial_{x_d} u_1^{FF} + \Lambda^k u_1^{FF} = 0 & on \quad \Sigma_L. \end{cases} \tag{59}$$

where c_1 *is given by* (57).

The far field u_1^{FF} is solution of the homogeneous Helmholtz equation in a half-space with a Dirichlet datum that is in $H^s(\Sigma_{\varepsilon H})$ for any $s > 0$. Using the Fourier transform, one can compute u_1^{FF} explicitly and deduce in particular that $u_1^{FF} \in H^s(\mathcal{B}_{\varepsilon H,L})$ for all $s > 0$.

5.3 Effective model

Either we build the far field terms incrementally, or we use the model that will approximate the truncated far field series. This can be useful for studying more general geometries. By using an approximated model, we can have a direct approximation up to a certain order.

The first model that we propose is built from u_0^{FF} satisfying (58). The particles and the impedance condition on the object are replaced by a conducting plane on $\Sigma_{\varepsilon H}$. This effective impedance condition does not take into account the particles. Therefore we propose a second model. This model is built from u_0^{FF} and u_1^{FF}. We want to have an approximation of $u_0^{FF} + \varepsilon u_1^{FF}$ up to the second order in ε. Since u_0^{FF} and u_1^{FF} satisfy the Helmholtz equation, we also have

$$-\Delta\left(u_0^{FF} + \varepsilon u_1^{FF}\right) - k^2 \left(u_0^{FF} + \varepsilon u_1^{FF}\right) = f \quad \text{in} \quad \mathbb{R}^{d-1} \times (\varepsilon H, +\infty). \tag{60}$$

Given the boundary conditions (39) and (56)

$$\left(u_0^{FF} + \varepsilon u_1^{FF}\right) = \varepsilon\, c_1\, \partial_{x_d} u_0^{FF} \quad \text{on} \quad \Sigma_{\varepsilon H}. \tag{61}$$

It is then natural to introduce the following effective model.

Definition. *Let v_ε satisfy*

$$\begin{cases} -\Delta v_\varepsilon - k^2 v_\varepsilon = f & \text{in } \mathcal{B}_{\varepsilon H,L}, \\ -\varepsilon c_1 \partial_{x_d} v_\varepsilon + v_\varepsilon = 0 & \text{on } \Sigma_{\varepsilon H}, \\ -\partial_{x_d} v_\varepsilon + \Lambda^k v_\varepsilon = 0 & \text{on } \Sigma_L, \end{cases} \tag{62}$$

where c_1 is given by (57).

If $c_1 > 0$, then the problem (62) is well posed in $H^1(\mathcal{B}_{\varepsilon H,L})$ [10]. Moreover, v_ε can be computed explicitly using the Fourier transform in the $\boldsymbol{y}_{\shortparallel}$-direction, and one can deduce easily that $v_\varepsilon \in H^s(\mathcal{B}_{\varepsilon H,L'})$ for any $s > 0$ with $L' < L$, with norms independent of ε, since the support of f lies far away from $\Sigma_{\varepsilon H}$. Let us prove that the parameter H can be chosen such that $c_1 > 0$.

Proposition 13. *There exists $H_0 \geq h$ such that for all $H \geq H_0$*

$$c_1(H) > 0, \tag{63}$$

where $c_1(H) := \mathbb{E}\left[W_1(H)\right]$ and $W_1(H)$ denotes the solution of (41) *with the normal derivative jump occurring at Σ_H.*

Proof. Let H'>H. One can easily establish thanks to proposition 5 that

$$W_1(H') = W_1(H) + (y_d - H)\chi_{(H,H')}(y_d) + (H' - H)\chi_{(H',+\infty)}(y_d).$$

Taking the limit as $y_d \to +\infty$, one gets

$$c_1(H') = c_1(H) + (H' - H).$$

Choosing H' large enough leads to (63). □

Finally, let us show that v_ε is indeed an approximation of $u_0^{FF} + \varepsilon u_1^{FF}$ up to an order 2 in ε.

Proposition 14. *For H large enough and $L > \varepsilon H$, the following estimate holds*

$$\left\| u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon \right\|_{H^2(\mathcal{B}_{\varepsilon H,L})} \underset{\varepsilon \to 0}{=} \mathcal{O}(\varepsilon^2). \tag{64}$$

Proof. Let $e_\varepsilon := u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon$. Applying the Fourier Transform in the $\boldsymbol{y}_{\shortparallel}$-direction to (58), (59) and (62), one obtains that

$$\widehat{e_\varepsilon}(\zeta, x_d) = \varepsilon^2 \frac{c_1 \partial_{x_d} \widehat{u}_1^{FF}(\zeta, \varepsilon H)}{1 - \mathrm{i}\varepsilon c_1 \sqrt{k^2 - |\zeta|^2}} e^{\mathrm{i}\sqrt{k^2 - |\zeta|^2} x_d}$$

where $\widehat{e_\varepsilon}$ (resp. $\partial_{x_d} \widehat{u}_1^{FF}$) denotes the Fourier Transform of e_ε (resp. $\partial_{x_d} u_1$) and by convention $\sqrt{k^2 - |\zeta|^2} = \mathrm{i}\sqrt{|\zeta|^2 - k^2}$. Using the characterization of Sobolev spaces from the Fourier Transform (see (8)), one easily shows that

$$\|e_\varepsilon\|_{H^2(\mathcal{B}_{\varepsilon H,L})} \lesssim \varepsilon^2 \|\partial_{x_d} u_1^{FF}\|_{H^{3/2}(\Sigma_{\varepsilon H})}.$$

□

6 Error estimates

In this section we estimate

- the error between u_ε and our two-scale asymptotic expansion in $H^1(\mathcal{B}_L^\omega)$ (Recall that $\mathcal{B}_L^\omega := \mathbb{R}^{d-1} \times (0, L) \setminus \mathcal{P}^\omega$ corresponds to the infinite strip (outside the particles) between Σ_0 and Σ_L with $L > \varepsilon H$);
- the error between u_ε and the far fields $u_0^{FF} + \varepsilon u_1^{FF}$ in $H^1(\mathcal{B}_{L',L})$. Recall that $\mathcal{B}_{L',L}$ is the infinite strip $\{(\boldsymbol{x}_{\shortparallel}, x_d),\ L' < x_d < L\}$ with $L' > \varepsilon H$. To estimate the latter it is necessary to consider a strip sufficiently far away from the layer (see Figure 2) so that the contribution of the near-fields in the asymptotic expansion are negligible. This motivates our choice of considering a strip at distance of order 1 from the small particles.

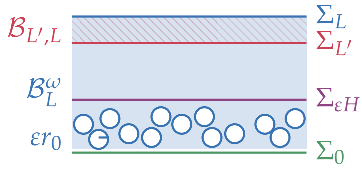


Figure 2: Illustration of the different domains where the errors are estimated in section 6

- the same two errors replacing the far fields $u_0^{FF} + \varepsilon u_1^{FF}$ by the effective solution v_ε.

We establish those error estimates in the specific case where the distance between two particles is bounded a.s. and a.e. in the layer. We assume indeed that

Hypothesis (L^∞). *The distance to the nearest particule R defined in* (14) *satisfies*

$$\sup_{y_d \in (0,h)} R(\cdot, y_d) \in L^\infty(\Omega, L^\infty(\mathbb{R}^{d-1})).$$

Remark 4. *Hypothesis (L^∞) corresponds to a stronger assumption than (H1) for the particle's distribution. A direct consequence is, since μ is bounded from below, that the near-fields solutions of problem of the form* (24) *are in*

$$\mathcal{H}_0(D) := \Big\{ a.s.\ V^\omega \in H^1_{loc}(D^\omega), V^\omega(\cdot, y_d) \text{ stationary for any } y_d > 0,$$
$$a.s.\ V^\omega = 0 \text{ on } \mathcal{P}^\omega,\ \mathbb{E}\left[\int_{\square_1}\int_0^{+\infty} \mathbb{1}_{D^\omega}\Big(\frac{|V^\omega|^2}{1+y_d^2} + |\nabla V^\omega|^2\Big)\,\mathrm{d}\boldsymbol{y}\right] < +\infty \Big\}.$$

We first prove in Section (6.2) estimates relying only on Hypothesis (L^∞). We then improve those estimates in subsection 6.3 by adding a quantitative mixing assumption (Hypothesis (Mix)) on the particle's distribution.

6.1 Technical lemmatas

Let us start with the following general lemma and its corollary that provide $L^2(\mathcal{B}_L^\omega)$- and $H^1(\mathcal{B}_{L',L})$-estimates of near-field terms depending on their behaviour as $y_d \to \infty$.

Lemma 15. *Let $u \in H^1(\mathbb{R}^{d-1})$ and $U \in \mathcal{H}_0(D)$ be such that*

$$\mathbb{E}[|U|^2(\cdot, y_d)] + y_d^2\,\mathbb{E}[|\nabla U|^2(\cdot, y_d)] \underset{y_d \to +\infty}{=} o(1). \tag{65}$$

Then $\mathcal{U}_\varepsilon : \boldsymbol{x} \mapsto u(\boldsymbol{x}_{\shortparallel})\, U(\dfrac{\boldsymbol{x}}{\varepsilon})$ satisfies

$$\|\mathcal{U}_\varepsilon\|_{L^2(\Omega, L^2(\mathcal{B}_L^\omega))} \le \eta(\varepsilon)\|u\|_{L^2(\mathbb{R}^{d-1})} \quad \text{and} \quad + \|\mathcal{U}_\varepsilon\|_{L^2(\Omega, H^1(\mathcal{B}_{L',L}))} \le \eta(\varepsilon)\|u\|_{H^1(\mathbb{R}^{d-1})}, \tag{66}$$

where $\eta(\varepsilon)$ tends to 0 as ε tends to 0.

Proof. By stationarity of U (and a change of variable in the first integral), we get immediately

$$\mathbb{E}\left[\int_0^L \int_{\mathbb{R}^{d-1}} |u(\boldsymbol{x}_{\shortparallel})|^2 \left|U\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\right|^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}\mathrm{d}x_d\right] = \varepsilon \int_0^{L/\varepsilon} \mathbb{E}\left[|U(\cdot, y_d)|^2\right] \mathrm{d}y_d \int_{\mathbb{R}^{d-1}} |u(\boldsymbol{x}_{\shortparallel})|^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}. \tag{67}$$

Let $\delta > 0$. By (65), there exists $L_0 \ge 0$ such that for all $y_d \ge L_0$,

$$\mathbb{E}[|U|^2(\cdot, y_d)] + y_d^2\mathbb{E}[|\nabla U|^2(\cdot, y_d)] \le \frac{\delta}{2L}. \tag{68}$$

We can then write

$$\int_0^{L/\varepsilon} \mathbb{E}\left[|U(\cdot,y_d)|^2\right] \mathrm{d}y_d \leq \int_0^{L_0} \mathbb{E}\left[|U(\cdot,y_d)|^2\right] \mathrm{d}y_d + \frac{\delta}{2\varepsilon}.$$

This and the previous relation prove that for ε small enough, $\|\mathcal{U}_\varepsilon\|^2_{L^2(\Omega,L^2(\mathcal{B}_L^\omega))} \leq \delta\,\|u\|^2_{L^2(\mathbb{R}^{d-1})}$. Moreover

$$\|\nabla\mathcal{U}_\varepsilon\|_{L^2(\Omega,L^2(\mathcal{B}_{L',L}))} \leq \|\nabla_{\boldsymbol{x}_\shortparallel} u\, U\left(\frac{\cdot}{\varepsilon}\right)\|_{L^2(\Omega,L^2(\mathcal{B}_{L',L}))} + \frac{1}{\varepsilon}\|u\,\nabla U\left(\frac{\cdot}{\varepsilon}\right)\|_{L^2(\Omega,L^2(\mathcal{B}_{L',L}))}. \tag{69}$$

The first term can be dealt with in the same way as $\|u(\cdot)U\left(\frac{\cdot}{\varepsilon}\right)\|^2_{L^2(\Omega,L^2(\mathcal{B}_L^\omega))}$ replacing u by $\nabla_{\boldsymbol{x}_\shortparallel} u$. For the second term, we use (68) to obtain

$$\begin{aligned}\frac{1}{\varepsilon^2}\|u(\cdot)\nabla U\left(\frac{\cdot}{\varepsilon}\right)\|^2_{L^2(\Omega,L^2(\mathcal{B}_{L',L}))} &= \frac{1}{\varepsilon}\|u\|^2_{L^2(\mathbb{R}^{d-1})}\int_{\frac{L'}{\varepsilon}}^{\frac{L}{\varepsilon}} \mathbb{E}\left[|\nabla U(\cdot,y_d)|^2\right]\mathrm{d}y_d \\ &\leq \frac{\delta}{2L\varepsilon}\|u\|^2_{L^2(\mathbb{R}^{d-1})}\int_{\frac{L'}{\varepsilon}}^{\frac{L}{\varepsilon}} \frac{1}{y_d^2}\mathrm{d}y_d \lesssim \delta\|u\|^2_{L^2(\mathbb{R}^{d-1})}.\end{aligned} \tag{70}$$

□

Corollary 16. *Let $u \in L^2(\mathbb{R}^{d-1})$. Consider $U \in \mathcal{H}_0(D)$ such that*

$$y_d^{d-1}\mathbb{E}[|U|^2(\cdot,y_d)] + y_d^{d+1}\mathbb{E}[|\nabla U|^2(\cdot,y_d)] \underset{y_d\to+\infty}{=} \mathcal{O}(1). \tag{71}$$

Then $\mathcal{U}_\varepsilon : \boldsymbol{x} \mapsto u(\boldsymbol{x}_\shortparallel)\,U(\frac{\boldsymbol{x}}{\varepsilon})$ satisfies (66) *with*

$$\eta(\varepsilon) \lesssim \begin{cases} \mathcal{O}\left(\varepsilon^{1/2}|\log\varepsilon|^{1/2}\right) & \textit{for } d=2, \\ \mathcal{O}(\varepsilon^{1/2}) & \textit{for } d=3. \end{cases}$$

Proof. We follow the same ideas as in the proof of Lemma 15. Thanks to (71) we can improve our initial estimate of $\int_0^{\frac{L}{\varepsilon}} \mathbb{E}\left[|U(\cdot,y_d)|^2\right]\mathrm{d}y_d$ as such

$$\begin{aligned}\int_0^{\frac{L}{\varepsilon}} \mathbb{E}\left[|U(\cdot,y_d)|^2\right]\mathrm{d}y_d &\leq \int_0^{L'} \mathbb{E}\left[|U(\cdot,y_d)|^2\right]\mathrm{d}y_d + C'\int_{L'}^{\frac{L}{\varepsilon}} y_d^{-(d-1)}\mathrm{d}y_d \\ &= \int_0^{L'} \mathbb{E}\left[|U(\cdot,y_d)|^2\right]\mathrm{d}y_d + C'\begin{cases} \log\frac{L}{\varepsilon} - \log L' & \text{for } d=2, \\ (L'^{-1} - \frac{\varepsilon}{L}) & \text{for } d=3. \end{cases}\end{aligned}$$

This inequality combined with (67) gives us the first result. Moreover, one has to adapt (70) using that

$$\int_{\frac{L'}{\varepsilon}}^{\frac{L}{\varepsilon}} y_d^{-(d+1)} = \mathcal{O}(\varepsilon^d).$$

□

6.2 Estimates under Hypothesis (L^∞)

Suppose that u_0^{FF} and u_1^{FF} satisfy respectively (58) and (59) then U_1^{NF} is given by (42) that rewrites for $\boldsymbol{x}_\shortparallel \in \mathbb{R}^{d-1}, \boldsymbol{y} \in D^\omega$,

$$U_1^{\omega,NF}(\boldsymbol{x}_\shortparallel;\boldsymbol{y}) = \partial_{x_d} u_0^{FF}\big|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_\shortparallel)V_1^\omega(\boldsymbol{y}) \quad \text{where } V_1^\omega(\boldsymbol{y}) := -c_1\mathbb{1}_{\mathcal{B}_H^\infty}(\boldsymbol{y}) + W_1^\omega(\boldsymbol{y}). \tag{72}$$

Consider now W_2 the unique solution in $\mathcal{H}_0(D)$ of the following near-field problem

$$\left|\begin{array}{l} -\Delta_{\boldsymbol{y}} W_2^\omega = 0 \quad \text{in} \quad \mathcal{B}_H^\omega \cup \mathcal{B}_H^\infty, \\ -\partial_{y_d} W_2^\omega = -ik\gamma V_1^\omega \ \text{ on } \Sigma_0, \quad \text{and} \quad W_2^\omega = 0 \text{ on } \partial\mathcal{P}^\omega, \\ \left[W_2^\omega\right]_H = 0 \quad \text{and} \quad \left[-\partial_{y_d} W_2^\omega\right]_H = 0. \end{array}\right. \tag{73}$$

Since $W_1|_{\Sigma_0} \in \mathcal{L}^2(\Sigma_0)$ under Hypothesis (L^∞), then $V_1|_{\Sigma_0} \in \mathcal{L}^2(\Sigma_0)$ and (73) is well-posed in $\mathcal{H}_0(D)$ (see Remark 4). Moreover we know that the results of Proposition 11 can be applied to W_2. Let us denote $c_2 := \mathbb{E}[W_2|_{\Sigma_L}]$, $V_2^\omega := -c_2 \mathbb{1}_{\mathcal{B}_H^\infty} + W_2^\omega$ and

$$\forall \boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}, \boldsymbol{y} \in D^\omega,\ \widetilde{U}_2^{\omega,NF}(\boldsymbol{x}_{\shortparallel};\boldsymbol{y}) := \partial_{x_d} u_1^{FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) V_1^\omega(\boldsymbol{y}) + \partial_{x_d} u_0^{FF}|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) V_2^\omega(\boldsymbol{y}). \tag{74}$$

Remark 5. *Note that we cannot prove that Problem (22) with $n = 2$ is well posed since it is not clear in general that $\mathbf{G}_1$, which depends on $\nabla_{\boldsymbol{x}_{\shortparallel}} U_1^{NF}$ (see (22)), is in $\mathcal{L}^2(D)$. The function $\widetilde{U}_2^{\omega,NF}$ is somehow the part of $U_2^{\omega,NF}$ that is well-defined and enables us to perform the error analysis.*

We estimate in the theorem below

- the error between u_ε and the second-order multi-scale asymptotic expansion

$$w_\varepsilon^\omega : \boldsymbol{x} \mapsto (u_0^{FF}(\boldsymbol{x}) + \varepsilon u_1^{FF}(\boldsymbol{x}))\mathbb{1}_{x_d \geq \varepsilon H}(\boldsymbol{x}) + \left[\varepsilon U_1^{\omega,NF} + \varepsilon^2 \widetilde{U}_2^{\omega,NF}\right]\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right) \in L^2(\Omega, H^1(\mathcal{B}_L^\omega));$$

- the error between u_ε and $u_0^{FF} + \varepsilon u_1^{FF}$ in $H^1(\mathcal{B}_{L',L})$.

Theorem 17. *Suppose that $\mathcal{P}$ verifies Hypothesis (L^∞). For all $L > L' > \varepsilon H$, the following estimates hold*

$$\|u_\varepsilon - w_\varepsilon\|_{L^2\left(\Omega, H^1(\mathcal{B}_L^\omega)\right)} \underset{\varepsilon \to 0}{=} o(\varepsilon), \tag{75}$$

and

$$\left\|u_\varepsilon - (u_0^{FF} + \varepsilon u_1^{FF})\right\|_{L^2\left(\Omega, H^1(\mathcal{B}_{L',L})\right)} \underset{\varepsilon \to 0}{=} o(\varepsilon). \tag{76}$$

Thanks to Proposition 14, similar error estimates can be proven replacing the far field expansion $u_0^{FF} + \varepsilon u_1$ by the effective solution v_ε. More precisely let us introduce for $\boldsymbol{x} \in D_\varepsilon^\omega$

$$\widetilde{w}_\varepsilon(\boldsymbol{x}) := v_\varepsilon(\boldsymbol{x})\mathbb{1}_{x_d \geq \varepsilon H}(\boldsymbol{x}) + \partial_{x_d} v_\varepsilon|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) V_1^\omega\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right) + \varepsilon^2 \partial_{x_d} v_\varepsilon|_{\Sigma_{\varepsilon H}}(\boldsymbol{x}_{\shortparallel}) V_2^\omega\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right).$$

In the following corollary we estimate $u_\varepsilon - \widetilde{w}_\varepsilon$ in $H^1(\mathcal{B}_L^\omega)$ and $u_\varepsilon - v_\varepsilon$ in $H^1(\mathcal{B}_{L',L})$.

Corollary 18. *For all $L > L' > \varepsilon H$, under Hypothesis (L^∞), the following estimates hold*

$$\|u_\varepsilon - \widetilde{w}_\varepsilon\|_{L^2\left(\Omega, H^1(\mathcal{B}_L^\omega)\right)} \underset{\varepsilon \to 0}{=} o(\varepsilon), \tag{77}$$

and

$$\|u_\varepsilon - v_\varepsilon\|_{L^2\left(\Omega, H^1(\mathcal{B}_{L',L})\right)} \underset{\varepsilon \to 0}{=} o(\varepsilon). \tag{78}$$

Proof of Theorem 17. Let us write the problem verified by w_ε^ω in $\mathcal{B}_L^\omega$. w_ε^ω satisfies on one hand for $\boldsymbol{x} \in \mathcal{B}_L^\omega$

$$\begin{aligned}
-(\Delta + k^2) w_\varepsilon^\omega(\boldsymbol{x}) = \quad & f(\boldsymbol{x}) - \varepsilon(\Delta + k^2)\left[(U_1^{\omega,NF} + \varepsilon \widehat{U}_2^{\omega,NF})\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right)\right] \\
= \quad & f(\boldsymbol{x}) + \left[-\frac{1}{\varepsilon}\Delta_{\boldsymbol{y}} - \nabla_{\boldsymbol{x}_{\shortparallel}} \cdot \nabla_{\boldsymbol{y}} - \varepsilon\Big(\Delta_{\boldsymbol{x}_{\shortparallel}} + k^2\Big)\right](U_1^{\omega,NF} + \varepsilon \widetilde{U}_2^{\omega,NF})\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right) \\
= \quad & f(\boldsymbol{x}) - \varepsilon \nabla \cdot \left[\nabla_{\boldsymbol{x}_{\shortparallel}}(U_1^{\omega,NF} + \varepsilon \widetilde{U}_2^{\omega,NF})\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right)\right] - \varepsilon k^2 (U_1^{\omega,NF} + \varepsilon \widetilde{U}_2^{\omega,NF})\left(\boldsymbol{x}_{\shortparallel}; \frac{\boldsymbol{x}}{\varepsilon}\right),
\end{aligned}$$

where we used that U_1^{NF} and $\widetilde{U}_2^{NF}$ are solutions of near-field problems with no volume source terms and that $\varepsilon\nabla[V(\boldsymbol{x}_{\shortparallel}; \boldsymbol{x}/\varepsilon)] = \varepsilon[\nabla_{\boldsymbol{x}_{\shortparallel}} V](\boldsymbol{x}_{\shortparallel}; \boldsymbol{x}/\varepsilon) + [\nabla_y V](\boldsymbol{x}_{\shortparallel}; \boldsymbol{x}/\varepsilon)$. On the other hand w_ε^ω verifies the following jumps and boundary conditions

$$\left|\begin{array}{l}
\left[w_\varepsilon^\omega\right]_{\varepsilon H} = 0, \text{ and } \left[\partial_{x_d} w_\varepsilon^\omega\right]_{\varepsilon H} = 0, \\
-\partial_{x_d} w_\varepsilon^\omega + ik\gamma w_\varepsilon^\omega = \varepsilon^2\, ik\gamma \widetilde{U}_2^{\omega,NF}|_{\Sigma_0} \quad \text{on } \Sigma_0, \quad \text{and} \quad w_\varepsilon^\omega = 0 \text{ on } \partial\mathcal{P}^\omega, \\
\partial_{x_d} w_\varepsilon^\omega + \Lambda^k w_\varepsilon^\omega = [\frac{1}{\varepsilon}\partial_{y_d} + \Lambda^k](\varepsilon U_1^{\omega,NF} + \varepsilon^2 \widetilde{U}_2^{\omega,NF}) \quad \text{on } \Sigma_L,
\end{array}\right.$$

where Λ^K denotes the DtN operator introduced in the definition .

We can now write the variational formulation verified by $e_\varepsilon := u_\varepsilon - w_\varepsilon$ in $L^2\big(\Omega, H^1(\mathcal{B}_L^\omega)\big)$. For all $v \in L^2\big(\Omega, H^1(\mathcal{B}_L^\omega)\big)$,

$$\mathbb{E}\left[a_\varepsilon^\omega(e_\varepsilon^\omega, v^\omega)\right] = \varepsilon \ell_\varepsilon^1(v) + \varepsilon^2 \ell_\varepsilon^2(v) + \varepsilon \ell_\varepsilon^3(v), \tag{79}$$

where a_ε^ω is defined in (12) and

$$\begin{aligned}
\ell_\varepsilon^1(v) &= -\mathbb{E}\left[\int_{\mathcal{B}_L^\omega} \mathbb{1}_{D^\omega} \nabla_{\boldsymbol{x}_\shortparallel}\left(U_1^{\omega,NF} + \varepsilon \widetilde{U}_2^{\omega,NF}\right)\left(\boldsymbol{x}_\shortparallel; \frac{\boldsymbol{x}}{\varepsilon}\right) \cdot \overline{\nabla v^\omega}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right. \\
&\qquad\qquad \left. + k^2 \int_{\mathcal{B}_L^\omega} \mathbb{1}_{D^\omega}\left(U_1^{\omega,NF} + \varepsilon \widetilde{U}_2^{\omega,NF}\right)\left(\boldsymbol{x}_\shortparallel; \frac{\boldsymbol{x}}{\varepsilon}\right)\overline{v^\omega}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\ \right] \\
\ell_\varepsilon^2(v) &= ik\gamma\, \mathbb{E}\left[\int_{\Sigma_0} \widetilde{U}_2^{\omega,NF}\Big|_{\Sigma_0}\left(\boldsymbol{x}_\shortparallel; \frac{\boldsymbol{x}_\shortparallel}{\varepsilon}\right)\overline{v^\omega}|_{\Sigma_0}(\boldsymbol{x}_\shortparallel)\mathrm{d}\boldsymbol{x}_\shortparallel\right] \\
\ell_\varepsilon^3(v) &= \mathbb{E}\left[\int_{\Sigma_L} \overline{v^\omega}|_{\Sigma_L}(\boldsymbol{x}_\shortparallel)\left[\frac{1}{\varepsilon}\partial_{y_d} + \Lambda^k\right]\left(U_1^{\omega,NF} + \varepsilon \widetilde{U}_2^{\omega,NF}\right)\Big|_{\Sigma_L}\left(\boldsymbol{x}_\shortparallel; \frac{\boldsymbol{x}_\shortparallel}{\varepsilon}\right)\mathrm{d}\boldsymbol{x}_\shortparallel\right].
\end{aligned} \tag{80}$$

Let us now bound the norm of each anti-linear form appearing in (79). For ℓ_ε^1, both volume integrals can be bounded applying Lemma 15 to $U_1^{NF}, \nabla_{\boldsymbol{x}_\shortparallel} U_1^{NF}, \widetilde{U}_2^{NF}$ or $\nabla_{\boldsymbol{x}_\shortparallel}\widetilde{U}_2^{NF}$ with U being $V_i, i = 1,2$ and u being $\partial_{x_d} u_i^{FF}|_{\Sigma_{\varepsilon H}}$ or $\nabla_{\boldsymbol{x}_\shortparallel}\partial_{x_d} u_i^{FF}|_{\Sigma_{\varepsilon H}}$. Note that $V_i, i = 1,2$ verify (65) by Proposition 11 and $\partial_{x_d} u_i^{FF}|_{\Sigma_{\varepsilon H}}, \nabla_{\boldsymbol{x}_\shortparallel}\partial_{x_d} u_i^{FF} \in H^1(\Sigma_{\varepsilon H}), i = 1,2$ ($\Sigma_{\varepsilon H}$ being identified to $\mathbb{R}^{d-1}$). By Cauchy-Schwarz inequality, we get

$$|\ell_\varepsilon^1(v)| \lesssim \eta(\varepsilon)\|v\|_{L^2(\Omega, H^1(\mathcal{B}_L^\omega))},$$

where $\eta(\varepsilon)$ tends to 0 as ε tends to 0.

For ℓ_ε^2, we use that under Hypothesis (L^∞) the following trace theorem (analogue to Lemma 7) holds for all $U \in \mathcal{H}_0(D)$

$$\mathbb{E}\left[\left|U\big|_{\Sigma_0}(\cdot)\right|^2\right] \lesssim \mathbb{E}\left[\int_0^{+\infty}\int_{\square_1} |\nabla U|^2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right].$$

We obtain for $u \in L^2(\mathbb{R}^{d-1})$ and $U \in \mathcal{H}_0(D)$

$$\mathbb{E}\left[\int_{\Sigma_0}\left|u(\boldsymbol{x}_\shortparallel)U\big|_{\Sigma_0}\left(\frac{\boldsymbol{x}_\shortparallel}{\varepsilon}\right)\right|^2 \mathrm{d}\boldsymbol{x}_\shortparallel\right] \lesssim \|u|_{\Sigma_0}\|^2_{L^2(\Sigma_0)}\mathbb{E}\left[\int_0^{+\infty}\int_{\square_1}|\nabla U|^2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right].$$

Applying this result to $\widetilde{U}_2^{NF}$ we get by Cauchy-Schwarz inequality

$$|\ell_\varepsilon^2(v)| \lesssim \|v|_{\Sigma_0}\|_{L^2(\Omega, L^2(\Sigma_0))}.$$

We conclude with a trace inequality in $L^2(\Omega, H^1(\mathcal{B}_L^\omega))$

$$\forall \alpha \in [0, L], \quad \|\mathbb{1}_D v|_{\Sigma_\alpha}\|_{L^2(\Omega, L^2(\Sigma_\alpha))} \lesssim \|\mathbb{1}_D v\|_{L^2(\Omega, H^1(\mathcal{B}_L^\omega))}. \tag{81}$$

Finally let us focus on ℓ_ε^3. Since by Proposition 11 $\|V_i(\cdot, y_d)\|_{L^2(\Omega)} + y_d\|\nabla V_i(\cdot, y_d)\|_{L^2(\Omega)} \underset{y_d\to+\infty}{=} o(1)$ for $i = 1$ and 2, we deduce that

$$\|V_i|_{\Sigma_{L/\varepsilon}}\|_{L^2(\Omega)} \underset{\varepsilon\to 0}{=} o(1), \quad \text{and} \quad \|\partial_{y_d} V_i|_{\Sigma_{L/\varepsilon}}\|_{L^2(\Omega)} \underset{\varepsilon\to 0}{=} o(\varepsilon).$$

By Definition (72) and (74) of U_1^{NF} and $\widetilde{U}_2^{NF}$ and thanks to the continuity of Λ^k and (81) we deduce that

$$|\ell_\varepsilon^3(v)| \lesssim \eta(\varepsilon)\, \|v\|_{L^2(\Omega, H^1(\mathcal{B}_L^\omega))},$$

Therefore, we have shown that, for any $v \in L^2\left(\Omega, H^1\left(\mathcal{B}_L^\omega\right)\right)$

$$\left|\varepsilon \ell_\varepsilon^1(v) + \varepsilon^2 \ell_\varepsilon^2(v) + \varepsilon \ell_\varepsilon^3(v)\right| \le \varepsilon\, \eta(\varepsilon)\, \|v\|_{L^2(\Omega, H^1(\mathcal{B}_L^\omega))}. \tag{82}$$

Thanks to the well-posedness result of Corollary 4 we deduce that

$$\|e_\varepsilon\|_{L^2(\Omega,H^1_0(\mathcal{B}^\omega_L))} \underset{\varepsilon\to 0}{=} \varepsilon\,\eta(\varepsilon).$$

This ends the proof of (75).

To prove (76) we decompose the error in two parts

$$\left\|u_\varepsilon - (u_0^{FF} + \varepsilon u_1^{FF})\right\|_{L^2\left(\Omega,H^1(\mathcal{B}_{L',L})\right)} \le \|u_\varepsilon - w_\varepsilon\|_{L^2\left(\Omega,H^1(\mathcal{B}_{L',L})\right)} + \left\|\varepsilon U_1^{NF} + \varepsilon^2 \widetilde{U}_2^{NF}\right\|_{L^2\left(\Omega,H^1(\mathcal{B}_{L',L})\right)}$$

From (75), we know that the first term of the right-hand side is a $o(\varepsilon)$. For the second term we use the result of Lemma 15. □

Proof of Corollary 18. To prove (77) we need to estimate the norm of $w_\varepsilon - \widetilde{w}_\varepsilon$ in $V_L := L^2(\Omega, H_1(\mathcal{B}^\omega_L))$. To do so we decompose the error in three parts

$$\|w_\varepsilon - \widetilde{w}_\varepsilon\|_{V_L} \le \left\|u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon\right\|_{V_L} + \varepsilon \left\|\partial_{x_d}(u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}} V_1\left(\frac{\cdot}{\varepsilon}\right)\right\|_{V_L} + \varepsilon^2 \left\|\partial_{x_d}(u_0^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}} V_2\left(\frac{\cdot}{\varepsilon}\right)\right\|_{V_L}. \quad (83)$$

Proposition 14 allows us to estimate the first term of the right-hand-side. Let us now deal with the two remaining terms. Since $V_i, i = 1,2$ verify (65) by Proposition 11 and $\partial_{x_d} u_i^{FF}|_{\Sigma_{\varepsilon H}}, \partial_{x_d} v_\varepsilon^{FF}|_{\Sigma_{\varepsilon H}} \in H^s(\Sigma_{\varepsilon H})$ for any s, we get deriving inequalities similar to (69) and (70), and from Lemma 15

$$\left\|\partial_{x_d}(u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}} V_1\left(\frac{\cdot}{\varepsilon}\right)\right\|_{V_L} \lesssim \frac{1}{\varepsilon}\|\partial_{x_d}(u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}}\|_{L^2(\Sigma_{\varepsilon H})} + \eta(\varepsilon)\|\partial_{x_d}(u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}}\|_{H^1(\Sigma_{\varepsilon H})}, \quad (84)$$

and

$$\left\|\partial_{x_d}(u_0^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}} V_2\left(\frac{\cdot}{\varepsilon}\right)\right\|_{V_L} \lesssim \frac{1}{\varepsilon}\|\partial_{x_d}(u_0^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}}\|_{L^2(\Sigma_{\varepsilon H})} + \eta(\varepsilon)\|\partial_{x_d}(u_0^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}}\|_{H^1(\Sigma_{\varepsilon H})} \quad (85)$$

where $\eta(\varepsilon)$ tends to 0 as ε tends to 0. From (64) we know moreover that

$$\|\partial_{x_d}(u_0^{FF} + \varepsilon u_1^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}}\|_{H^1(\Sigma_{\varepsilon H})} \lesssim \varepsilon^2, \quad \text{and} \quad \|\partial_{x_d}(u_0^{FF} - v_\varepsilon)|_{\Sigma_{\varepsilon H}}\|_{H^1(\Sigma_{\varepsilon H})} \lesssim \varepsilon. \quad (86)$$

Inserting (86) into (84) and (85) allows us to conclude. Finally, to prove (78) we combine (76) and (64). □

6.3 Improved estimates under a quantitative mixing assumption on $\mathcal{P}$

The limiting factor in the two estimates of Theorem 17 is that contrarily to the periodic case, we cannot without an additional quantitative mixing assumption on $\mathcal{P}$ quantify the rate of convergence of V_1 and $y_d^2|\nabla V_1|$ to 0. Under such an assumption (Hypothesis (Mix)) we prove in this subsection that for $R \ge 1$

$$\mathbb{E}\left[\left|\fint_{\square_R} V_1^\omega\Big|_{\Sigma_L}\right|^2\right] = \mathbb{E}\left[\left|\mathbb{E}[W_1^\omega|_{\Sigma_L}] - \fint_{\square_R} W_1^\omega|_{\Sigma_L}\right|^2\right] \lesssim R^{-(d-1)}, \quad (87)$$

so that the results of proposition 12 hold true. More specifically we establish the following result

Theorem 19 (Fluctuations of $W_1^\omega|_{\Sigma_L}$)**.** *Let $L \ge \varepsilon H$. Providing that $\mathcal{P}$ verifies (L^∞) and the mixing hypothesis (Mix), $W_1^\omega|_{\Sigma_L}$ verifies for all $f \in L^\infty(\Sigma_L)$ with compact support*

$$Var\left[\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\right] \lesssim \|f\|^2_{L^2(\Sigma_L)}. \quad (88)$$

For $f := R^{-\frac{d-1}{2}}\mathbb{1}_{\square_R}$ with $R \geq 1$, we directly retrieve (87). Consequently the $L^2(\mathcal{B}_L^\omega)$- and $H^1(\mathcal{B}_{L',L})$-estimates of corollary 16 can be applied to the near-field U_1^{NF} and we can then prove the following improved error estimates

Theorem 20. *Suppose that $\mathcal{P}$ verifies (L^∞) and (Mix) . For all $R > 0$ and $L > L' > \varepsilon H$, the following estimates hold*

$$\|u_\varepsilon - w_\varepsilon\|^2_{L^2\left(\Omega, H^1(\mathcal{B}_L^\omega)\right)} + \left\|u_\varepsilon - (u_0^{FF} + \varepsilon u_1^{FF})\right\|^2_{L^2\left(\Omega, H^1(\mathcal{B}_{L',L})\right)} \underset{\varepsilon\to 0}{=} \begin{cases} \mathcal{O}\left(\varepsilon^3|\log\varepsilon|\right) & \text{for } d = 2, \\ \mathcal{O}(\varepsilon^3) & \text{for } d = 3, \end{cases} \tag{89}$$

Corollary 21. *Suppose that $\mathcal{P}$ verifies (L^∞) and (Mix). For all $R > 0$ and $L > L' > \varepsilon H$, the following estimates hold*

$$\|u_\varepsilon - \widetilde{w}_\varepsilon\|^2_{L^2\left(\Omega, H^1(\mathcal{B}_L^\omega)\right)} + \|u_\varepsilon - v_\varepsilon\|^2_{L^2\left(\Omega, H^1(\mathcal{B}_{L',L})\right)} \underset{\varepsilon\to 0}{=} \begin{cases} \mathcal{O}\left(\varepsilon^3|\log\varepsilon|\right) & \text{for } d = 2, \\ \mathcal{O}(\varepsilon^3) & \text{for } d = 3, \end{cases}$$

Proof of Theorem 20. By Theorem 19, under Hypotheses (Mix) and (L^∞), for all $R \geq 1$

$$\mathbb{E}\left[\left|\mathbb{E}[W_1^\omega|_{\Sigma_L}] - \fint_{\square_R} W_1^\omega|_{\Sigma_L}\right|^2\right] \lesssim R^{-(d-1)}.$$

Consequently by Proposition 12 we know that V_1, defined in (72), verifies

$$y_d^{d-1}\mathbb{E}[|V_1|^2(\cdot, y_d)] + y_d^{d+1}\mathbb{E}[|\nabla V_1|^2(\cdot, y_d)] \underset{y_d\to+\infty}{=} \mathcal{O}(1).$$

We then obtain for $L > \varepsilon H$

$$\|V_1|_{\Sigma_{L/\varepsilon}}\|_{L^2(\Omega)} \underset{\varepsilon\to 0}{=} \mathcal{O}\left(\varepsilon^{\frac{d-1}{2}}\right), \quad \text{and} \quad \|\partial_{y_d} V_1|_{\Sigma_{L/\varepsilon}}\|_{L^2(\Omega)} \underset{\varepsilon\to 0}{=} \mathcal{O}\left(\varepsilon^{\frac{d+1}{2}}\right). \tag{90}$$

Inserting (90) and the result of Corollary 16 in the proof of Theorem 17 yields (89).

□

Proof of Corollary 21. The proof is identical as the proof of Corollary 18 but we take advantage here of the improved estimate of Corollary 16.

□

The rest of this section is dedicated to the proof of Theorem 19.

6.3.1 The quantitative mixing assumption

To prove Theorem 19 we need to assume that the point process associated to the distribution of particles $\mathcal{P}$ verifies a quantitative mixing condition. Since the dependency of W_1^ω in $\mathcal{P}$ is highly non-linear, we choose to write the hypothesis as a variance inequality on any function F of $\mathcal{P}$. As proven in [23], most common hard-core point processes such as random parking or hard-core Poisson point processes verify the following variance inequality.

Hypothesis (Mix)**.** *There exists a non-increasing weight function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ with super-algebraic decay such that $\mathcal{P}$ verifies for all $\sigma(\mathcal{P})$- measurable random variable $F(\mathcal{P})$,*

$$Var\left[F(\mathcal{P})\right] \leq \mathbb{E}\left[\int_1^{+\infty}\int_{\mathbb{R}^{d-1}}\left(\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}F(\mathcal{P})\right)^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}\ell^{-(d-1)}\pi(\ell-1)d\ell\right], \tag{91}$$

where $\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel}) := \square_\ell(\boldsymbol{x}_{\shortparallel}) \times (0, h)$ is the portion of the layer with width $\ell \geq 0$ centered at $(\boldsymbol{x}_{\shortparallel}, h/2)$ and the oscillation $\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}F(\mathcal{P})$ of $F(\mathcal{P})$ with respect to $\mathcal{P}$ on $\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})$ is defined by:

$$\begin{aligned}\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}F(\mathcal{P}) := sup\ ess &\left\{F(\mathcal{P}'), \mathcal{P}' \cap \left(\mathbb{R}^d \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right) = \mathcal{P} \cap \left(\mathbb{R}^d \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right)\right\} \\ &- inf\ ess\left\{F(\mathcal{P}'), \mathcal{P}' \cap \left(\mathbb{R}^d \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right) = \mathcal{P} \cap \left(\mathbb{R}^d \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right)\right\}.\end{aligned} \tag{92}$$

The structure of the proof follows the classical structure of proofs of quantitative stochastic homogenization estimates [28]. For a particularly pedagogical proof of the same nature but in a Gaussian setting we refer to [34]. More precisely to prove Theorem 19 we adapted [24] that study random suspensions of rigid particles in a steady Stokes flow to our setting of a thin layer of randomly distributed particles.

The starting point is to apply (91) to $\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}$ for $f \in L^\infty(\Sigma_L)$ with compact support. A preliminary step of the proof consists then in bounding a.s. the oscillation of the integral. This step is done in Section 6.3.4. The analysis requires the introduction of a (well-posed) auxiliary problem and the control of high stochastic moments of W_1. These constitute the respective subjects of Sections 6.3.2 and 6.3.3. Theorem 19 is then finally proven in Section 6.3.5.

6.3.2 Auxiliary problem

Let

$$W_0(D^\omega) := \left\{ v \in H^1_{loc}(D^\omega), v = 0 \,\text{on}\, \partial\mathcal{P}^\omega, \int_{\mathbb{R}^{d-1}} \int_0^{+\infty} \mathbb{1}_D \,(\mu|v|^2 + |\nabla v|^2)\mathrm{d}\boldsymbol{y} < +\infty \right\}. \tag{93}$$

Equipped with the norm

$$\forall v \in W_0(D^\omega), \quad \|v\|^2_{W_0(D^\omega)} := \int_{\mathbb{R}^{d-1}} \int_0^{+\infty} \mathbb{1}_D \,(\mu|v|^2 + |\nabla v|^2)\mathrm{d}\boldsymbol{y},$$

the weighted Sobolev space $W_0(D^\omega)$ is a Hilbert space. Recall that we proved in Proposition 8 (see Remark 3) that if $v \in W_0(D^\omega)$ for $\omega \in \Omega$, then

$$\int_{D^\omega} \mu^\omega |v^\omega|^2 \lesssim \int_{D^\omega} |\nabla v^\omega|^2, \tag{94}$$

so that the $W_0(D^\omega)$-norm and the H^1 semi-norm are equivalent in $W_0(D^\omega)$.

Consider now the following adjoint auxiliary problem

$$\begin{cases} -\Delta u^\omega_{f,\boldsymbol{g}} = \nabla \cdot \boldsymbol{g} \text{ in } D^\omega \setminus (\Sigma_H \cup \Sigma_L), \\ -\partial_{y_d} u^\omega_{f,\boldsymbol{g}} = 0 \text{ on } \Sigma_0, \quad \text{and} \quad u^\omega_{f,\boldsymbol{g}} = 0 \text{ on } \partial\mathcal{P}^\omega, \\ \left[u^\omega_{f,\boldsymbol{g}}\right]_H = 0 \quad \text{and} \quad \left[-\partial_{y_d} u^\omega_{f,\boldsymbol{g}}\right]_H = 0, \\ \left[u^\omega_{f,\boldsymbol{g}}\right]_L = 0 \quad \text{and} \quad \left[-\partial_{y_d} u^\omega_{f,\boldsymbol{g}}\right]_L = f, \end{cases} \tag{95}$$

for $f \in L^\infty(\Sigma_L)$ and $\boldsymbol{g} \in L^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^+)^d$ both with compact support. Problem (95) is well-posed a.s. in $W_0(D^\omega)$ as stated by the following proposition.

Proposition 22. *Let $f \in L^2(\Sigma_L)$ and $\boldsymbol{g} \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+)^d$ with compact support. There exists a unique process $u_{f,\boldsymbol{g}}$ such that a.s. $u^\omega_{f,\boldsymbol{g}} \in W_0(D^\omega)$ is a weak solution of* (95) *and*

$$\|\nabla u_{f,\boldsymbol{g}}\|_{L^2(D^\omega)} \lesssim \|\mu^{-\frac{1}{2}} f\|_{L^2(\Sigma_L)} + \|\boldsymbol{g}\|_{L^2(D^\omega)^d}. \tag{96}$$

Proof of Proposition 22. It suffices to use Lax-Milgram Theorem. The variational formulation reads for

$$\forall v^\omega \in W_0(D^\omega), \quad \int_{D^\omega} \nabla u^\omega_{f,\boldsymbol{g}} \cdot \nabla \overline{v}^\omega = \int_{D^\omega} \boldsymbol{g} \cdot \nabla \overline{v}^\omega + \int_{\Sigma_L} f \overline{v^\omega}. \tag{97}$$

The coercivity of the sesquilinear form is ensured by (94). Since $f \in L^2(\Sigma_L)$ has compact support, $\mu^{-\frac{1}{2}} f \in L^2(\Sigma_L)$ and

$$\left| \int_{\Sigma_L} f \overline{v}^\omega \right| \le \|\mu^{-\frac{1}{2}} f\|_{L^2(\Sigma_L)} \|v^\omega\|_{W_0(D^\omega)}.$$

□

Theorem 23 (L^p-regularity of $u_{f,\boldsymbol{g}}$). *Suppose that $\mathcal{P}$ verifies (L^∞) and (Mix). There exists $C_0 \simeq 1$ such that for all $f \in L^\infty(\Sigma_L)$ and $\boldsymbol{g} \in L^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^+)^d$ with compact support, the solution $u_{f,\boldsymbol{g}}$ of the auxiliary problem* (95) *statisfies*

$$\|\mathbb{1}_D \nabla u_{f,\boldsymbol{g}}\|_{L^p(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^q(\Omega))} \lesssim \|f\|_{L^p(\Sigma_L)} + \|\boldsymbol{g}\|_{L^p(\mathbb{R}^{d-1}\times\mathbb{R}^+)^d}, \tag{98}$$

for all $|p-2|, |q-2| \leq \frac{1}{C_0}$.

The proof of the theorem is postponed to Appendix B.

6.3.3 Higher stochastic moments of ∇W_1

The last ingredient we need is a control in $L^q(\Omega)$-norm of $\|\mathbb{1}_D \nabla W_1\|_{L^2(\square_1(\boldsymbol{x}_{\shortparallel})\times(0,L))}$ for L large enough.

Theorem 24 (Stochastic moments of ∇W_1). *Assume that $\mathcal{P}$ verifies hypotheses (L^∞) and (Mix). For all $q \geq 2$, $R_1 \geq 1$, the following estimate holds for all L large enough*

$$\mathbb{E}\left[\left(\fint_{\square_{R_1}} \int_0^L \mathbb{1}_D |\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \lesssim 1, \tag{99}$$

where the constant depends on R_1 and L.

To prove Theorem 24 we adapt the proof of [24, Theorem 4.2] to our problem. The proof relies on two arguments that we state in the two propositions below. The first result is a control in the $L^q(\Omega)$-norm of the fluctuations of ∇W_1.

Proposition 25 (Fluctuations of ∇W_1). *Providing that $\mathcal{P}$ verifies hypotheses (L^∞) and (Mix), ∇W_1^ω verifies for $\boldsymbol{g} \in L^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^+)^d$ with compact support, $q \gg 2$ and $R_1 \geq 1$*

$$\mathbb{E}\left[\left|\int_0^{+\infty}\int_{\mathbb{R}^{d-1}} \mathbb{1}_{D^\omega} \boldsymbol{g} \cdot \nabla W_1^\omega \mathrm{d}\boldsymbol{y}\right|^q\right]^{\frac{1}{q}} \lesssim \|\boldsymbol{g}\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)^d} \mathbb{E}\left[\left(\int_{\square_{R_1}}\int_0^L \mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}. \tag{100}$$

The second argument consists in bounding local norms of ∇W_1 (such as the one appearing in the right-hand side of (100)) with large scale averages of ∇W_1.

Proposition 26. *Suppose that $\mathcal{P}$ verifies Hypotheses (L^∞) and (Mix). Consider $\chi \in C_c^\infty(\square_1 \times (0,1))$ such that $\int_{\square_1\times(0,1)} \chi = 1$ and define for $r > 0$, $\chi_r(\boldsymbol{x}) := r^{-d}\chi\left(\frac{\boldsymbol{x}}{r}\right)$. For all $R \gg 1$, $L \gg 1$, $\alpha \in (0,1)$ and $q \geq 2$*

$$\mathbb{E}\left[\left(\fint_{\square_R}\int_0^L \mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \lesssim 1 + \left(\int_0^{2L} \mathbb{E}\left[\left|\int_{\mathbb{R}^{d-1}}\int_0^{+\infty} \mathbb{1}_{D^\omega}\chi_{R^\alpha}(\boldsymbol{y}_{\shortparallel}, x_d - y_d)\nabla W_1^\omega(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^q\right]^{\frac{2}{q}} \mathrm{d}x_d\right)^{\frac{1}{2}},$$

where the constant only depends on χ.

We prove below Theorem 24. The proof of Propositions 25 and 26 is postponed to Appendix C.

Proof of Theorem 24. We prove here (99) for $q \gg 2$. By Jensen's inequality the result can then be extended to all $q \geq 2$. By stationarity of W_1 we have on one hand for all $R_1, R_2 \geq 1$, $L \geq H$

$$\begin{aligned}
\mathbb{E}\left[\left(\fint_{\square_{R_1}}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} &= \mathbb{E}\left[\fint_{\square_{R_2}}\left(\fint_{\square_{R_1}(\boldsymbol{x}_{\shortparallel})}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{q-1}{2}}\left(\fint_{\square_{R_1}(\boldsymbol{x}_{\shortparallel})}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{1}{2}}\mathrm{d}\boldsymbol{x}_{\shortparallel}\right]^{\frac{1}{q}}\\
&\leq \left(\frac{R_1+R_2}{R_1}\right)^{\frac{(d-1)(q-1)}{2q}}\mathbb{E}\left[\left(\fint_{\square_{R_1+R_2}}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{q-1}{2}}\fint_{\square_{R_2}}\left(\fint_{\square_{R_1}(\boldsymbol{x}_{\shortparallel})}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{1}{2}}\mathrm{d}\boldsymbol{x}_{\shortparallel}\right]^{\frac{1}{q}}\\
&\leq \left(\frac{R_1+R_2}{R_1}\right)^{\frac{(d-1)(q-1)}{2q}}\frac{1}{R_2^{\frac{d-1}{q}}}\mathbb{E}\left[\left(\fint_{\square_{R_1+R_2}}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{q-1}{2}}\left(\int_{\square_{R_2}}\fint_{\square_{R_1}(\boldsymbol{x}_{\shortparallel})}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{1}{2}}\mathrm{d}\boldsymbol{x}_{\shortparallel}\right]^{\frac{1}{q}}\\
&\leq \left(\frac{R_1+R_2}{R_1}\right)^{\frac{(d-1)}{2}}\frac{1}{R_2^{\frac{d-1}{q}}}\mathbb{E}\left[\left(\fint_{\square_{R_1+R_2}}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}},
\end{aligned}\tag{101}$$

where we used Jensen's inequality since $x \mapsto \sqrt{x}$ is concave. On the other hand according first to Proposition 26 for $R \gg 1$, $L \gg 1$ and $q \gg 2$ and second to Proposition 25

$$\begin{aligned}
\mathbb{E}\left[\left(\fint_{\square_{R}}\int_0^L |\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} &\lesssim 1+\left(\int_0^{2L}\mathbb{E}\left[\left|\int_{\mathbb{R}^{d-1}}\int_0^{+\infty}\chi_{R^\alpha}(\boldsymbol{y},x_d-y_d)\nabla W_1(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^q\right]^{\frac{2}{q}}\mathrm{d}x_d\right)^{\frac{1}{2}}\\
&\lesssim 1+\left(\int_0^{2L}\|\chi_{R^\alpha}(x_d-\cdot)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)}\mathrm{d}x_d\right)^{\frac{1}{2}}\mathbb{E}\left[\left(\int_{\square_{R_1}}\int_0^L |\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\\
&\lesssim 1+R^{-\frac{\alpha d}{2}}R_1^{\frac{d-1}{2}}\mathbb{E}\left[\left(\fint_{\square_{R_1}}\int_0^L |\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}
\end{aligned}\tag{102}$$

Injecting (102) with $R = R_1 + R_2$ into the final inequality of (101) yields

$$\mathbb{E}\left[\left(\fint_{\square_{R_1}}\int_0^L \mathbb{1}_D|\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \lesssim \left(\frac{R_1+R_2}{R_1}\right)^{\frac{d-1}{2}}\frac{1}{R_2^{\frac{d-1}{q}}}+\frac{(R_1+R_2)^{\frac{(1-\alpha)d-1}{2}}}{R_2^{\frac{d-1}{q}}}\mathbb{E}\left[\left(\fint_{\square_{R_1}}\int_0^L |\nabla W_1|^2\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}$$

With α chosen so that $(1-\alpha)d < 1$ and R_2 large enough, the second term of the right-hand side can be absorbed into the left hand-side and we get the result. $\square$

6.3.4 A.s. estimate of the oscillation

We prove that the oscillation of $\displaystyle\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel})W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}$ in $\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})$ can be controlled by the L^2-norms of ∇W_1^ω and $\nabla u^\omega_{f,\mathbf{0}}$ in a layer slightly larger (represented in orange on Figure 3) than $\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})$ (represented in red on Figure 3).

Proposition 27. *Let $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$ and $\ell \geq 1$. For $f \in L^\infty(\Sigma_L)$ with compact support*

$$\left|\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel})W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\right| \lesssim (\ell^\omega_+(\boldsymbol{x}_{\shortparallel}))^2\|\mathbb{1}_{D^\omega}\nabla u^\omega_f\|_{L^2(\mathcal{L}_{\ell^\omega_+}(\boldsymbol{x}_{\shortparallel}))}\|\mathbb{1}_{D^\omega}\nabla W^\omega_1\|_{L^2(\mathcal{L}_{\ell^\omega_+}(\boldsymbol{x}_{\shortparallel}))},$$

where $u^\omega_f := u^\omega_{f,\mathbf{0}}$ and $\ell^\omega_+(\boldsymbol{x}_{\shortparallel})$ is the smallest distance such that $\mathcal{L}_{\ell^\omega_+}(\boldsymbol{x}_{\shortparallel}) \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}$ contains at least one particle (see Figure 3), i.e.

$$\ell^\omega_+(\boldsymbol{x}_{\shortparallel}) := \mathit{argmin}\left\{d > \ell, \quad \exists j \in \mathbb{N}, B(\boldsymbol{x}^\omega_j) \subset \mathcal{L}_d(\boldsymbol{x}_{\shortparallel}) \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right\}. \tag{103}$$

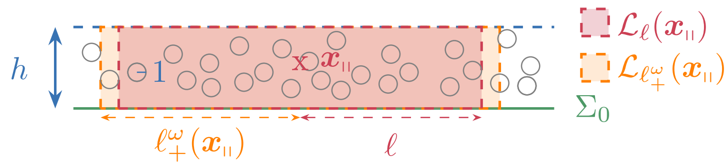


Figure 3: Notations used in Section 6.3.4

Proof of proposition 27. Let $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$ and $\ell \geq 1$. Consider $\mathcal{P}^\omega$ a realization of the particle distribution in the layer and let $\mathcal{P}'$ be a repartition of particles satisfying the hard-core assumption and such that

$$\mathcal{P}' \cap \left(\mathbb{R}^d \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right) = \mathcal{P}^\omega \cap \left(\mathbb{R}^d \setminus \overline{\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\right).$$

We let W_1' denote the solution of problem (41) where $\mathcal{P}^\omega$ (resp. D^ω) is replaced by $\mathcal{P}'$ (resp. $D' := \mathbb{R}^{d-1} \times \mathbb{R}^+ \setminus \overline{\mathcal{P}'}$).

Step 1: We first prove that W_1' is well-defined and verifies

$$\|\mathbb{1}_{D'} \nabla W_1'\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))} \lesssim \ell_+^\omega(\boldsymbol{x}_{\shortparallel}) \, \|\mathbb{1}_{D^\omega} \nabla W_1^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}. \tag{104}$$

Consider the set of particles of $\mathcal{P}^\omega$ (resp. $\mathcal{P}'$) intersecting with $\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})$, *i.e.*

$$\mathcal{P}_\ell^\omega := \bigcup_{i \in \mathcal{I}_\ell^\omega(\boldsymbol{x}_{\shortparallel})} B(\boldsymbol{x}_i^\omega), \text{ with } \quad \mathcal{I}_\ell^\omega(\boldsymbol{x}_{\shortparallel}) := \{i \in \mathbb{N}, \quad \boldsymbol{x}_i^\omega \in \mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})\},$$

$$\left(\text{resp. } \mathcal{P}_\ell' := \bigcup_{i \in \mathcal{I}_\ell'(\boldsymbol{x}_{\shortparallel})} B(\boldsymbol{x}_i'), \text{ with } \quad \mathcal{I}_\ell'(\boldsymbol{x}_{\shortparallel}) := \left\{i \in \mathbb{N}, \quad \boldsymbol{x}_i' \in \mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})\right\}\right).$$

Let $\mathcal{S} := \text{Hull}\left(\mathcal{P}_\ell^\omega\right)$ where Hull denotes the convex hull. Let χ be a smooth cut-off function such that $\text{supp}\chi \subset \mathcal{S}$ then $X^\omega := W_1' - \mathbb{1}_{\mathcal{P}_\ell^\omega} W_1^\omega(1-\chi)$ verifies the following problem in D'

$$\begin{cases} -\Delta X^\omega = \Delta(\chi W_1^\omega) \text{ in } \quad D' \setminus \Sigma_H, \\ -\partial_{y_d} X^\omega = 0 \text{ on } \quad \Sigma_0, \quad \text{and} \quad X^\omega = 0 \text{ on } \quad \mathcal{P}', \\ \Big[X^\omega\Big]_H = 0 \quad \text{and} \quad \Big[-\partial_{y_d} X^\omega\Big]_H = 0. \end{cases} \tag{105}$$

Since $\nabla(\chi W_1) \in [L^2(D')]^d$ with compact support, there exists a unique $X^\omega \in W_0(D')$ by proposition 22. Hence W_1' is well-defined. Let us now show that it verifies estimate (104).

Let $\widetilde{D}^\omega := D^\omega \cup \overline{\mathcal{P}_\ell^\omega}$. $Z^\omega := \mathbb{1}_{\mathcal{P}_\ell'} W_1' - \mathbb{1}_{\mathcal{P}_\ell^\omega} W_1^\omega$ verifies

$$\forall v \in W_0(\widetilde{D}^\omega), \quad \int_{\widetilde{D}^\omega} \nabla Z^\omega \cdot \nabla \overline{v} = \int_{\partial \mathcal{P}_\ell^\omega} \nabla W_1^\omega \cdot \boldsymbol{n} \, \overline{v} + \int_{\partial \mathcal{P}_\ell'} \nabla W_1' \cdot \boldsymbol{n} \, \overline{v}. \tag{106}$$

To simplify notations we use here integrals on the boundary of $\mathcal{P}_\ell^\omega$ and $\mathcal{P}_\ell'$ to denote the duality product $H^{-1/2}, H^{1/2}$.

The coercivity of the associated sesquilinear form is a direct consequence of the weighted Poincaré inequality (94). The continuity of the linear form is ensured by the trace theorem and a Poincare's inequality that holds with a constant proportional to $\ell_+^\omega(\boldsymbol{x}_{\shortparallel})$ since at least one particle lies in $\mathcal{L}_{\ell_+^\omega(\boldsymbol{x}_{\shortparallel})}(\boldsymbol{x}_{\shortparallel})$. We obtain for all $i \in \mathcal{I}_\ell'(\boldsymbol{x}_{\shortparallel}), j \in \mathcal{I}_\ell^\omega(\boldsymbol{x}_{\shortparallel}), v \in W_0(\widetilde{D}^\omega)$

$$\|v\|_{H^{1/2}(\partial B(\boldsymbol{x}_j^\omega))} + \|v\|_{H^{1/2}(\partial B(\boldsymbol{x}_i'))} \lesssim \ell_+^\omega(\boldsymbol{x}_{\shortparallel}) \, \|\nabla v\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}. \tag{107}$$

Let us now estimate $\|\nabla W_1^\omega \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial B(\boldsymbol{x}_i^\omega))}$ for $i \in \mathcal{I}_\ell^\omega(\boldsymbol{x}_{\shortparallel})$. We proceed by duality. Let $\psi_i \in H^{1/2}(\partial B(\boldsymbol{x}_i^\omega))$ and $\Psi_i \in H^1(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))$ a lifting of ψ such that $\text{supp}\Psi_i \subset \mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel})$, $\Psi_i|_{\partial B(\boldsymbol{x}_i^\omega)} = \psi_i \delta_{i,j}$ for all $j \in \mathcal{I}_\ell^\omega(\boldsymbol{x}_{\shortparallel})$ and

$$\|\Psi_i\|_{H^1(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))} \leq \|\psi_i\|_{H^{1/2}(\partial B(\boldsymbol{x}_i^\omega))}.$$

We write the weak formulation associated to (41) in $\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel})$

$$\int_{\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel})} \mathbb{1}_{D^\omega} \nabla W_1^\omega \cdot \nabla \overline{\Psi_i} = \int_{\partial B(\boldsymbol{x}_i^\omega)} \nabla W_1^\omega \cdot \boldsymbol{n}\, \overline{\psi_i}.$$

By definition of the $H^{-1/2}$-norm we deduce that

$$\|\nabla W_1^\omega \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial B(\boldsymbol{x}_i^\omega))} \leq \|\mathbb{1}_{D^\omega} \nabla W_1^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}.$$

We can prove similarly for $i \in \mathcal{I}'_\ell(\boldsymbol{x}_{\shortparallel})$, $\|\nabla W_1^{'} \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial B(\boldsymbol{x}'_i))} \leq \|\mathbb{1}_{D'} \nabla W'_1\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}$.

Testing (106) with $v = Z^\omega$, we get

$$\int_{\widetilde{D}^\omega} |\nabla Z^\omega|^2 = -\int_{\partial \mathcal{P}_\ell^\omega} \nabla W_1^\omega \cdot \boldsymbol{n}\, \mathbb{1}_{D'} \overline{W'_1} - \int_{\partial \mathcal{P}'_\ell} \nabla W_1^{'} \cdot \boldsymbol{n}\, \mathbb{1}_{D^\omega} \overline{W_1^\omega},$$

and therefore

$$\int_{\widetilde{D}^\omega} |\nabla Z^\omega|^2 \lesssim \ell_+^\omega(\boldsymbol{x}_{\shortparallel}) \|\mathbb{1}_{D^\omega} \nabla W_1^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))} \|\mathbb{1}_{D'} \nabla W_1^{'}\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}.$$

Claim (104) follows from Young's inequality.

Step 2: Sensitivity of $W_1|_{\Sigma_L}$

Recall that $u_f := u_{f,\mathbf{0}}$ denotes the unique weak solution in $W_0(D^\omega)$ of (95) with $\boldsymbol{g} = \mathbf{0}$. The weak formulation associated to (95) reads for all $v \in W_0(\widetilde{D}^\omega)$

$$\int_{\widetilde{D}^\omega} \mathbb{1}_{D^\omega} \nabla u_f^\omega \cdot \nabla \overline{v} = \int_{\Sigma_L} f \overline{v} + \int_{\partial \mathcal{P}_\ell^\omega} \nabla u_f^\omega \cdot \boldsymbol{n}\, \overline{v}. \tag{108}$$

We evaluate (106) for $v = \mathbb{1}_{D^\omega} \overline{u_f^\omega}$ and (108) for $v = \overline{Z^\omega}$. After subtracting both expressions and since $u_f^\omega = W_1^\omega = 0$ on $\partial \mathcal{P}^\omega$, we obtain

$$\int_{\Sigma_L} f Z^\omega = -\int_{\partial \mathcal{P}_\ell^\omega} \nabla u_f^\omega \cdot \boldsymbol{n}\, \mathbb{1}_{D'} W_1^{'} - \int_{\partial \mathcal{P}'_\ell} \nabla W_1^{'} \cdot \boldsymbol{n}\, \mathbb{1}_{D^\omega} u_f^\omega.$$

Next we can estimate $\|\nabla u_f^\omega \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial B(\boldsymbol{x}_i^\omega))}$ for $i \in \mathcal{I}_\ell(\boldsymbol{x}_{\shortparallel})$ using similar steps as in the estimate of $\|\nabla W_1^\omega \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial B(\boldsymbol{x}_j^\omega))}$. We deduce that

$$\|\nabla u_f \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial B(\boldsymbol{x}_i^\omega))} \leq \|\mathbb{1}_{D^\omega} \nabla u_f^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}.$$

Therefore

$$\left| \int_{\Sigma_L} f(\mathbb{1}_{\mathcal{P}'_\ell} W'_1 - \mathbb{1}_{\mathcal{P}_\ell^\omega} W_1^\omega) \right| \lesssim \ell_+^\omega(\boldsymbol{x}_{\shortparallel})\, \|\mathbb{1}_{D^\omega} \nabla u_f^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}\, \|\mathbb{1}_{D'} \nabla W_1^{'}\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}.$$

Using (104) we deduce that

$$\left| \int_{\Sigma_L} f(\mathbb{1}_{\mathcal{P}'_\ell} W'_1 - \mathbb{1}_{\mathcal{P}''_\ell} W_1^{''}) \right| \lesssim \ell_+^\omega(\boldsymbol{x}_{\shortparallel})\, \|\mathbb{1}_{D^\omega} \nabla u_f^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}\, \|\mathbb{1}_{D^\omega} \nabla W_1^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}.$$

where $\mathcal{P}''_\ell$ satisfies the same assumption than $\mathcal{P}'_\ell$ and W''_1 is the associated solution. By definition of (92), this allows us to conclude. □

6.3.5 Proof of Theorem 19

Let $f \in L^\infty(\Sigma_L)$ with compact support. The first step consists in applying the mixing hypothesis (Hypothesis (Mix)) to $\displaystyle\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}$ and use Proposition 27.

$$\begin{aligned}\mathrm{Var}\left[\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\right] &\leq \mathbb{E}\left[\int_1^{+\infty}\int_{\mathbb{R}^{d-1}}\left|\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\right|^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}\ell^{-(d-1)}\pi(\ell-1)\mathrm{d}\ell\right]\\ &\lesssim \mathbb{E}\left[\int_1^{+\infty}\int_{\mathbb{R}^{d-1}}(\ell_+^\omega(\boldsymbol{x}_{\shortparallel}))^4\|\mathbb{1}_{D^\omega}\nabla u_f^\omega\|^2_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}\|\mathbb{1}_{D^\omega}\nabla W_1^\omega\|^2_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}\mathrm{d}\boldsymbol{x}_{\shortparallel}\ell^{-(d-1)}\pi(\ell-1)\mathrm{d}\ell\right]\end{aligned}$$

Next we use Hypothesis (L^∞) to bound a.s. and for a.e. $\boldsymbol{x}_{\shortparallel} \in R^{d-1}$, $\ell_+^\omega(\boldsymbol{x}_{\shortparallel})$ by $\ell + R^+$.

$$\begin{aligned}\mathrm{Var}\left[\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\right] \lesssim \int_1^{+\infty}\int_{\mathbb{R}^{d-1}}(\ell+R^+)^{2(d+1)}\mathbb{E}\Bigg[&\left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{1}_{D^\omega}|\nabla u_f^\omega|^2\right)\\ &\left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)\Bigg]\mathrm{d}\boldsymbol{x}_{\shortparallel}\ell^{-(d-1)}\pi(\ell-1)\mathrm{d}\ell\end{aligned}$$

By Hölder's inequality, we get for q such that $|q-1| \leq \frac{1}{2C_0}$ where C_0 is defined in Theorem 23 and $q' := q/(q-1)$

$$\begin{aligned}\mathrm{Var}\left[\int_{\Sigma_L} f(\boldsymbol{y}_{\shortparallel}) W_1^\omega|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\right] \lesssim \int_1^{+\infty}\int_{\mathbb{R}^{d-1}}(\ell+R^+)^{2(d+1)}&\mathbb{E}\left[\left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{1}_{D^\omega}|\nabla u_f^\omega|^2\right)^q\right]^{1/q}\\ &\mathbb{E}\left[\left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{q'}\right]^{1/q'}\mathrm{d}\boldsymbol{x}_{\shortparallel}\ell^{-(d-1)}\pi(\ell-1)\mathrm{d}\ell.\end{aligned}$$

We use the stationarity of W_1 and Theorem 24 with $R_1 = \ell + R^+$ to deal with the near-field term

$$\mathbb{E}\left[\left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{q'}\right]^{1/q'} = \mathbb{E}\left[\left(\fint_{\square_{\ell+R^+}(0)}\int_0^h \mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{q'}\right]^{1/q'} \lesssim 1. \tag{109}$$

Since $X \mapsto \mathbb{E}[|X|^q]^{1/q}$ is a norm we get by Jensen's inequality

$$\begin{aligned}\int_{\mathbb{R}^{d-1}}\mathbb{E}\left[\left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{1}_{D^\omega}|\nabla u_f^\omega|^2\right)^q\right]^{1/q}\mathrm{d}\boldsymbol{x}_{\shortparallel} &\leq \int_{\mathbb{R}^{d-1}}\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})}\int_0^h \mathbb{E}\left[\mathbb{1}_{D^\omega}|\nabla u_f^\omega|^{2q}\right]^{1/q}\\ &= \int_{\mathbb{R}^{d-1}}\int_0^h \mathbb{E}\left[\mathbb{1}_{D^\omega}|\nabla u_f^\omega|^{2q}\right]^{1/q}\\ &= \|\mathbb{1}_D\nabla u_f^\omega\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^{2q}(\Omega))}.\end{aligned} \tag{110}$$

On the penultimate row we simplified the spatial average in the second term of the right-hand side by noticing that by Fubini-Tonelli's Theorem for $h \in L^1(\mathbb{R}^{d-1})$

$$\int_{\mathbb{R}^{d-1}}\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} h(\boldsymbol{y}_{\shortparallel})\mathrm{d}\boldsymbol{y}_{\shortparallel}\mathrm{d}\boldsymbol{x}_{\shortparallel} = \int_{\mathbb{R}^{d-1}} h(\boldsymbol{x}_{\shortparallel})\mathrm{d}\boldsymbol{x}_{\shortparallel}.$$

We conclude with the regularity result on the adjoint problem (see Theorem 23). Finally since π is super-algebraic the integral over ℓ is finite.

$$\int_1^{+\infty}(\ell+R^+)^{2(d+1)}\ell^{-(d-1)}\pi(\ell-1)\mathrm{d}\ell < +\infty. \tag{111}$$

Combining (109), (110), (111) yields the desired result.

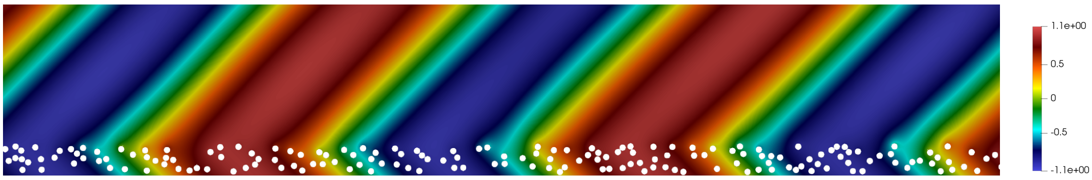


Figure 4: Real part of the total field u_ε for $\rho = 0.4$, frequency= 2GHz, $\theta = \pi/4$, $k_2\varepsilon = 10^{-1}$m, $\gamma = 1+i$

7 Numerical results

In this section we present numerical simulations that illustrate our theoretical results and more specifically the error estimates derived in Corollary 21. All simulations were conducted using XLife++ an open source FEM-BEM solver [35].

Instead of considering a compactly supported source we study the scattering by an incident plane wave $u_{inc} := e^{i(k_1x_1+k_2x_2)}$. We solve then the following problem

$$\begin{cases} -\Delta u_\varepsilon - k^2 u_\varepsilon = 0 & \text{in } \mathcal{B}_{\varepsilon H,L}, \\ -\partial_{x_d} u_\varepsilon + ik\gamma u_\varepsilon = 0 & \text{on } \Sigma_{\varepsilon H}, \\ -\partial_{x_d} u_\varepsilon + \Lambda^k u_\varepsilon = -\partial_{x_d} u_{inc} + \Lambda^k u_{inc} & \text{on } \Sigma_L, \end{cases}$$

Since the problem satisfied by u_ε is unbounded in the longitudinal directions, we approximate it by truncating the layer $L_T := \square_T \times (0, \varepsilon h)$ with $T = 400$ and impose that the field, still denoted u_ε, is $k_1 T$-quasi-periodic, *i.e.* $x_1 \mapsto e^{-k_1 x_1} u_\varepsilon(x_1, x_2)$ is T-periodic. Thanks to the quasi-periodicity the Dirichlet-to-Neumann operator on $\Sigma_L^T := \square_T \times \{y_d = L\}\}$ admits the following modal decomposition

$$\forall v \in H^{1/2}(\Sigma_L^T), \quad \Lambda^k v(\cdot) = i \sum_{m\in\mathbb{Z}} \beta_m \langle v, \varphi_m \rangle_{L^2(\Sigma_L^T)} \varphi_m(\cdot),$$

where $\beta_m^2 := k^2 - (2m\pi/T + k_1)^2$ and $\varphi_m(x_1) := 1/\sqrt{T} e^{i(2m\pi/T+k_1)x_1}$ for $x_1 \in \mathbb{R}$. To implement this operator we truncate the series to $-N \le m \le N$ where $N > 0$ is chosen such that

$$e^{-\sqrt{(2N\pi/T+k_1)^2-k^2}(L-\varepsilon H)} < \eta$$

with η being a small threshold parameter ($\eta = 10^{-6}$ in the simulations). The centers of the particles are sampled according to a Matèrn point process [32, Section 6.5.2] restricted to the truncated layer L_T. The algorithm follows the following steps :

1. for a given initial density $\rho \in (0,1)$ we sample uniformly in L_T, N_p centers where N_p is drawn according to a Poisson distribution of parameter $\nu^\varepsilon := \rho T \varepsilon h / \pi \varepsilon^2$;
2. we assign to each center independently and randomly a score between 0 and 1;
3. we remove the centers violating the hard-core assumption with the centers with a lower score.

On Figure 4, the real part of the total field u_ε^ω for a given realization ω is displayed.

To implement the effective model (62) the first step is to compute c_1. By stationarity of W_1 for all $R > 0$, it holds that

$$c_1 = \mathbb{E}[W_1\big|_{\Sigma_L}] = \mathbb{E}\left[\fint_{\square_R} W_1\big|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel}) \mathrm{d}\boldsymbol{y}_{\shortparallel} \right].$$

To compute the expectation we use a Monte-Carlo algorithm so that

$$c_1 \sim \frac{1}{N} \sum_{j=1}^{N} \left[\fint_{\square_R} W_1^{\omega_j}\big|_{\Sigma_L}(\boldsymbol{y}_{\shortparallel})\, \mathrm{d}\boldsymbol{y}_{\shortparallel} \right]. \tag{112}$$

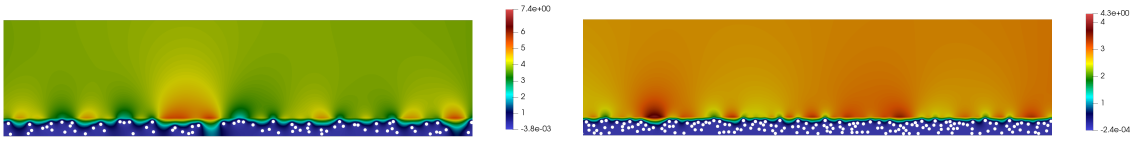


Figure 5: Profile function W_1 for $\rho = 0.1$ (left) and $\rho = 0.4$ (right)

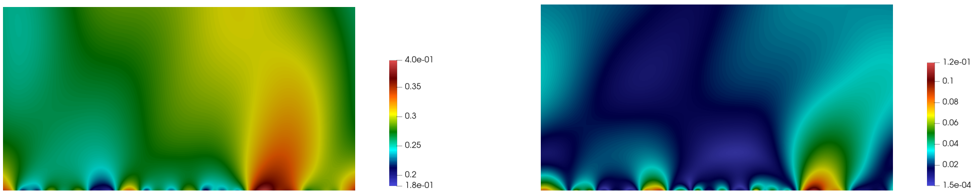


Figure 6: $|u_\varepsilon^\omega - u_0^{FF}|$ (left) and $|u_\varepsilon^\omega - v_\varepsilon|$ (right) on $(\varepsilon H, \varepsilon L)$ for a given ω and $L = 70$

W_1^ω is solution of a Laplace-type problem in $\mathcal{B}_L^\omega$, an unbounded random domain. To compute it numerically, we restrict the domain to $\square_R \times [0, L] \setminus \mathcal{P}^\omega$ and impose periodic boundary conditions at $x_1 = -R/2, R/2$ as it is customary in stochastic homogenization (see [16]). Since $y_1 \to W_1^\omega(y_1, y_2)$ is R-periodic, the DtN operator on $\Sigma_L^\sharp := \{(y_1, L), y_1 \in (-R/2, R/2)\}$ admits the following modal decomposition

$$\forall v \in H_\sharp^{1/2}(\Sigma_L^\sharp), \quad \Lambda v(\cdot) = \sum_{m \in \mathbb{Z}} \frac{2|m|\pi}{R} \langle v, \phi_m \rangle_{L^2(\Sigma_L^\sharp)} \phi_m(\cdot),$$

where $\phi_m(y_1) := 1/\sqrt{R} e^{\imath 2m\pi y_1/R}$ for $y_1 \in \mathbb{R}$.To implement this operator we truncate the series to $-N \leq m \leq N$ where $N > 0$ is chosen such that

$$e^{-2N\pi(L-H)/R} < \eta.$$

The centers of the particles are sampled according to a Matèrn point process restricted to the truncated layer $\square_R \times (0, h)$. Keep in mind that contrarily to the simulation of u_ε, the particles are now of size 1 so that the parameter of the Poisson process for a given ρ is $\nu := \rho R h/\pi$. On Figure 5 two plots of W_1^ω for two different realizations and two different ρ are displayed. We see that the density of particles plays a role in the convergence rate of W_1^ω when $y_2 \to +\infty$.

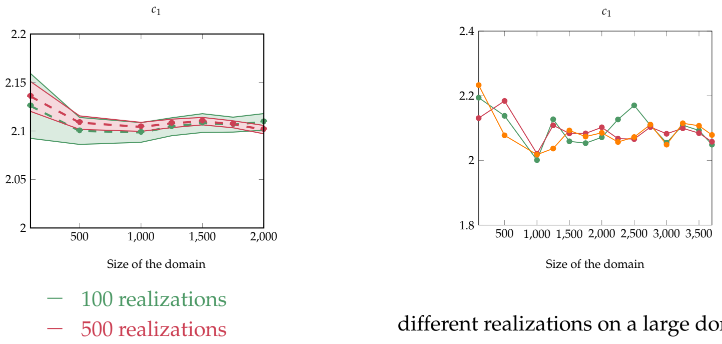
In (112) we can adjust the size of the domain R and the number of realizations N to improve the rate of convergence. On Figure 7 on the left, we have plotted the value of the computed constant for different sizes R of the computational domain. In green, the average is computed over 100 realizations and in red over 500 realizations. The two averages are displayed as the dotted lines and the colored zones correspond to the 95%-confidence interval. On Figure 7 on the right, the constant is computed with one realization for three different realizations thanks to the ergodicity. We notice that the computed coefficient does converge to the same limit as the ones computed with Monte-Carlo algorithm but as expected it requires a much larger domain to achieve convergence (3200 vs 1700).

Finally we verify the convergence rate of the error between the effective model and the reference solution. In the case of an incident plane wave $u_{inc} := e^{i(k_1 x_1 + k_2 x_2)}$ the solution v_ε to the effective model problem

$$\begin{cases} -\Delta v_\varepsilon - k^2 v_\varepsilon = 0 & \text{in } \mathcal{B}_{\varepsilon H, L}, \\ -\varepsilon c_1 \partial_{x_d} v_\varepsilon + v_\varepsilon = 0 & \text{on } \Sigma_{\varepsilon H}, \\ -\partial_{x_d} v_\varepsilon + \Lambda^k v_\varepsilon = -\partial_{x_d} u_{inc} + \Lambda^k u_{inc} & \text{on } \quad \Sigma_L, \end{cases}$$

is explicit:

$$v_\varepsilon(x_1, x_2) = e^{i(k_1 x_1 + k_2 x_2)} + r_\varepsilon^2 e^{i(k_1 x_1 - k_2 x_2)}$$



different realizations on a large domain

Figure 7: Computation of the effective model's coefficient

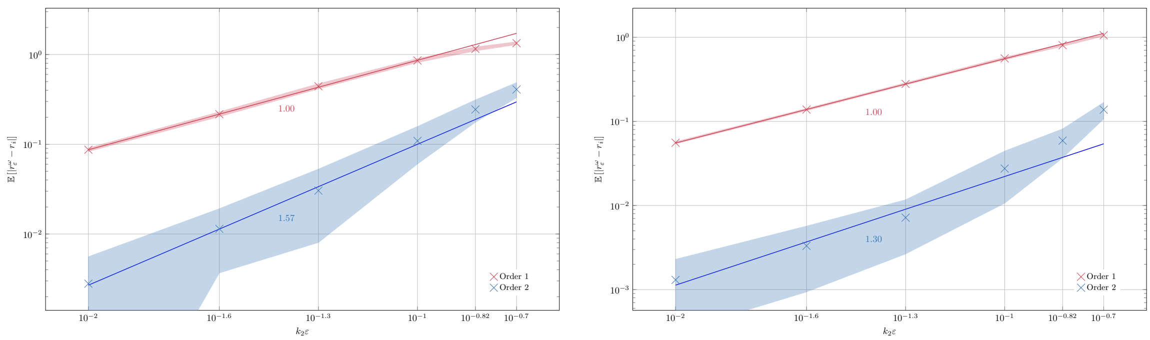


Figure 8: Error between the reflection coefficients for the reference solution and the effective model for $\rho = 0.1$ (left) and $\rho = 0.4$ (right)

for $x_1 \in \mathbb{R}, x_2 \geq \varepsilon H$ where the reflection coefficient r_ε^2 is given by

$$r_\varepsilon^2 := \frac{ik_2\varepsilon c_1 - 1}{ik_2\varepsilon c_1 + 1} e^{2ik_2\varepsilon H}.$$

Similarly $u_0^{FF} = e^{i(k_1x_1+k_2x_2)} + r_\varepsilon^1 e^{i(k_1x_1-k_2x_2)}$ with $r_\varepsilon^1 := -e^{2ik_2\varepsilon H}$. The reflection coefficient of the computed reference solution u_ε^ω for a given realization ω can be computed as

$$r_{ref}^\omega := \frac{e^{ik_2L}}{R} \int_{\Sigma_L^\sharp} (u_\varepsilon^\omega - u_{inc})(x_1, L) e^{-k_1x_1} \, \mathrm{d}x_1.$$

On Figure 8 we plot the errors $\mathbb{E}[|r_{ref}^\omega - r_\varepsilon^i|]$ for $i = 1, 2$ with respect to $k_2\varepsilon$ for two different densities of particles $\rho = 0.1$ and 0.4. The expectation is computed over 50 samples. The shadowed areas correspond to the confidence intervals using one empirical standard deviation. For each ρ we recover the expected rate of convergence of 1 for $i = 1$. For $i = 2$, the convergence rate is 1.57 for $\rho = 0.1$ and 1.3 for $\rho = 0.4$ (recall that the theoretical rate is 1.5). Note first that as expected the larger the density the closer the solution is to the solution of the Dirichlet problem. For $k_2\varepsilon = 0.025$ the first order model gives a precision of 0.22 for $\rho = 0.1$ and 0.14 for $\rho = 0.4$. For this value of $k_2\varepsilon$, it is imperative to use the second order model that is more accurate. At order 2 for $k_2\varepsilon = 0.025$ the error is indeed about 0.01 for $\rho = 0.1$ and 0.003 for $\rho = 0.4$. Again, the error at order 2 becomes smaller as ρ increases. This explains in our point of view why the convergence rate for the order 2 degrades as ρ increases. Indeed, since the error is smaller it is more sensible to the approximation of the reference solution. As a result to witness the theoretical convergence rates one needs to solve for the reference solution with higher accuracy (*i.e.* smaller ε, larger T) which makes the computation more costly. Note finally that this last observation justifies directly the need for effective boundary conditions.

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A Integral representation of the near-field

Let us introduce the space of stationary traces

$$\mathcal{W}^{\frac{1}{2}}(\Sigma_L) := \Big\{ \psi \in L^2(\Omega, H^{\frac{1}{2}}_{loc}(\Sigma_L)), \psi \text{ stationary}, \quad \mathbb{E}\Big[\| h^{\frac{1}{2}}_{|\Sigma_L} \psi \|^2_{L^2(\square_1)} \Big] < +\infty \Big\}$$

where we have identified Σ_L with $\mathbb{R}^{d-1}$ for the definition of the stationarity.

For any φ such that $\varphi \in \mathcal{W}^{\frac{1}{2}}(\Sigma_L)$, we consider the following half-space problem

$$\begin{cases} -\Delta U^\omega = 0 & \text{in} \quad \mathcal{B}_L^\infty\,, \\ \quad U^\omega = \varphi^\omega & \text{on} \quad \Sigma_L. \end{cases} \tag{113}$$

The appropriate functional framework is defined as follows

$$\mathcal{W}(\mathcal{B}_L^\infty) := \Big\{ V \in L^2(\Omega, H^1_{loc}(\mathcal{B}_L^\infty), V(\cdot, y_d) \text{ stationary for any } y_d > L, \\ \mathbb{E}\Big[\int_{\square_1} \int_L^{+\infty} |\nabla V^\omega|^2 \mathrm{d}\boldsymbol{y} \Big] < +\infty \Big\}.$$

Let $\widehat{U}^\omega$ be the unique solution of (24) in $\mathcal{W}_0(D)$ for $F = 0$, $G = 0$,$\mu^{-\frac{1}{2}}\Psi \in \mathcal{L}^2(\Sigma_0)$ and $\mu^{-\frac{1}{2}}\alpha^N \in \mathcal{L}^2(\Sigma_H)$. Note that $\widehat{U}^\omega|_{\mathcal{B}_L^\infty} \in \mathcal{W}(\mathcal{B}_L^\infty)$ verifies (113) with $\varphi = \widehat{U}^\omega|_{\Sigma_L}$. The outline of the proof is the following : we first show that there exists a unique solution $V \in \mathcal{W}(\mathcal{B}_L^\infty)$ to (113). We then prove that the integral representation (44) is in $\mathcal{W}(\mathcal{B}_L^\infty)$ and verifies problem (113).

Let us first prove the well-posedness result.

Proposition 28. *For all φ such that $\varphi \in \mathcal{W}^{\frac{1}{2}}(\Sigma_L)$, there exists a unique $V \in \mathcal{W}(\mathcal{B}_L^\infty)$ that is a.s. solution of* (113).

To prove this result that is similar to the analysis of the half-space problem for the Stokes equation with a trace $H^{\frac{1}{2}}_{uloc}$ by D. Gerard-Varet and N.Masmoudi in [27], we follow the following steps: for all φ such that $\varphi \in \mathcal{W}^{\frac{1}{2}}(\Sigma_L)$,

Step 1 we construct a regular stationary lift of φ;

Step 2 we reduce problem (113) to the problem with a vanishing Dirichlet boundary condition and prove well-posedness using an appropriate Hardy inequality.

In the rest of the proof φ is fixed. *Step 1: construction of a stationary and regular lifting of φ.* In order to derive a lifting, we introduce the Green's function associated to the regularized operator $-\Delta + T^{-1}$ in $\mathbb{R}^d$ for $T > 0$, namely

$$G_T(\boldsymbol{z}) := -\frac{i}{4} H_0^{(1)}\big(i\sqrt{T}^{-1}|\boldsymbol{z}|\big) \text{ for } d = 2, \quad \frac{e^{-\sqrt{T}^{-1}|\boldsymbol{z}|}}{4\pi|\boldsymbol{z}|} \text{ for } d = 3, \quad \boldsymbol{z} \in \mathbb{R}^d. \tag{114}$$

Note that, using [2, Equation (9.2.30)], one can show that for $d = 2$ there exists a constant (whose value is not relevant for the sequel) such that

$$G_T(\boldsymbol{z}) = C \frac{e^{-\sqrt{T}^{-1}|\boldsymbol{z}|}}{|\boldsymbol{z}|^{\frac{1}{2}}} \Big(1 + \mathcal{O}(|\boldsymbol{z}|^{-1})\Big) \quad \text{as } |\boldsymbol{z}| \to +\infty. \tag{115}$$

As we shall prove in the next proposition, a natural lifting of φ is given by the integral formula

$$\text{a.s.} \quad V_T^{\omega}\big(\boldsymbol{y}_{\shortparallel}, y_d\big) = -2 \int_{\mathbb{R}^{d-1}} \partial_{y_d} G_T\big(\boldsymbol{y}_{\shortparallel} - \boldsymbol{z}_{\shortparallel}, y_d - L\big)\, \varphi^{\omega}(\boldsymbol{z}_{\shortparallel})\, \mathrm{d}\boldsymbol{z}_{\shortparallel}, \quad \boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1},\ y_d > L \tag{116}$$

that is in the functional space

$$\mathcal{H}^1(\mathcal{B}_L^{\infty}) := \Big\{ V \in L^2\big(\Omega, H^1_{loc}(\mathcal{B}_L^{\infty})\big), V(\cdot, y_d) \text{ stationary for any } y_d > L, \\ \mathbb{E}\Big[\int_{\square_1}\int_L^{+\infty} |V^{\omega}|^2 + |\nabla V^{\omega}|^2 \,\mathrm{d}\boldsymbol{y}\Big] < +\infty \Big\};$$

Lemma 29. *The function V_T given by* (116) *is in $\mathcal{H}^1(\mathcal{B}_L^{\infty})$ and satisfies a.s.*

$$\begin{cases} -\Delta V_T^{\omega} + T^{-1} V_T^{\omega} = 0 & \text{in} \quad \mathcal{B}_L^{\infty}, \\ \qquad\qquad\quad V_T^{\omega} = \varphi^{\omega} & \text{on} \quad \Sigma_L. \end{cases} \tag{117}$$

Remark 6. *Even if it is not fundamental for the sequel, note that the function V_T given by* (116) *is even the unique solution of* (117) *in $\mathcal{H}^1(\mathcal{B}_L^{\infty})$*

Let us also mention that if $\varphi^{\omega} \in H^{\frac{1}{2}}(\Sigma_L)$ then it is easy to show using Lax Milgram's Theorem that $V^{\omega} \in H^1(B_L)$. The difficulty in our setting comes from the fact that $\varphi \in \mathcal{W}^{\frac{1}{2}}(\Sigma_L)$ but as we shall see the stationarity plays a fundamental role.

Proof of Lemma 29. Let V_T be given by the integral expression of (116). It is easy to show (using similar arguments than in [17] or in [10, Theorem 3.2] for instance) that a.s. for any $\boldsymbol{y}_{\shortparallel} \in \mathbb{R}^{d-1}$, $y_d > L$, $\widetilde{V}_T^{\omega}$ is in $C^2(\mathcal{B}_L^{\infty})$, it satisfies the first equation of (117) and the boundary condition of (117) in the trace sense. This implies in particular that a.s. $V_T^{\omega} \in H^1_{loc}(\mathcal{B}_L^{\infty})$. By stationarity of φ, we have for $\boldsymbol{y}_{\shortparallel}$, $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$ and $y_d > L$,

$$\begin{aligned} \widetilde{V}_T^{\omega}\big(\boldsymbol{y}_{\shortparallel} + \boldsymbol{x}_{\shortparallel}, y_d\big) &= -2 \int_{\mathbb{R}^{d-1}} \partial_{y_d} G_T^{\omega}\big(\boldsymbol{y}_{\shortparallel} - \boldsymbol{z}_{\shortparallel}, y_d - L\big)\, \varphi^{\omega}(\boldsymbol{z}_{\shortparallel} + \boldsymbol{x}_{\shortparallel})\, \mathrm{d}\boldsymbol{z}_{\shortparallel}, \\ &= -2 \int_{\mathbb{R}^{d-1}} \partial_{y_d} G_T^{\omega}\big(\boldsymbol{y}_{\shortparallel} - \boldsymbol{z}_{\shortparallel}, y_d - L\big)\, \varphi^{\tau_{\boldsymbol{x}_{\shortparallel}}\omega}(\boldsymbol{z}_{\shortparallel})\, \mathrm{d}\boldsymbol{z}_{\shortparallel} = \widetilde{V}_T^{\tau_{\boldsymbol{x}_{\shortparallel}}\omega}\big(\boldsymbol{y}_{\shortparallel}, y_d\big). \end{aligned}$$

which shows that $\big(\omega, \boldsymbol{y}_{\shortparallel}\big) \mapsto V^{\omega}\big(\boldsymbol{y}_{\shortparallel}, y_d\big)$is stationary for any $y_d > L$.

Moreover, we have on one hand for $L' > L$,

$$\begin{aligned}
\mathbb{E}\left[\int_{\square_1}\int_{L'}^{+\infty}\left|V_T^\omega(\boldsymbol{y}_{\shortparallel},y_d)\right|^2\mathrm{d}\boldsymbol{y}\right] &= 4\,\mathbb{E}\left[\int_{\square_1}\int_{L'}^{+\infty}\left|\int_{\mathbb{R}^{d-1}}\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\;\varphi^\omega(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel})\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right|^2\mathrm{d}\boldsymbol{y}\right],\\
&\lesssim \int_{\square_1}\int_{L'}^{+\infty}\mathbb{E}\left[\int_{\mathbb{R}^{d-1}}(\mu^\omega)^{-1}(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel},L)\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right]\\
&\quad\mathbb{E}\left[\int_{\mathbb{R}^{d-1}}\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\left|(\mu^\omega_{|\Sigma_L})^{\frac12}\varphi^\omega(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel})\right|^2\mathrm{d}\boldsymbol{z}_{\shortparallel}\right]\mathrm{d}\boldsymbol{y},\\
&\lesssim \int_{L'}^{+\infty}\left(\int_{\mathbb{R}^{d-1}}\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right)^2 dy_d\,\mathbb{E}\left[h_{|\Sigma_L}|\varphi|^2\right]\\
&\lesssim \|\mu^{\frac12}\varphi\|^2_{\mathcal{L}^2(\Sigma_L)},
\end{aligned}$$

where we used that $\mu(\cdot,L)$ and φ are stationary, $\mathbb{E}[\mu^{-1}(\cdot,L)]\lesssim L^2$ and that (thanks to (114) and (115)) for L' large enough

$$\int_{\mathbb{R}^{d-1}}\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\lesssim\frac{1}{y_d-L}\quad\text{for } y_d>L'.$$

On the other hand, for $\chi\in\mathcal{C}_c^\infty(\mathbb{R}^{d-1})$ such that $0\le\chi\le1$ and $\chi=1$ on $\left(-\frac32,\frac32\right)^{d-1}$, we can write

$$\begin{aligned}
\mathbb{E}\left[\int_{\square_1}\int_{L}^{L'}\left|V_T^\omega(\boldsymbol{y}_{\shortparallel},y_d)\right|^2\mathrm{d}y_d\mathrm{d}\boldsymbol{y}_{\shortparallel}\right] &\lesssim \mathbb{E}\left[\int_{\square_1}\int_{L}^{L'}\left|\int_{\mathbb{R}^{d-1}}\partial_{y_d}G_T(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel},y_d-L)\;\chi\varphi^\omega(\boldsymbol{z}_{\shortparallel})\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right|^2\mathrm{d}\boldsymbol{y}\right]\\
&+\mathbb{E}\left[\int_{\square_1}\int_{L}^{L'}\left|\int_{\mathbb{R}^{d-1}}\partial_{y_d}G_T(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel},y_d-L)\;(1-\chi)\varphi^\omega(\boldsymbol{z}_{\shortparallel})\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right|^2\mathrm{d}\boldsymbol{y}\right].
\end{aligned}$$

Since a.s. $\chi\varphi^\omega\in H^{\frac12}(\Sigma_L)$ it is easy to show (see Remark 6) that a.s.

$$\int_{\square_1}\int_{L}^{L'}\left|\int_{\mathbb{R}^{d-1}}\partial_{y_d}G_T(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel},y_d-L)\;\chi\,\varphi^\omega(\boldsymbol{z}_{\shortparallel})\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right|^2\mathrm{d}\boldsymbol{y}<+\infty$$

Moreover, using that when $\boldsymbol{y}_{\shortparallel}\in\square_1$, $(1-\chi)(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel})=0$ when $\boldsymbol{z}_{\shortparallel}\in\square_1$, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\int_{\square_1}\int_{L}^{L'}\left|\int_{\mathbb{R}^{d-1}}\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\;(1-\chi)\varphi^\omega(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel})\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right|^2\mathrm{d}\boldsymbol{y}\right],\\
&\lesssim \int_{\square_1}\int_{L}^{L'}\mathbb{E}\left[\int_{\mathbb{R}^{d-1}\setminus\square_1}(\mu^\omega)^{-1}(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel},L)\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right]\\
&\qquad\mathbb{E}\left[\int_{\mathbb{R}^{d-1}\setminus\square_1}\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\,\left|(\mu^\omega_{|\Sigma_L})^{\frac12}\varphi^\omega(\boldsymbol{y}_{\shortparallel}-\boldsymbol{z}_{\shortparallel})\right|^2\mathrm{d}\boldsymbol{z}_{\shortparallel}\right]\mathrm{d}\boldsymbol{y},\\
&\lesssim \int_{L}^{L'}\left(\int_{\mathbb{R}^{d-1}\setminus\square_1}\left|\partial_{y_d}G_T(\boldsymbol{z}_{\shortparallel},y_d-L)\right|\,\mathrm{d}\boldsymbol{z}_{\shortparallel}\right)^2\mathrm{d}y_d\,\mathbb{E}\left[h_{|\Sigma_L}|\varphi|^2\right],\\
&\lesssim \|\mu^{\frac12}\varphi\|^2_{\mathcal{L}^2(\Sigma_L)}.
\end{aligned}$$

Note that $\partial_{y_d}G_T\in L^1(\mathbb{R}^{d-1}\setminus\square_1\times(L,L'))$ is a regular function with an exponential decay at infinity (see (114)-(115)).

Similar results can be derived for ∇V_T^ω.

$\square$

Step 2 : Laplace Problem with a vanishing Dirichlet boundary condition. Now consider $\widetilde{V}=V-V_T$. By Lemma 29, $\widetilde{V}$ verifies a.s.

$$\begin{cases}-\Delta\widetilde{V}^\omega=-T^{-1}V_T^\omega & \text{in}\quad\mathcal{B}_L^\infty\,,\\ \widetilde{V}^\omega=0 & \text{in}\quad\Sigma_L\,.\end{cases}\tag{118}$$

We introduce the following functional space

$$\mathcal{W}_0(\mathcal{B}_L^\infty) := \Big\{\text{a.s. } V^\omega \in H^1_{loc}(\mathcal{B}_L^\infty), V(\cdot, y_d) \text{ stationary for any } y_d > L,$$
$$\text{a.s. } V^\omega = 0 \text{ on } \Sigma_L,\ \mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}\Big(\frac{|V^\omega|^2}{1+y_d^2} + |\nabla V^\omega|^2\Big)\,\mathrm{d}\boldsymbol{y}\right] < +\infty\Big\}.$$

equipped with the norm

$$\|V\|^2_{\mathcal{W}^0} := \mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}\Big(\frac{|V^\omega|^2}{1+y_d^2} + |\nabla V^\omega|^2\Big)\,\mathrm{d}\boldsymbol{y}\right].$$

A classical Hardy inequality gives that for all $V \in \mathcal{W}_0(\mathcal{B}_L^\infty)$ and since $V^\omega = 0$ on Σ_L a.s., we have a.s.

$$\int_{\square_1}\int_L^{+\infty}\frac{|V^\omega|^2}{1+y_d^2}\,\mathrm{d}\boldsymbol{y} \le \int_{\square_1}\int_L^{+\infty}|\nabla V^\omega|^2\,\mathrm{d}\boldsymbol{y}.$$

By taking the expectation of the last inequality, we obtain that the following Hardy inequality is verified in $\mathcal{W}_0(\mathcal{B}_L^\infty)$

$$\forall V \in \mathcal{W}_0(\mathcal{B}_L^\infty)\,, \quad \mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}\frac{|V^\omega|^2}{1+y_d^2}\,\mathrm{d}\boldsymbol{y}\right] \lesssim \mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}|\nabla V^\omega|^2\,\mathrm{d}\boldsymbol{y}\right]. \tag{119}$$

Lemma 30. *Problem* (118) *is well posed in* $\mathcal{W}_0(\mathcal{B}_L^\infty)$.

Proof of Lemma 119. The variational formulation associated with (118) is given by

$$\forall W \in \mathcal{W}_0(\mathcal{B}_L^\infty), \quad \mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}\nabla\widetilde{V}\cdot\nabla\overline{W}\,\mathrm{d}\boldsymbol{y}\right] = -T^{-1}\mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}V_T\overline{W}\mathrm{d}\boldsymbol{y}\right]. \tag{120}$$

The coercivity of the sesquilinear form can be directly deducted from Hardy's inequality (119). From the integral representation of V_T, we deduce that $y_d \mapsto V_T(\cdot, y_d)$ is exponentially decaying and in particular

$$\mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}(1+y_d^2)|V_T|^2\,\mathrm{d}\boldsymbol{y}\right] < +\infty\,.$$

We have then by Cauchy-Schwarz inequality

$$T^{-1}\left|\mathbb{E}\left[\int_{\square_1}\int_L^{+\infty}V_T\overline{W}\mathrm{d}\boldsymbol{y}\right]\right| \lesssim T^{-1}\|\sqrt{1+y_d^2}V_T\|_{\mathcal{L}^2(\mathcal{B}_L^\infty)}\|W\|_{\mathcal{W}^0}\,.$$

The linear form in the r.h.s of (120) is then continuous in $\mathcal{W}^0$. We conclude by Lax-Milgram's Theorem. □

We deduce finally proposition 28 : since there exists a unique $\widetilde{V} \in \mathcal{W}_0(\mathcal{B}_L^\infty)$ solution of (118), $V := \widetilde{V} + V_T \in \mathcal{W}(\mathcal{B}_L^\infty)$ satisfies (113). The uniqueness result for (113) is the consequence of the one for(118).

Let us finish the proof of Proposition 9 by proving that the integral representation given by (44) is in $\mathcal{W}(\mathcal{B}_L^\infty)$ and verifies problem (113). By uniqueness of the solution this shows that $\widehat{U}|_{\mathcal{B}_L^\infty}$ can be represented by (44) .

Proposition 31. *Let* $Z : \mathbb{R}^{d-1} \times (L, +\infty)$ *be defined as*

$$Z(\boldsymbol{y}) = -2\int_{\mathbb{R}^{d-1}}\partial_{y_d}\Gamma\big(\boldsymbol{y}_{\shortparallel} - \boldsymbol{z}_{\shortparallel}, y_d - L\big)\ \varphi^\omega\big(\boldsymbol{z}_{\shortparallel}\big)\mathrm{d}\boldsymbol{z}_{\shortparallel},$$

Then $Z \in \mathcal{W}(\mathcal{B}_L^\infty)$ *and* Z^ω *is a.s. solution of* (113).

Proof of proposition 31. We follow the same steps as in the proof of Lemma 29 to show that Z satisfies (113), that a.s. it belongs to $H^1_{\mathrm{loc}}(\mathcal{B}_L^\infty)$ and that $Z(\cdot, y_d)$ is stationary for $y_d > L$. To prove that

$$\mathbb{E}\Big[\int_{\square_1}\int_L^{+\infty} |\nabla \widehat{U}^\omega|^2\, \mathrm{d}\boldsymbol{y}\Big] < +\infty$$

the calculations are similar to the proof of Lemma 29, provided one uses that

$$\partial_{y_d}\nabla_{\boldsymbol{z}}\Gamma(\boldsymbol{z}) = \mathcal{O}\left(\frac{1}{((z_d)^2 + |\boldsymbol{z}_{\shortparallel}|^2)^{d/2}}\right) \quad \text{as } |\boldsymbol{z}| \to +\infty$$

and

$$\int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}\boldsymbol{z}_{\shortparallel}}{\left(\left(y_d - L\right)^2 + |\boldsymbol{z}_{\shortparallel}|^2\right)^{d/2}} \lesssim \frac{1}{y_d - L} \quad \text{for } y_d > L' \quad \text{and} \quad \int_{\mathbb{R}^{d-1}\setminus\square_1} \frac{\mathrm{d}\boldsymbol{z}_{\shortparallel}}{\left(\left(y_d - L\right)^2 + |\boldsymbol{z}_{\shortparallel}|^2\right)^{d/2}} < +\infty.$$

□

B L^p-regularity of the auxiliary problem

To prove Theorem 23, we adapt the proof of [24, Theorem 3.1] to our problem. To this effect we start by reproducing two technical lemmatas that are key in the proof of the theorem. The first one is Gehring's Lemma [25] in the form of [40, proposition 5.1].

Lemma 32 (Gehring's Lemma)**.** *Let $1 < q < s$ and Q_0 be a reference cube in $\mathbb{R}^d$. Take $F \in L^s(2Q_0)$ and $G \in L^q(2Q_0)$ two non-negative functions. Suppose that for every cube $Q \subset Q_0$*

$$\left(\fint_Q G^q\right)^{\frac{1}{q}} \leq b\fint_{2Q} G + b\left(\fint_{2Q} F^q\right)^{\frac{1}{q}} + \theta\left(\fint_{2Q} G^q\right)^{\frac{1}{q}},$$

where $b > 1$ and $\theta \geq 0$. There exists $\theta_0 = \theta_0(q,s,d)$ and $\eta_0 = \eta_0(b,d,q,s) > 0$ such that if $\theta \leq \theta_0$, then for all $q \leq p \leq q + \eta_0$

$$\left(\fint_{Q_0} G^p\right)^{\frac{1}{p}} \lesssim \left(\fint_{2Q_0} F^p\right)^{\frac{1}{p}} + \fint_{2Q_0} G,$$

The second lemma corresponds to the dual version of Calderón-Zygmund Lemma due to Shen [41, Theorem 3.2].

Lemma 33 (Dual Calderón-Zygmund Lemma)**.** *Let $1 \leq p_0 < p_1$, $C \in [\frac{1}{2}, 2]$ and Q_0 be a reference cube in $\mathbb{R}^d$. Take $F \in L^{p_1}(2Q_0)$ and $G \in L^{p_1}(2Q_0)$. Suppose that for every cube $Q \subset Q_0$, there exists two measurable functions $F_{Q,0}$ and $F_{Q,1}$ such that $|F| \leq |F_{Q,0}| + |F_{Q,1}|$ and $|F_{Q,1}| \leq |F| + |F_{Q,0}|$ on $2Q_0$*

$$\left(\fint_Q |F_{Q,0}|^{p_0}\right)^{\frac{1}{p_0}} \leq C_1\left(\fint_{CQ} |G|^{p_0}\right)^{\frac{1}{p_0}},$$

$$\left(\fint_{\frac{1}{C}Q} |F_{Q,1}|^{p_1}\right)^{\frac{1}{p_1}} \leq C_2\left(\fint_Q |F_{Q,1}|^{p_0}\right)^{\frac{1}{p_0}},$$

with $C_1, C_2 > 0$. Then for all $p_0 < p < p_1$

$$\left(\fint_{Q_0} |F|^p\right)^{\frac{1}{p}} \lesssim \fint_{2Q_0} |F| + \left(\fint_{2Q_0} |G|^p\right)^{\frac{1}{p}}.$$

Proof of Theorem 23. Step 1. We prove that a.s. for all $2 \leq p \leq 2 + \frac{1}{C_0}$

$$\left(\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^+} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^p\right)^{\frac{1}{p}} \lesssim \left(\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^+} |\boldsymbol{g}|^p\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^{d-1}} \mu^{-\frac{p}{2}}|f|^p\right)^{\frac{1}{p}}. \tag{121}$$

Let $R > 0$. Take χ a smooth cut-off function such that $\chi|_{\square_R\times(0,R)} \equiv 1$, $\chi \geq 0$, $\mathrm{supp}\chi \subset \square_{2R} \times (0,2R)$ and $|\nabla\chi| \lesssim \dfrac{1}{R}$. We test (97) with $v := \chi(u_{f,\boldsymbol{g}} - c)$ where c is a constant. We obtain

$$\int_{\square_R\times(0,R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^2 \leq \int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}\,\boldsymbol{g}\cdot(\chi\nabla u^\omega_{f,\boldsymbol{g}} + (u^\omega_{f,\boldsymbol{g}} - c)\nabla\chi) + \mathbb{1}_{R\geq L}\int_{\square_{2R}\cap\Sigma_L} f\chi(u^\omega_{f,\boldsymbol{g}} - c) - \int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}\,\nabla u^\omega_{f,\boldsymbol{g}}\cdot(u^\omega_{f,\boldsymbol{g}} - c)\nabla\chi.$$

Using Young's inequality with $K \geq 1$ we get

$$\int_{\square_R\times(0,R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^2 \lesssim \frac{1}{R^2}\left(\frac{K^2}{2} + \frac{1}{2K^2}\right)\int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}|u^\omega_{f,\boldsymbol{g}} - c|^2 + \frac{1}{K^2}\int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^2 + \frac{1}{2K^2}\int_{\square_{2R}\cap\Sigma_L} \mu|u^\omega_{f,\boldsymbol{g}} - c|^2 + \frac{K^2}{2}\int_{\square_{2R}\times(0,2R)} |\boldsymbol{g}|^2 + \frac{K^2}{2}\int_{\square_{2R}\cap\Sigma_L} \mu^{-1}|f|^2.$$

Choosing $c := \fint_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega} u^\omega_{f,\boldsymbol{g}}$, we apply Poincaré-Sobolev inequality to the first term and the trace Theorem to the third term of the right-hand side

$$\left(\int_{\square_R\times(0,R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^2\right)^{\frac{1}{2}} \lesssim K\left(\int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^{p^\star}\right)^{\frac{1}{p^\star}} + \frac{1}{K}\left(\int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^2\right)^{\frac{1}{2}} + K\left(\int_{\square_{2R}\times(0,2R)} \left(|\boldsymbol{g}|^2 + (2R\mu)^{-1}|f|^2\right)\right)^{\frac{1}{2}},$$

with $p^\star = 2d/(d+2)$ and where we have identified the function $\mu^{-1}|f|^2 \in L^1(\square_{2R}\cap\Sigma_L)$ to a function in $L^1(\square_{2R}\times(0,2R))$ that is independent of x_d. We conclude by applying Gehring's Lemma (Lemma 32) with $q = 2/p^\star$, $F := \left(|\boldsymbol{g}|^2 + (2R\mu)^{-1}|f|^2\right)^{p^\star/2}$ and $G := |\nabla u^\omega_{f,\boldsymbol{g}}|^{p^\star}$ that for all $2 \leq p \leq 2 + 1/C_0$

$$\left(\fint_{\square_R\times(0,R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^p\right)^{\frac{1}{p}} \lesssim \left(\fint_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^{p^\star}\right)^{\frac{1}{p^\star}} + \left(\fint_{\square_{2R}\times(0,2R)} |\boldsymbol{g}|^p + \fint_{\square_{2R}} (2R\mu)^{-\frac{p}{2}}|f|^p\right)^{\frac{1}{p}}, \tag{122}$$

where $C_0 := \eta_0 p^\star$. By Jensen's inequality and since $2 > p^*$, (122) yields

$$\left(\int_{\square_R\times(0,R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^p\right)^{\frac{1}{p}} \lesssim R^{(\frac{1}{p}-\frac{1}{2})d}\left(\int_{\square_{2R}\times(0,2R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^2\right)^{\frac{1}{2}} + \left(\int_{\square_{2R}\times(0,2R)} |\boldsymbol{g}|^p + \int_{\square_{2R}} \mu^{-\frac{p}{2}}|f|^p\right)^{\frac{1}{p}}.$$

(96) implies

$$\left(\int_{\square_R\times(0,R)} \mathbb{1}_{D^\omega}|\nabla u^\omega_{f,\boldsymbol{g}}|^p\right)^{\frac{1}{p}} \lesssim R^{(\frac{1}{p}-\frac{1}{2})d}\left(\left(\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+} |\boldsymbol{g}|^2\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^{d-1}} \mu^{-1}|f|^2\right)^{\frac{1}{2}}\right) + \left(\int_{\square_{2R}\times(0,2R)} |\boldsymbol{g}|^p + \int_{\square_{2R}} \mu^{-\frac{p}{2}}|f|^p\right)^{\frac{1}{p}}.$$

Next we let $R \to +\infty$ and obtain (121) for $2 \leq p \leq 2 + \frac{1}{C_0}$.

Step 2: Proof of (98) *for* $2 \le q < p \le 2 + \frac{1}{C_0}$ Let $2 \le p_0 \le p_1 \le 2 + \frac{1}{C_0}$ and $R > 0$. Let $u^{\omega,0}_{f,\boldsymbol{g}}$ be the unique solution in $W_0(D^\omega)$ of

$$\begin{cases} -\Delta u^{\omega,0}_{f,\boldsymbol{g}} = \nabla \cdot (\boldsymbol{g} \mathbb{1}_{\square_R \times (0,R)}) \text{ in } D^\omega \setminus (\Sigma_H \cup \Sigma_L), \\ -\partial_{y_d} u^{\omega,0}_{f,\boldsymbol{g}} = 0 \text{ on } \Sigma_0, \quad \text{and} \quad u^{\omega,0}_{f,\boldsymbol{g}} = 0 \text{ on } \partial \mathcal{P}^\omega, \\ \left[u^{\omega,0}_{f,\boldsymbol{g}}\right]_H = 0 \quad \text{and} \quad \left[-\partial_{y_d} u^{\omega,0}_{f,\boldsymbol{g}}\right]_H = 0, \\ \left[u^{\omega,0}_{f,\boldsymbol{g}}\right]_L = 0 \quad \text{and} \quad \left[-\partial_{y_d} u^{\omega,0}_{f,\boldsymbol{g}}\right]_L = f \mathbb{1}_{\square_R}. \end{cases}$$

We apply (121) to $u^{\omega,0}_{f,\boldsymbol{g}}$

$$\begin{aligned} \int_{\square_R} \int_{(0,R)} \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^{\omega,0}_{f,\boldsymbol{g}}|^{p_0}\right] &\le \mathbb{E}\left[\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^+} \mathbb{1}_{D^\omega} |\nabla u^{\omega,0}_{f,\boldsymbol{g}}|^{p_0}\right] \\ &\lesssim \mathbb{E}\left[\int_{\square_R} \int_{(0,R)} |\boldsymbol{g}|^{p_0} + \int_{\square_R} \mu^{-\frac{p_0}{2}} |f|^{p_0}\right] \\ &\lesssim \mathbb{E}\left[\int_{\square_{2R}} \int_{(0,2R)} |\boldsymbol{g}|^{p_0} + \int_{\square_{2R}} \mu^{-\frac{p_0}{2}} |f|^{p_0}\right]. \end{aligned}$$

On the other hand $u^{\omega,1}_{f,\boldsymbol{g}} := u^{\omega}_{f,\boldsymbol{g}} - u^{\omega,0}_{f,\boldsymbol{g}}$ verifies in $D^\omega_R := \square_R \times (0,R) \setminus \overline{\mathcal{P}^\omega}$

$$\begin{cases} -\Delta u^{\omega,1}_{f,\boldsymbol{g}} = 0 \text{ in } D^\omega_{R,R} \setminus (\Sigma_H \cup \Sigma_L), \\ -\partial_{y_d} u^{\omega,1}_{f,\boldsymbol{g}} = 0 \text{ on } \Sigma_0 \cap \square_R, \quad \text{and} \quad u^{\omega,1}_{f,\boldsymbol{g}} = 0 \text{ on } \partial \mathcal{P}^\omega, \\ \left[u^{\omega,1}_{f,\boldsymbol{g}}\right]_{\Sigma_H \cap \square_R} = 0 \quad \text{and} \quad \left[-\partial_{y_d} u^{\omega,1}_{f,\boldsymbol{g}}\right]_{\Sigma_H \cap \square_R} = 0, \\ \left[u^{\omega,1}_{f,\boldsymbol{g}}\right]_{\Sigma_L \cap \square_R} = 0 \quad \text{and} \quad \left[-\partial_{y_d} u^{\omega,1}_{f,\boldsymbol{g}}\right]_{\Sigma_L \cap \square_R} = 0. \end{cases}$$

We first deduce from the triangle inequality in $L^{\frac{p_1}{p_0}}(\square_{\frac{R}{2}} \times (0, \frac{R}{2}))$ the following

$$\left(\fint_{\square_{\frac{R}{2}}} \fint_{(0,\frac{R}{2})} \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^{\omega,1}_{f,\boldsymbol{g}}|^{p_0}\right]^{\frac{p_1}{p_0}} \right)^{\frac{1}{p_1}} \le \mathbb{E}\left[\left(\fint_{\square_{\frac{R}{2}}} \fint_{(0,\frac{R}{2})} \mathbb{1}_{D^\omega} |\nabla u^{\omega,1}_{f,\boldsymbol{g}}|^{p_1} \right)^{\frac{p_0}{p_1}} \right]^{\frac{1}{p_0}}.$$

From (122) with $g = 0$ on $\square_R \times (0,R)$ and $f = 0$ on $\Sigma_L \cap \square_R$, we obtain

$$\left(\fint_{\square_{\frac{R}{2}}} \fint_{(0,\frac{R}{2})} \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^{\omega,1}_{f,\boldsymbol{g}}|^{p_0}\right]^{\frac{p_1}{p_0}} \right)^{\frac{1}{p_1}} \lesssim \mathbb{E}\left[\left(\fint_{\square_R} \fint_{(0,R)} \mathbb{1}_{D^\omega} |\nabla u^{\omega,1}_{f,\boldsymbol{g}}|^{p^\star} \right)^{\frac{p_0}{p^\star}} \right]^{\frac{1}{p_0}}.$$

Finally with Jensen's inequality (since $p_0 \ge p^\star$) we conclude that

$$\left(\fint_{\square_{\frac{R}{2}}} \fint_{(0,\frac{R}{2})} \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^{\omega,1}_{f,\boldsymbol{g}}|^{p_0}\right]^{\frac{p_1}{p_0}} \right)^{\frac{1}{p_1}} \lesssim \mathbb{E}\left[\fint_{\square_R} \fint_{(0,R)} \mathbb{1}_{D^\omega} |\nabla u^{\omega,1}_{f,\boldsymbol{g}}|^{p_0} \right]^{\frac{1}{p_0}}.$$

We are now in position to apply Lemma 33 with $C = 2$,

$$\begin{aligned} F &:= \mathbb{E}[|\nabla u_{f,\boldsymbol{g}}|^{p_0}|]^{\frac{1}{p_0}}, \quad G := \mathbb{E}[|\boldsymbol{g}|^{p_0} + \mu^{-\frac{p_0}{2}} |f|^{p_0}]^{\frac{1}{p_0}}, \\ F^0 &:= \mathbb{E}[|\nabla u^0_{f,\boldsymbol{g}}|^{p_0}|]^{\frac{1}{p_0}}, \quad F^1 := \mathbb{E}[|\nabla u^1_{f,\boldsymbol{g}}|^{p_0}|]^{\frac{1}{p_0}}. \end{aligned}$$

We obtain for all $p_0 < p < p_1$, $R_0 > 0$

$$\left(\fint_{\square_{R_0}}\fint_{(0,R_0)}\mathbb{E}[|\nabla u_{f,\boldsymbol{g}}|^{p_0}|]^{\frac{p}{p_0}}\right)^{\frac{1}{p}} \lesssim \fint_{\square_{R_0}}\fint_{(0,R_0)}\mathbb{E}[|\nabla u_{f,\boldsymbol{g}}|^{p_0}| + \left(\fint_{\square_{2R_0}}\fint_{(0,2R_0)}\mathbb{E}[|\boldsymbol{g}|^{p_0}]^{\frac{p}{p_0}} + \fint_{\square_{2R_0}}\mu^{-\frac{p_0}{2}}|f|^{p_0}]^{\frac{p}{p_0}}\right)^{\frac{1}{p}}.$$

We let $R_0 \to +\infty$ and obtain (98) for $2 \le p \le 2 + \frac{1}{C_0}$.
Step 3: Proof of (98) *for* $2 - 1/2C_0 \le p < q \le 2$. We proceed by duality:

$$\|\nabla u_{f,\boldsymbol{g}}\|_{L^p(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^q(\Omega))} = \sup_{\boldsymbol{h}\in L^{p'}(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^{q'}(\Omega))}\left\{\mathbb{E}\left[\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+} h\cdot\nabla u_{f,\boldsymbol{g}}\right] \,|\, \|h\|_{L^{p'}(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^{q'}(\Omega))} = 1\right\}.$$

Moreover testing the variational formulation verified by $u_{0,\boldsymbol{h}}$ with $v := \overline{u_{f,\boldsymbol{g}}}$ yields

$$\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}\mathbb{1}_{D^\omega}\nabla u_{0,\boldsymbol{h}}\cdot\nabla u_{f,\boldsymbol{g}} = \int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}\boldsymbol{h}\cdot\nabla u_{f,\boldsymbol{g}}. \tag{123}$$

Testing the variational formulation verified by $u_{f,\boldsymbol{g}}$ with $v := \overline{u_{0,\boldsymbol{h}}}$ and subtracting (123) gives

$$\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}\boldsymbol{h}\cdot\nabla u_{f,\boldsymbol{g}} = \int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}\boldsymbol{g}\cdot\nabla u_{0,\boldsymbol{h}} + \int_{\mathbb{R}^{d-1}} f u_{0,\boldsymbol{h}}.$$

Moreover

$$\sup\left\{\mathbb{E}\left[\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}\boldsymbol{g}\cdot\nabla u_{0,\boldsymbol{h}} + \int_{\mathbb{R}^{d-1}} f u_{0,\boldsymbol{h}}\right] \,|\, \|h\|_{L^{p'}(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^{q'}(\Omega))} = 1\right\} \le \Big(\|\boldsymbol{g}\|_{L^p(\mathbb{R}^{d-1},L^q(\Omega))} + \|\mu^{-\frac{1}{2}}f\|_{L^p(\mathbb{R}^{d-1},L^q(\Omega))}\Big)\sup\left\{\|\nabla u_{0,\boldsymbol{h}}\|_{L^{p'}(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^{q'}(\Omega))} \,|\, \|h\|_{L^{p'}(\mathbb{R}^{d-1}\times\mathbb{R}^+,L^{q'}(\Omega))} = 1\right\}.$$

Since $2 \le q' < p' \le 2 + \frac{1}{C_0}$ we conclude thanks to step 2.2. By interpolation estimate (98) holds then for all $|p-2|, |q-2| < \frac{1}{C_0'}$ with $C_0' = 8C_0$. □

C Proof of propositions 25 and 26

To prove Proposition 25, we use the following corollary of Hypothesis (Mix) [22, proposition 1.10].

Lemma 34 (Control of high moments)**.** *If $\mathcal{P}$ satisfies Hypothesis (Mix), then for all $q < +\infty$ and all $\sigma(\mathcal{P})$-measurable random variable $F(\mathcal{P})$*

$$\mathbb{E}\left[|F(\mathcal{P}) - \mathbb{E}[F(\mathcal{P})]|^q\right]^{\frac{1}{q}} \lesssim q^2\mathbb{E}\left[\int_1^{+\infty}\left(\int_{\mathbb{R}^{d-1}}\left(\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_{\shortparallel})}F(\mathcal{P})\right)^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}\right)^{\frac{q}{2}}\ell^{-\frac{(d-1)q}{2}}\pi(\ell-1)d\ell\right]^{\frac{1}{q}}.$$

Proof of Proposition 25. Step 1: Sensitivity of the ∇W_1 Let $\boldsymbol{x}_{\shortparallel} \in \mathbb{R}^{d-1}$ and $\ell \ge 1$. In what follows we use the same notations as in the proof of Proposition 27. Let us first prove that

$$\left|\int_{\widetilde{D}^\omega}\boldsymbol{g}\cdot\nabla Z^\omega\right| \lesssim (\ell_+^\omega(\boldsymbol{x}_{\shortparallel}))^2\|\mathbb{1}_{D^\omega}\nabla u_{\boldsymbol{g}}^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}\|\mathbb{1}_{D^\omega}\nabla W_1^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_{\shortparallel}))}. \tag{124}$$

Let $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^{d-1}\times\mathbb{R}^+,\mathbb{R}^d)$. We introduce the auxiliary problem $u_{\boldsymbol{g}}$ solution of (95) with $f := 0$. The weak formulation associated to (95) reads for all $\varphi \in W_0(\widetilde{D}^\omega)$

$$\int_{\widetilde{D}^\omega}\mathbb{1}_{D^\omega}\nabla u_{\boldsymbol{g}}\cdot\overline{\nabla\varphi} = \int_{\widetilde{D}^\omega}\mathbb{1}_{D^\omega}\boldsymbol{g}\cdot\overline{\nabla\varphi} + \int_{\partial\mathcal{P}_\ell^\omega}\mathbb{1}_{D^\omega}\nabla u_{\boldsymbol{g}}\cdot\boldsymbol{n}\,\overline{\varphi}. \tag{125}$$

We evaluate (106) for $\varphi = \overline{u_{\boldsymbol{g}}}$ and (125) for $\varphi = \overline{Z}$. After substracting both expressions we obtain

$$\int_{\widetilde{D}^\omega} \boldsymbol{g} \cdot \nabla Z^\omega = -\int_{\partial \mathcal{P}_\ell^\omega} \nabla u_{\boldsymbol{g}}^\omega \cdot \boldsymbol{n}\, \mathbb{1}_{D'} W_1' - \int_{\partial \mathcal{P}_\ell'} \nabla W_1' \cdot \boldsymbol{n}\, \mathbb{1}_{D^\omega} u_{\boldsymbol{g}}^\omega.$$

Using the same arguments as in the proof of proposition 27, we can estimate by duality $\|\nabla u_{\boldsymbol{g}} \cdot \boldsymbol{n}\|_{H^{-\frac{1}{2}}(\partial B(\boldsymbol{x}_i^\omega))}$ and obtain,

$$\|\nabla u_{\boldsymbol{g}} \cdot \boldsymbol{n}\|_{H^{-\frac{1}{2}}(\partial B(\boldsymbol{x}_i^\omega))} \leq \|\mathbb{1}_{D^\omega} \nabla u_{\boldsymbol{g}}^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_\shortparallel))}. \tag{126}$$

Combining (107) and (126) we conclude that

$$\left| \int_{\widetilde{D}^\omega} \boldsymbol{g} \cdot \nabla Z^\omega \right| \lesssim 2(\ell_+^\omega(\boldsymbol{x}_\shortparallel)) \|\mathbb{1}_{D^\omega} \nabla u_{\boldsymbol{g}}^\omega\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_\shortparallel))} \|\mathbb{1}_{D'} \nabla W_1'\|_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_\shortparallel))}.$$

We get (124) using (104).

Step 2: Let $g \in \mathcal{C}_c^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^+)$ and $2 \leq q < +\infty$. The second step consists in applying the higher-order version (Lemma 34) of the mixing hypothesis (Hypothesis (Mix)) to $\int_0^{+\infty} \int_{\mathbb{R}^{d-1}} \mathbb{1}_{D^\omega} \boldsymbol{g} \cdot \nabla W_1(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$.

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} \mathbb{1}_{D^\omega} \boldsymbol{g} \cdot \nabla W_1(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right|^q \right]^{\frac{2}{q}} \\
&\leq q^2 \mathbb{E}\left[\int_1^{+\infty} \left(\int_{\mathbb{R}^{d-1}} \left(\partial^{osc}_{\mathcal{P},\mathcal{L}_\ell(\boldsymbol{x}_\shortparallel)} \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} \mathbb{1}_{D^\omega} \boldsymbol{g} \cdot \nabla W_1(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right)^2 \mathrm{d}\boldsymbol{x}_\shortparallel \right)^{\frac{q}{2}} \ell^{-\frac{q(d-1)}{2}} \pi(\ell-1) \mathrm{d}\ell \right]^{\frac{2}{q}} \\
&\lesssim q^2 \mathbb{E}\left[\int_1^{+\infty} \left(\int_{\mathbb{R}^{d-1}} (\ell_+^\omega(\boldsymbol{x}_\shortparallel))^4 \|\mathbb{1}_{D^\omega} \nabla u_{\boldsymbol{g}}^\omega\|^2_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_\shortparallel))} \|\mathbb{1}_{D^\omega} \nabla W_1^\omega\|^2_{L^2(\mathcal{L}_{\ell_+^\omega}(\boldsymbol{x}_\shortparallel))} \mathrm{d}\boldsymbol{x}_\shortparallel \right)^{\frac{q}{2}} \ell^{-\frac{q(d-1)}{2}} \pi(\ell-1) \mathrm{d}\ell \right]^{\frac{2}{q}}.
\end{aligned}$$

Next we use Hypothesis (L^∞) to bound a.s. and for a.e. $\boldsymbol{x}_\shortparallel \in R^{d-1}$, $\ell_+^\omega(\boldsymbol{x}_\shortparallel)$ by $\ell + R^+$.

$$\begin{aligned}
\mathbb{E}\left[\left| \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} \mathbb{1}_{D^\omega} \boldsymbol{g} \cdot \nabla W_1(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right|^q \right]^{\frac{2}{q}} &\lesssim q^2 \mathbb{E}\Bigg[\int_1^{+\infty} \Bigg(\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2 \right) \\
&\qquad \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2 \right) \mathrm{d}\boldsymbol{x}_\shortparallel \Bigg)^{\frac{q}{2}} (\ell+R^+)^{(d+1)q} \ell^{-\frac{q(d-1)}{2}} \pi(\ell-1) \mathrm{d}\ell \Bigg]^{\frac{2}{q}} \\
&\lesssim q^2 \Bigg(\int_1^{+\infty} \mathbb{E}\Bigg[\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2 \right) \\
&\qquad \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2 \right) \mathrm{d}\boldsymbol{x}_\shortparallel \Bigg]^{\frac{q}{2}} (\ell+R^+)^{(d+1)q} \ell^{-\frac{q(d-1)}{2}} \pi(\ell-1) \mathrm{d}\ell \Bigg)^{\frac{2}{q}}
\end{aligned} \tag{127}$$

By duality we can write

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2 \right) \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2 \right) \mathrm{d}\boldsymbol{x}_\shortparallel \right)^{\frac{q}{2}} \right]^{\frac{2}{q}} \\
&\quad = \sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \mathbb{E}\left[\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2 \right) \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_\shortparallel)} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2 \right) \mathrm{d}\boldsymbol{x}_\shortparallel X \right],
\end{aligned}$$

where $\tilde{q} = \frac{q}{q-2}$ and the supremum is computed over positive random variables X independent of $\boldsymbol{x}$. We artificially introduce a spatial average over $\square_{R_1}$ for a given $R_1 \geq 1$ by noticing that by Fubini-Tonelli's Theorem for $h \in L^1(\mathbb{R}^{d-1})$

$$\int_{\mathbb{R}^{d-1}} \fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} h(\boldsymbol{y}_{\shortparallel}) \mathrm{d}\boldsymbol{y}_{\shortparallel} \mathrm{d}\boldsymbol{x}_{\shortparallel} = \int_{\mathbb{R}^{d-1}} h(\boldsymbol{x}_{\shortparallel}) \mathrm{d}\boldsymbol{x}_{\shortparallel}. \tag{128}$$

We get

$$\begin{aligned}
&\sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \mathbb{E}\left[\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right) \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2\right) \mathrm{d}\boldsymbol{x}_{\shortparallel} X\right] \\
&= \sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \mathbb{E}\left[\fint_{\square_{R_1}(\boldsymbol{y}_{\shortparallel})} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right) \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2\right) \mathrm{d}\boldsymbol{x}_{\shortparallel} \mathrm{d}\boldsymbol{y}_{\shortparallel} X\right] \\
&\leq \sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \mathbb{E}\left[\left(\left(1 + \frac{R_1}{\ell + R^+}\right)^{\frac{d-1}{2}} \fint_{\square_{\ell+R^+ + R_1}(\boldsymbol{y}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right)\right. \\
&\qquad\qquad \left.\left(\fint_{\square_{R_1}(\boldsymbol{y}_{\shortparallel})} \fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}\right) \mathrm{d}\boldsymbol{y}_{\shortparallel} X\right]
\end{aligned}$$

By Hölder's inequality and the stationarity of ∇W_1 we obtain then for $q \gg 2$ such that $|\tilde{q} - 1| \leq \frac{1}{2C_0}$

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right) \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2\right) \mathrm{d}\boldsymbol{x}_{\shortparallel}\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \\
&\lesssim \left(1 + \frac{R_1}{\ell + R^+}\right)^{\frac{d-1}{2}} \mathbb{E}\left[\left(\fint_{\square_{\ell+R^+ + R_1}(0)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \\
&\qquad \sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \mathbb{E}\left[\left(\fint_{\square_{R_1}(\boldsymbol{y}_{\shortparallel})} \fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u_{\boldsymbol{g}}^\omega|^2 X \mathrm{d}\boldsymbol{x}_{\shortparallel}\right)^{\tilde{q}}\right]^{\frac{1}{\tilde{q}}} \mathrm{d}\boldsymbol{y}_{\shortparallel},.
\end{aligned}$$

Since $X \mapsto \mathbb{E}[|X|^{\frac{q}{2}}]^{\frac{1}{\frac{q}{2}}}$ is a norm we have by Jensen's inequality and the stationarity of ∇W_1

$$\begin{aligned}
&\frac{1}{(\ell + R^+)^{\frac{d-1}{2}}} \mathbb{E}\left[\left(\int_{\square_{\ell+R^+ + R_1}(0)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \\
&\leq \frac{1}{(\ell + R^+)^{\frac{d-1}{2}}} \mathbb{E}\left[\left(\int_{\square_{\ell+R^+}(0)} \int_{\square_{R_1}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2 \mathrm{d}\boldsymbol{x}_{\shortparallel}\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \\
&\leq \frac{1}{(\ell + R^+)^{\frac{d-1}{2}}} \int_{\square_{\ell+R^+}(0)} \mathbb{E}\left[\left(\int_{\square_{R_1}(0)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \mathrm{d}\boldsymbol{x}_{\shortparallel} = \mathbb{E}\left[\left(\int_{\square_{R_1}(0)} \int_0^h \mathbb{1}_{D^\omega} |\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}
\end{aligned} \tag{129}$$

Since $X \mapsto \mathbb{E}[|X|^{\tilde{q}}]^{\frac{1}{\tilde{q}}}$ is a norm we get by Jensen's inequality

$$\begin{aligned}
&\sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \mathbb{E}\left[\left(\fint_{\square_{R_1}(\boldsymbol{y}_{\shortparallel})} \fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u^\omega_{\boldsymbol{g}}|^2 X \mathrm{d}\boldsymbol{x}_{\shortparallel}\right)^{\tilde{q}}\right]^{\frac{1}{\tilde{q}}} \mathrm{d}\boldsymbol{y}_{\shortparallel} \\
&\le \sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \fint_{\square_{R_1}(\boldsymbol{y}_{\shortparallel})} \fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^\omega_{\boldsymbol{g}}|^{2\tilde{q}} |X|^{\tilde{q}}\right]^{\frac{1}{\tilde{q}}} \mathrm{d}\boldsymbol{x}_{\shortparallel} \mathrm{d}\boldsymbol{y}_{\shortparallel} \\
&= \sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \int_0^h \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^\omega_{\boldsymbol{g}}|^{2\tilde{q}} |X|^{\tilde{q}}\right]^{\frac{1}{\tilde{q}}} \mathrm{d}\boldsymbol{x}_{\shortparallel}.
\end{aligned}$$

In the last inequality we simplified the spatial averages in the right-hand side thanks to (128). Moreover since the supremum is computed over positive X we can reformulate it as follows

$$\begin{aligned}
\sup_{\|X\|_{L^{\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \int_0^h \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^\omega_{\boldsymbol{g}}|^{2\tilde{q}} |X|^{\tilde{q}}\right]^{\frac{1}{\tilde{q}}} \mathrm{d}\boldsymbol{x}_{\shortparallel} &= \sup_{\|Y\|_{L^{2\tilde{q}}(\Omega)}=1} \int_{\mathbb{R}^{d-1}} \int_0^h \mathbb{E}\left[\mathbb{1}_{D^\omega} |\nabla u^\omega_{\boldsymbol{g}} Y|^{2\tilde{q}}\right]^{\frac{1}{\tilde{q}}} \mathrm{d}\boldsymbol{x}_{\shortparallel} \\
&= \sup_{\|Y\|_{L^{2\tilde{q}}(\Omega)}=1} \|\mathbb{1}_{D^\omega} \nabla u^\omega_{\boldsymbol{g}} Y\|^2_{L^2(\mathbb{R}^{d-1}\times(0,h), L^{2\tilde{q}}(\Omega))}
\end{aligned}$$

Appealing to Theorem 23 we get

$$\begin{aligned}
\|\mathbb{1}_{D^\omega} \nabla u^\omega_{\boldsymbol{g}} Y\|_{L^2(\mathbb{R}^{d-1}\times(0,h), L^{2\tilde{q}}(\Omega))} = \|\mathbb{1}_{D^\omega} \nabla u^\omega_{Y\boldsymbol{g}}\|_{L^2(\mathbb{R}^{d-1}\times(0,h), L^{2\tilde{q}}(\Omega))} &\lesssim \|\boldsymbol{g} Y\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+, L^{2\tilde{q}}(\Omega))} \\
&= \|Y\|_{L^{2\tilde{q}}(\Omega)} \|\boldsymbol{g}\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)^d}.
\end{aligned}$$

Finally we proved that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^{d-1}} \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2\right) \left(\fint_{\square_{\ell+R^+}(\boldsymbol{x}_{\shortparallel})} \int_0^h \mathbb{1}_{D^\omega} |\nabla u^\omega_{\boldsymbol{g}}|^2\right) \mathrm{d}\boldsymbol{x}_{\shortparallel}\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \\
&\qquad \lesssim \|\boldsymbol{g}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)^d} \mathbb{E}\left[\left(\int_{\square_{R_1}} \int_0^h \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}.
\end{aligned} \tag{130}$$

We inject (130) into (127) and conclude thanks to the weight π with super-algebraic decay. □

Proof of proposition 26. Let $R \ge R^+$ and $L \ge R + H$. Take χ a smooth cut-off function such that $0 \le \chi \le 1$, $\chi|_{\square_R\times(0,L)} \equiv 1$, $\mathrm{supp}\chi \subset \square_{2R} \times (0, 2L)$ and $|\nabla\chi(\boldsymbol{x})| \lesssim \frac{1}{R}$ for all $\boldsymbol{x} \in \mathbb{R}^{d-1} \times \mathbb{R}$. We test (41) with $v := \chi W_1$. We obtain

$$\int_{\square_R} \int_0^L \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2 \le -\int_{\square_{2R}} \int_0^{2L} \nabla W^\omega_1 \cdot W^\omega_1 \nabla\chi + \int_{\square_{2R}\cap\Sigma_H} \chi W^\omega_1.$$

Using Young's inequality with $K \ge 1$ we get

$$\begin{aligned}
\int_{\square_R} \int_0^L \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2 \lesssim \frac{1}{R^2}\frac{K^2}{2} \int_{\square_{2R}} \int_0^{2L} \mathbb{1}_{D^\omega} |W^\omega_1|^2 + \frac{1}{2K^2} \int_{\square_{2R}} \int_0^{2L} \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2 \\
+ \frac{1}{2K^2} \int_{\square_{2R}\cap\Sigma_H} |W^\omega_1|^2 + K^2 R^{d-1}.
\end{aligned}$$

Next we apply the trace Theorem to the third term of the right-hand side

$$\begin{aligned}
\int_{\square_R} \int_0^L \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2 \lesssim K^2 \left(R^{d-1} + \frac{1}{R^2} \int_{\square_{2R}} \int_0^{2L} \mathbb{1}_{D^\omega} |W^\omega_1|^2\right) \\
+ \frac{1}{K^2} \left(\int_{\square_{2R}} \int_0^{2L} \mathbb{1}_{D^\omega} |\nabla W^\omega_1|^2\right).
\end{aligned} \tag{131}$$

Recall that under Hypothesis (L^∞) Poincaré's inequality holds in boxes larger than R^+ with a constant linear in R^+. We decompose the second term of the right-hand side as follows

$$\int_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|W_1^\omega|^2 \leq \int_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|W_1^\omega-\chi_r * W_1^\omega|^2+\int_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|\chi_r * W_1^\omega|^2.$$

where $\chi_r(\boldsymbol{x}) := r^{-d}\chi\left(\frac{\boldsymbol{x}}{r}\right)$. On one hand, we obtain by Poincaré's inequality under Hypothesis (L^∞)

$$\int_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|\chi_r * W_1^\omega|^2 \lesssim (R^+)^2\int_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|\chi_r * \nabla W_1^\omega|^2. \tag{132}$$

On the other hand, we write

$$\mathbb{1}_{D^\omega}\left(W_1^\omega-\chi_r * W_1^\omega\right)(\boldsymbol{x}_{\shortparallel},x_d)=\int_0^r\int_{\square_r}\int_0^1\mathbb{1}_{D^\omega}\chi_r(\boldsymbol{y})\nabla W_1^\omega(\boldsymbol{x}+t\boldsymbol{y})\cdot\boldsymbol{y}\mathrm{d}t\mathrm{d}\boldsymbol{y}_{\shortparallel}\mathrm{d}y_d.$$

Using Jensen's inequality, we get

$$\begin{aligned}\int_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|W_1^\omega-\chi_r * W_1^\omega|^2 &\leq \int_{\square_{2R}}\int_0^{2L}\int_0^r\int_{\square_r}\int_0^1\mathbb{1}_{D^\omega}|\chi_r(\boldsymbol{y})|^2|\nabla W_1^\omega(\boldsymbol{x}+t\boldsymbol{y})|^2\,|\boldsymbol{y}|^2\,\mathrm{d}t\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\ &\lesssim r^2\int_{\square_{2R+r}}\int_0^{2L+r}\mathbb{1}_{D^\omega}|\nabla W_1^\omega(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{133}$$

Plugging back (132) and (133) into (131) we finally proved the following

$$\begin{aligned}\fint_{\square_R}\int_0^L\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2 \lesssim K^2\left(1+\fint_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|\chi_r*\nabla W_1^\omega|^2\right)&\\ +\left(1+\frac{r}{2R}\right)^{d-1}\left(\frac{1}{K^2}+K^2\frac{r^2}{R^2}\right)\fint_{\square_{2R+r}}\int_0^{2L+r}\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2.&\end{aligned}\tag{134}$$

Next we take the $L^{\frac{q}{2}}(\Omega)$ norm of both sides of (134) for $q\geq 2$. By stationarity and Jensen's inequality we have on one hand

$$\begin{aligned}\mathbb{E}\left[\left(\fint_{\square_{2R}}\int_0^{2L}\mathbb{1}_{D^\omega}|\chi_r*\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} &\leq \int_0^{2L}\mathbb{E}\left[\mathbb{1}_{D^\omega}|\chi_r*\nabla W_1^\omega|^q\right]^{\frac{2}{q}}\\ &\leq \int_0^{2L}\mathbb{E}\left[\left|\int_{\mathbb{R}^{d-1}\times\mathbb{R}^+}\mathbb{1}_{D^\omega}\chi_r(\boldsymbol{y}_{\shortparallel},x_d-y_d)\nabla W_1^\omega(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^q\right]^{\frac{2}{q}}\mathrm{d}x_d,\end{aligned}$$

and on the other hand similarly as in (129)

$$\left(1+\frac{r}{2R}\right)^{d-1}\mathbb{E}\left[\left(\fint_{\square_{2R+r}}\int_0^{2L+r}\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}\lesssim\mathbb{E}\left[\left(\fint_{\square_R}\int_0^{2L+r}\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}. \tag{135}$$

To conclude we want to absorb the last term of the right-hand side of (134) into the left-hand side. In order to do so we decompose the integral in y_d in two parts: $(0,L)$ and $(L,2L+r)$. Thanks to (135) choosing $K\gg 1$ and $r=R^\alpha$ for some $\alpha\in(0,1)$ so that $\frac{1}{K^2}+K^2\frac{r^2}{R^2}$ is small enough we can absorb $\mathbb{E}\left[\left(\fint_{\square_R}\int_0^L\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}$ into the left-hand side.

Let us deal with the last term $\mathbb{E}\left[\left(\fint_{\square_R}\int_L^{2L+r}\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}$. We plug in W_1^ω's integral representation (44) and use Jensen's inequality

$$\begin{aligned}
&\mathbb{E}\left[\left(\fint_{\square_R}\int_L^{2L+r}\mathbb{1}_{D^\omega}|\nabla W_1^\omega|^2\right)^{\frac{q}{2}}\right]^{\frac{2}{q}} \lesssim \fint_{\square_R}\int_L^{2L+r}\mathbb{1}_{D^\omega}\mathbb{E}\left[|\nabla W_1^\omega|^q\right]^{\frac{2}{q}}\\
&\lesssim \fint_{\square_R}\int_L^{2L+r}\mathbb{1}_{D^\omega}\left|\int_{\mathbb{R}^{d-1}}|\nabla\partial_d\Gamma(\boldsymbol{z}_\shortparallel, x_d-L)|\mathbb{E}\left[\left(\int_0^{\boldsymbol{z}_\shortparallel}|\varphi^\omega|^2(\boldsymbol{x}_\shortparallel-\boldsymbol{w})\mathrm{d}\boldsymbol{w}\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\mathrm{d}\boldsymbol{z}_\shortparallel\right|^2,
\end{aligned}$$

where $\varphi := W_1^\omega|_{\Sigma_{L'}}$ with $L > L'$ and $\partial_d\Gamma$ is defined in (45). By the trace Theorem we deduce

$$\begin{aligned}
&\fint_{\square_R}\int_0^{2L+l}\mathbb{1}_{D^\omega}\left|\int_{\mathbb{R}^{d-1}}|\nabla\partial_d\Gamma(\boldsymbol{z}_\shortparallel, x_d-L)|\mathbb{E}\left[\left(\int_0^{\boldsymbol{z}_\shortparallel}|\varphi^\omega|^2(\boldsymbol{x}_\shortparallel-\boldsymbol{w})\mathrm{d}\boldsymbol{w}\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\mathrm{d}\boldsymbol{z}_\shortparallel\right|^2\\
&\lesssim \fint_{\square_R}\int_L^{2L+l}\mathbb{1}_{D^\omega}\left|\int_{\mathbb{R}^{d-1}}|\nabla\partial_d\Gamma(\boldsymbol{z}_\shortparallel, x_d-L')|(2R)^{\frac{d-1}{2}}\mathbb{E}\left[\left(\fint_{\square_{2R}}\int_0^{L'}|\nabla W_1^\omega|^2(\boldsymbol{w})\mathrm{d}\boldsymbol{w}\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\mathrm{d}\boldsymbol{z}_\shortparallel\right|^2,
\end{aligned}$$

where the constant is proportional to R^+L'. We obtain finally

$$\begin{aligned}
&\fint_{\square_R}\int_L^{2L+l}\mathbb{1}_{D^\omega}\left|\int_{\mathbb{R}^{d-1}}|\nabla\partial_d\Gamma(\boldsymbol{z}_\shortparallel, x_d-L')|(2R)^{\frac{d-1}{2}}\mathbb{E}\left[\left(\fint_{\square_{2R}}\int_0^{L'}|\nabla W_1^\omega|^2(\boldsymbol{w})\mathrm{d}\boldsymbol{w}\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\mathrm{d}\boldsymbol{z}_\shortparallel\right|^2\\
&\lesssim (2R)^{d-1}\mathbb{E}\left[\left(\fint_{\square_{2R}}\int_0^{L}|\nabla W_1^\omega|^2(\boldsymbol{w})\mathrm{d}\boldsymbol{w}\right)^{\frac{q}{2}}\right]^{\frac{2}{q}}\int_L^{2L+l}\left|\int_{\square_R}|\nabla\partial_d\Gamma(\boldsymbol{z}_\shortparallel, x_d-L')|\mathrm{d}\boldsymbol{z}_\shortparallel\right|^2.
\end{aligned}$$

We conclude once again by stationarity of W_1 and thanks to the integrability property of $\nabla\partial_d\Gamma$ that ensures that both in 2 and 3 dimensions

$$R^{d-1}\int_L^{2L+l}\left|\int_{\square_R}|\nabla\partial_d\Gamma(\boldsymbol{z}_\shortparallel, x_d-L')|\mathrm{d}\boldsymbol{z}_\shortparallel\right|^2 = o(R,L).$$

□