INSTITUT POLYTECHNIQUE DE PARIS

NNT : 2024IPPAE004

Thèse de doctorat

ENSTA

IP PARIS

Simulation and analysis of sign-changing Maxwell's equations in cold plasma

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École nationale supérieure de techniques avancées

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 4 avril 2024, par

Étienne PEILLON

Composition du Jury :

Kim Pham Professeur associé, ENSTA Paris	Président du jury
Martin Campos Pinto Professeur, Max Planck Institute, Allemagne	Rapporteur
Ralf Hiptmair Professeur, ETH Zürich, Suisse	Rapporteur
Simon Labrunie Maître de conférence, Université de Lorraine	Examinateur
Claire Scheid Maître de conférence, Université Côte-d'Azur	Examinatrice
Patrick Ciarlet Professeur, ENSTA Paris	Directeur de thèse
Maryna Kachanovska Professeur, Inria Saclay	Co-encadrante de thèse

Remerciements

Ce mémoire de thèse est le fruit d'un travail de longue haleine, qui a été rendu possible par le soutien de nombreuses personnes. Il me semble juste de souligner qu'il n'aurait pas pu être achevé sans le concours de certaines d'entre elles, et je leur en suis infiniment reconnaissant.

Mes premiers remerciements sont destinés à mes encadrants de thèse, Maryna et Patrick. On dit que les thésards tombent parfois amoureux de leurs encadrants, et, à défaut de l'être, j'ai une profonde admiration envers vous qui avez su guider mon effort au cours de la thèse. Maryna, tes idées qui sortent à toute vitesse et ta maîtrise technique ont été inspirante. J'ai été le premier doctorant que tu as encadré, et je n'ai été que le premier de la brillante carrière qui t'attend. Patrick, ta patience et ta rigueur à toute épreuve m'impressionne. J'ai beaucoup apprécié les discussions et la vision que tu m'as apporté lorque on cherche des réponses[1]. Je vous dis simplement : MERCI.

Je remercie chaleureusement les membres de mon jury. Je suis très reconnaissant envers Martin Campos Pinto et Ralf Hiptmair, qui avez accepté de lire, commenter et rapporter ce manuscrit. Je suis également honoré que Simon Labrunie, Claire Scheid et Kim Pham ayez accepté de faire partie de mon jury de thèse. Les discussions qui ont suivi ma présentation furent hautement stimulantes, à tel point que je n'ai pas vu le temps passer.

J'ai une pensée particulière pour Anouk Nicolopoulos, dont les échanges ont permis de grandement faciliter le travail durant la thèse.

Le laboratoire qui m'a accueilli, l'UMA, a également été d'un grand soutient d'une part matérielle, mais également par les personnes qui le constituent. Malgré mon air maussade, j'ai eu un réel plaisir à m'y rendre et y partager de nombreux cafés et autres discussions très diverses. Il s'agit pour moi d'un exemple de convivialité ! Je remercie notamment Éliane, Axel, Anne-Sophie, Lucas, Nicolas, Christophe, Patrick J., Pierre M., Zoïs, Frédéric et Laure pour votre écoute et vos conseils.

Il me semble qu'un bon indicateur de la vitalité du laboratoire est le nombre de doctorants et postdoctorants qui arrivent et partent chaque année. Zoé, Laura, Alice, Pierre et moi avons d'abord été accueilli par Mahran, Jean-François, Émile, Othmane, Akram et Damien. Puis, les années suivantes, ce fut notre tour d'accueillir de nouvelles personnes Rose-Cloé, Adrian, Farah, Quentin, Louise, Simone, Antonin, Timothée, Sarah, Cédric, Natalia, Thibaut et Raphaël. L'arrivée de l'équipe IDEFIX fut également un grand moment avec Morgane et Fabien. Et puis je tiens à remercier la gentillesse et l'écoute de mes collègues de bureau Simone et Rose-Cloé à l'égard à mon égard.

[1] Réponse : trouver la question.

J'ai une pensée particulière pour mes amis, qui m'ont apporté un réconfort inestimable. Pierre, nous avons commencé notre thèse ensemble et nous avons surmonté nos démons ensembles. Tu n'as cessé de me supporter et je suis envieux de la force qui t'anime ! Mathieu, les discussions que nous avons eues au cours des trop rares pauses déjeuner que nous avons eu ensemble m'auront permis de mieux comprendre notre société et nos luttes. Alice, nous nous suivons depuis le premier TD de la première année à Versailles, jusqu'à l'ENSTA. Élie, Paul, David, Victor vous avez toujours été attentionné à mon égard et apporté chance avec vos dés. Je pense également à mes anciens camarades de l'ENSTA Morgane, Alexandre, Baptiste et Thomas, et mes anciens camarades de Versailles Gilles, Agnès, Alexandre, Edwyn, Aurélie, Wesley, Samy et Emil.

Enfin, mes dernières pensées vont à ma famille, qui me supporte (sans doute) depuis ma naissance. Ma mère Isabelle et mon père Marc m'ont apporté un soutien généreux, attentif et indéfectible tout à long de mes études, et ce, jusqu'à la préparation du pot de thèse. J'ai également pu compter sur l'aide de ma sœur Madeleine et mon frère Émile. Cette thèse vous est dédiée.

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1.1 Généralités sur les plasmas

Le plasma est le quatrième état de la matière et la forme de matière la plus abondante dans l'univers. Il se caractérise par la présence de particules chargées et d'ions, dans des proportions et des densités qui peuvent varier dans l'espace et le temps. La température d'un plasma est généralement beaucoup plus élevée que la température ambiante, dépassant souvent plusieurs milliers de kelvins. Sur Terre, les plasmas sont principalement utilisés à des fins industrielles. L'un des exemples d'utilisation industrielle les plus fréquemment cités est la production d'énergie électrique par le biais de réacteurs nucléaires à fusion. En fait, cela motive de nombreux aspects de la recherche universitaire et industrielle. Différents types de réacteurs sont étudiés de nos jours, comme les Tokamaks [37], ou les Stellarators [39, Chapitre 17].

Il y a plusieurs défis à relever pour obtenir une réaction de fusion stable à l'intérieur de ces réacteurs. Nous en présentons ici trois. Le premier est le confinement du plasma à l'intérieur du réacteur. Pour ce faire, plusieurs dispositifs sont disposés de manière à imposer des champs magnétiques poloïdaux et toroïdaux et un courant électrique toroïdal [36], cf. figure 1.1. Il en résulte qu'un champ magnétique hélicoïdal est imposé au plasma. Néanmoins, ce type d'installation n'est pas suffisant pour empêcher les instabilités du plasma. Afin de contenir correctement le plasma, des mesures de la densité du plasma doivent être effectuées. En raison de la température extrême, une mesure intrusive est impossible. Ensuite, le deuxième défi consiste à contrôler la densité du plasma par des méthodes de réflectométrie [43, 40, 30, 33]. Pour ce faire, des ondes électromagnétiques de différentes fréquences sont envoyées, puis on mesure la réponse. Enfin, le dernier défi est le chauffage du plasma, et se fait en envoyant des ondes électromagnétiques à des fréquences et des directions spécifiques en fonction des caractéristiques du plasma. Théoriquement, il existe actuellement trois types d'ondes utilisables [39, Chapitre 12] : les ondes de fréquence cyclotron ionique, les ondes de fréquence cyclotron électronique et les ondes « lower-hybrid ».

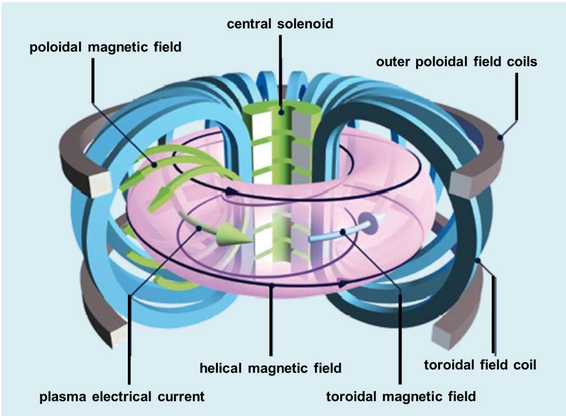


Fig. 1.1 : Représentation des champs magnétiques et du courant à l'intérieur d'un tokamak[1].

Pour relever ces défis, une description fine des champs électromagnétiques à l'intérieur du plasma est nécessaire. Ce rapport décrit le travail effectué sur deux configurations très simplifiées. Les deux configurations considèrent un plasma magnétisé avec en arrière-plan un champ magnétique constant $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$. La première configuration donne lieu à une équation aux dérivées partielles hyperbolique non standard dans l'espace. La seconde configuration consiste en l'étude de la résonance « lower-hybrid », qui conduit à une EDP dégénérée à changement de signe.

1.2 Ondes dans les plasmas

Les champs électromagnétiques dans un plasma sont décrits par quatre fonctions vectorielles dans l'espace-temps :

- le champ électrique $\boldsymbol{E}$,
- le déplacement électrique $\boldsymbol{D}$,
- le champ magnétique $\boldsymbol{H}$,
- le champ d'induction magnétique ou champ magnétisant $\boldsymbol{B}$.

Ces quatre champs sont liés par les bien connues équations de Maxwell :

$$\left|\begin{aligned} &\boldsymbol{curl}\,\boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t},\\ &\boldsymbol{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},\\ &\operatorname{div}\boldsymbol{D} = \rho,\\ &\operatorname{div}\boldsymbol{B} = 0 \end{aligned}\right. \tag{1.1}$$

[1]Source : https://www.iter.org/newsline/-/3037

où $\boldsymbol{j}$ est le vecteur de densité de courant et ρ est la densité de charge. En outre, elle est complétée par les deux relations constitutives qui s'appliquent à l'échelle microscopique :

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}, \qquad \boldsymbol{B} = \mu_0 \boldsymbol{H},$$

où ε_0 est la permittivité du vide et μ_0 est la perméabilité du vide. D'autre part, le milieu considéré est un plasma, qui se caractérise par la présence de particules libres chargées électriquement, telles que des électrons ou des ions. Par conséquent, une densité de courant $\boldsymbol{j}$ est induite par le déplacement des charges à l'intérieur du plasma. La présence d'une telle densité de courant au sein du plasma fait une différence majeure avec les milieux classiques. Plus précisément, nous considérons un plasma *froid magnétisé sans collision.*

Le plasma peut être décrit par deux approches qui ne sont pas équivalentes : l'approche fluide avec la densité des particules et des électrons et l'approche cinétique avec la fonction de distribution des particules. Notre plasma étant considéré comme froid, l'approche fluide est pertinente. L'approche cinétique repose sur la théorie de Boltzmann et ne sera pas notre sujet d'intérêt. Le lecteur intéressé pourra se référer à [55, Chapitre 8, 56, Chapitre 4].

Par conséquent, étant donné l'ensemble des différentes espèces d'ions S, nous pouvons décomposer le courant de plasma comme suit

$$\boldsymbol{j} = \sum_{s\in S} \boldsymbol{j}_s = \sum_{s\in S} \mathcal{N}_s q_s \boldsymbol{v}_s$$

où, étant donné une espèce ionique $s \in S$, $\mathcal{N}_s$ est la densité ionique, c'est-à-dire le nombre d'ions par unité de volume, q_s est la charge ionique et $\boldsymbol{v}_s$ est la vitesse. La vitesse et les champs électromagnétiques sont liés par l'équation de Navier-Stokes et la force de Lorentz :

$$\mathcal{N}_s m_s \left(\frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s \cdot \nabla)\, \boldsymbol{v}_s \right) = \mathcal{N}_s q_s \left(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B} \right) - \operatorname{div} \mathbb{T},$$

où $\mathbb{T}$ est le tenseur de contrainte du fluide.

À ce stade, plusieurs hypothèses de simplification sont faites. Soit $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ une base orthonormée de $\mathbb{R}^3$, avec (x_1, x_2, x_3) les coordonnées associées. Tout d'abord, le plasma est *froid* et *sans collision*. Par conséquent, le tenseur de contrainte fluide $\mathbb{T}$ est négligé. Ensuite, nous *linéarisons* l'équation autour de l'équilibre $(\boldsymbol{v}_s, \boldsymbol{E}, \boldsymbol{B}) = (0, 0, \boldsymbol{B}_0)$ où $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_3$ est le *champ magnétique en arrière-plan* imposé au plasma. Nous supposons également que les densités d'ions $\mathcal{N}_s$ ne varient pas dans le temps. Par conséquent, en développant $\boldsymbol{v}_s$, $\boldsymbol{E}$, $\boldsymbol{B}$ au premier ordre et en substituant ces quantités dans l'équation de Navier-Stokes, on obtient

$$\frac{\partial \boldsymbol{v}_s}{\partial t} = \frac{q_s}{m_s} \left(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}_0 \right).$$

Enfin, nous supposons que nous sommes en régime harmonique, c'est-à-dire que toutes les quantités $a(\boldsymbol{x}, t)$ qui dépendent du temps peuvent être développées comme $a(\boldsymbol{x}, t) = \operatorname{Re}\left(\hat{a}(\boldsymbol{x}) e^{-i\omega t}\right)$. Les équations ci-dessus deviennent alors

$$-i\omega \hat{\boldsymbol{v}}_s = \frac{q_s}{m_s} \left(\hat{\boldsymbol{E}} + \hat{\boldsymbol{v}}_s \times (B_0 \boldsymbol{e}_3) \right).$$

Les vecteurs propres de l'opérateur $\boldsymbol{e}_3 \times \cdot$ sont $(\boldsymbol{e}_+, \boldsymbol{e}_-, \boldsymbol{e}_3)$ avec $\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}} (\boldsymbol{e}_1 \mp i \boldsymbol{e}_2)$, et ils constituent une base orthonormée de l'espace vectoriel complexe $\mathbb{C}^3$. Dans cette base, nous avons

$$\hat{v}_{s,\pm} = \left(\frac{q_s}{m_s} \right) \frac{i}{\omega \mp \omega_c} \hat{E}_\pm, \qquad \hat{v}_{s,z} = \left(\frac{q_s}{m_s} \right) \frac{i}{\omega} \hat{E}_z,$$

avec $\omega_{c,s} = \frac{q_s B_0}{m_s}$, la *fréquence cyclotron* associée à l'espèce ionique s. On remarque que $\hat{v}_{s,\pm} = \frac{1}{\sqrt{2}}\left(\hat{v}_{s,x} \pm i\hat{v}_{s,y}\right)$ et $\hat{E}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{E}_x \pm i\hat{E}_y\right)$. Ensuite, la densité de courant $\hat{\boldsymbol{j}}$ peut être exprimée en fonction du champ électrique $\hat{\boldsymbol{E}}$ dans la base $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$:

$$\hat{\boldsymbol{j}}_s = \mathcal{N}_s q_s \hat{\boldsymbol{v}}_s = \sigma_s \hat{\boldsymbol{E}}, \quad \text{avec} \quad \sigma_s = i\omega\varepsilon_0\chi_s \quad \text{et} \quad \chi_s = \begin{pmatrix} -\frac{\omega_{p,s}^2}{\omega^2-\omega_{c,s}^2} & -i\frac{\omega_{c,s}\omega_{p,s}^2}{\omega(\omega^2-\omega_{c,s}^2)} & 0 \\ i\frac{\omega_{c,s}\omega_{p,s}^2}{\omega(\omega^2-\omega_{c,s}^2)} & -\frac{\omega_{p,s}^2}{\omega^2-\omega_{c,s}^2} & 0 \\ 0 & 0 & \frac{\omega_{p,s}^2}{\omega^2} \end{pmatrix}.$$

Les matrices σ_s et χ_s sont respectivement appelées tenseur de conductivité et tenseur de susceptibilité électrique. Nous introduisons également la *fréquence du plasma* $\omega_{p,s} = \sqrt{\frac{\mathcal{N}_s q_s^2}{m_s \varepsilon_0}}$.

Remark 1.2.1. L'approche peut être généralisée à n'importe quel champ magnétique en arrière-plan $\boldsymbol{B}_0(\boldsymbol{x})$. En fait, étant donné un point $\boldsymbol{x}\, dans \mathbb{R}^3$, le tenseur de susceptibilité électrique $\chi_s(\boldsymbol{x})$ est toujours diagonal lorsqu'il est exprimé dans la base constituée par les vecteurs propres de l'opérateur $\boldsymbol{B}_0(\boldsymbol{x}) \times \cdot$, cf. [31, Chapitre 2].

Enfin, si l'on revient aux équations de Maxwell exprimées en régime harmonique, on obtient

$$\left| \begin{aligned} &\mathbf{curl}\, \hat{\boldsymbol{B}} = -\frac{i\omega}{c^2}\epsilon\hat{\boldsymbol{E}}, \\ &\mathbf{curl}\, \hat{\boldsymbol{E}} = i\omega\hat{\boldsymbol{B}}, \end{aligned} \right. \tag{1.2}$$

où $c = (\mu_0\varepsilon_0)^{-1/2}$, et le *tenseur diélectrique du plasma froid* est donné par

$$\begin{gathered} \epsilon = \mathbb{I}_3 + \sum_{s\in S}\chi_s = \begin{pmatrix} \alpha & -i\delta & 0 \\ i\delta & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \\ \alpha = 1 - \sum_{s\in S}\frac{\omega_{p,s}^2}{\omega^2-\omega_{c,s}^2}, \quad \delta = \frac{1}{\omega}\sum_{s\in S}\frac{\omega_{c,s}\omega_{p,s}^2}{\omega^2-\omega_{c,s}^2}, \quad \beta = 1 - \frac{\sum_{s\in S}\omega_{p,s}^2}{\omega^2}, \\ \omega_{c,s} = \frac{q_s B_0}{m_s}, \qquad \omega_{p,s}^2 = \frac{\mathcal{N}_s q_s^2}{m_s\varepsilon_0}. \end{gathered} \tag{1.3}$$

Faisons quelques commentaires sur le modèle ci-dessus. Tout d'abord, le tenseur diélectrique du plasma froid varie en fonction de la variable d'espace $\boldsymbol{x}$ et de la fréquence du régime harmonique ω. En effet, les fréquences du plasma $\omega_{p,s}$ dépendent des densités d'ions $\mathcal{N}_s$[2], et la dépendance en la fréquence ω indique que le modèle est manifestement dispersif. Deuxièmement, ϵ n'est pas nécessairement positif pour toute fréquence en tout point de l'espace. Cette observation est à la base de cette thèse et sera discutée dans les deux paragraphes suivants. D'autre part, le cas ϵ uniformément positif ou uniformément négatif correspond aux équations de Maxwell classiques qui ont déjà été étudiées du point de vue mathématique depuis l'établissement de ces équations.

Le modèle ci-dessus a été largement étudié par la communauté des physiciens, et nous renvoyons aux monographies suivantes [55, 56, 39].

[2] $\omega_{p,s}$ ne dépend pas de la fréquence ω, car $\mathcal{N}_s$ ne dépend pas du temps.

1.3 Plasma dans un champ magnétique fort

Comme indiqué dans le dernier paragraphe, le tenseur diélectrique $\mathbb{e}$ n'est pas nécessairement positif. Si nous considérons la fréquence globale du plasma $\omega_p^2 = \sum_{s\in S} \omega_{p,s}^2$, alors nous avons $\beta = 1 - \frac{\omega_p^2}{\omega^2}$ négatif chaque fois que $\omega < \omega_p$. D'autre part, nous pouvons clairement trouver les fréquences $\omega_{p,s}$ et $\omega_{c,s}$ de telle sorte que le bloc 2×2

$$\begin{pmatrix} \alpha & -i\delta \\ i\delta & \alpha \end{pmatrix}$$

est défini positif, ce qui est équivalent à $\alpha > |\delta|$. Afin de simplifier l'analyse, nous supposerons que le champ magnétique en arrière-plan $\boldsymbol{B}_0$ a une magnitude très importante, de telle sorte que les fréquences cyclotron $\omega_{c,s}$ sont très grandes par rapport à la fréquence globale du plasma ω_p. Ensuite, compte tenu des expressions (1.3) de α et δ, le tenseur diélectrique peut être approximé comme suit

$$\mathbb{e} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{pmatrix}. \tag{1.4}$$

Nous supposons dans la suite que cette approximation, que nous appelons *limite du champ magnétique fort*, est valable dans tout l'espace libre, et que ω_p ne varie pas. L'objectif de la première partie de cette thèse est donc d'étudier le problème posé dans *l'espace libre* :

$$\left|\begin{array}{l} \text{Trouver } \hat{\boldsymbol{E}}, \widehat{\boldsymbol{B}} \text{ tels que} \\ \textbf{\textit{curl}}\, \widehat{\boldsymbol{B}} + \dfrac{i\omega}{c^2} \mathbb{e} \hat{\boldsymbol{E}} = \hat{\boldsymbol{j}}, \\ \textbf{\textit{curl}}\, \hat{\boldsymbol{E}} - i\omega \widehat{\boldsymbol{B}} = \widehat{\boldsymbol{m}}, \end{array}\right. \tag{1.5}$$

où certains termes sources $\boldsymbol{j}, \boldsymbol{m}$ ont été ajoutés.

Alors que de nombreux travaux sont consacrés aux modèles isotropes, c'est-à-dire aux modèles dans lesquels la permittivité diélectrique et la perméabilité magnétique sont toutes deux des scalaires [10, 47, 11, 15, 16, 44, 8], peu de travaux sont consacrés aux milieux anisotropes, surtout si le tenseur de permittivité diélectrique ou de perméabilité magnétique n'est plus de signe défini. Remarquons qu'il existe des travaux consacrés à l'étude des équations de Maxwell avec des tenseurs de perméabilité magnétique et de permittivité diélectrique elliptiques anisotropes [19, 21, 20], ce qui n'est malheureusement pas notre cas. Nous étudions ici ce qui est appelé des métamatériaux hyperboliques, voir [53] et les références à l'intérieur. À notre connaissance, peu d'analyses mathématiques [12, 22] sont consacrées à l'étude des métamatériaux hyperboliques. On notera que les problèmes hyperboliques dans le domaine des fréquences peuvent apparaître dans la dynamique des fluides, voir [28, 29]. En outre, les problèmes hyperboliques en domaines bornés ont été étudiés pour la première fois, à notre connaissance, dans [38].

Notre modèle a déjà été étudié dans le cas de champs électromagnétiques 2D, c'est-à-dire que le champ magnétique et le champ électrique sont indépendants de la variable y dans [22] dans le régime harmonique ; voir [7, 6] pour son équivalent dans le domaine temporel.

Remark 1.3.1. Le système peut être réécrit dans le domaine temporel comme suit :

$$\begin{aligned} \boldsymbol{curl}\,\boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E} &= \mu_0 \boldsymbol{J}_p, \\ \boldsymbol{curl}\,\boldsymbol{E} + \partial_t \boldsymbol{B} &= 0. \end{aligned}$$

où $\partial_t \boldsymbol{J}_p = \omega_p^2 \varepsilon_0 (\boldsymbol{E} \cdot \boldsymbol{e}_3)\boldsymbol{e}_3$.

1.4 Plasma avec une densité variable

1.4.1 Plasma à une espèce

Lorsqu'une onde électromagnétique est envoyée à l'intérieur d'un plasma, elle peut transférer de l'énergie aux particules et produire un chauffage du plasma dans une région localisée. Ce phénomène est lié à ce que l'on appelle les ondes résonantes et apparaît avec la variation dans l'espace des densités d'ions $\mathcal{N}_s(\boldsymbol{x})$. Pour simplifier, nous supposons que le plasma est constitué d'une seule espèce d'ions de densité $\mathcal{N}_e(\boldsymbol{x})$. Par conséquent, le tenseur diélectrique

$$\varepsilon(\boldsymbol{x}) = \begin{pmatrix} \alpha(\boldsymbol{x}) & -i\delta(\boldsymbol{x}) & 0 \\ i\delta(\boldsymbol{x}) & \alpha(\boldsymbol{x}) & 0 \\ 0 & 0 & \beta(\boldsymbol{x}) \end{pmatrix},$$

varie également dans l'espace, où α, δ, β sont donnés par (1.3), ce qui peut être écrit dans notre cas comme

$$\alpha = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \delta = \frac{1}{\omega}\frac{\omega_c \omega_p^2}{\omega^2 - \omega_c^2}, \quad \beta = 1 - \frac{\omega_p^2}{\omega^2},$$
$$\omega_c = \frac{q_e B_0}{m_c}, \qquad \omega_p^2 = \frac{q_e^2}{m_e \varepsilon_0}\mathcal{N}_e.$$

Nous supposons que la densité du plasma $\mathcal{N}_e$ varie dans l'espace de sorte que nous ayons

$$\alpha(\boldsymbol{x}) = 1 - C_{\alpha,\omega}\mathcal{N}_e(\boldsymbol{x}), \qquad \delta(\boldsymbol{x}) = C_{\delta,\omega}\mathcal{N}_e(\boldsymbol{x}), \qquad \beta(\boldsymbol{x}) = 1 - C_{\beta,\omega}\mathcal{N}_e(\boldsymbol{x}). \tag{1.6}$$

Nous considérons une résonance « lower-hybrid » dans le plasma, voir [55, Chapter 2-6] et les récents travaux [35, 26, 13, 14, 27, 25, 48, 50, 49], qui est caractérisée par le fait que $\alpha = 0$ sur une courbe à l'intérieur de la région. Comme dans les travaux cités ci-dessus, nous nous intéresserons particulièrement aux cas où la densité $\mathcal{N}_e(\boldsymbol{x})$ est telle que le signe de α change continûment entre deux sous-régions séparées par une interface.

Avec les notations évidentes, $\boldsymbol{E} = E_1\boldsymbol{e}_1 + E_2\boldsymbol{e}_2 + E_3\boldsymbol{e}_3$, etc. Nous supposerons dans cette partie que toutes les quantités sont indépendantes de x_3, la variable correspondant à la direction du champ magnétique extérieur, de sorte que $E_1 = E_1(x_1, x_2)$, etc. Alors, en développant la première équation de (1.2), on a

$$\begin{pmatrix} \partial_2 B_3 \\ -\partial_1 B_3 \\ \partial_1 B_2 - \partial_2 B_1 \end{pmatrix} = -\frac{i\omega}{c^2}\begin{pmatrix} \alpha E_1 - i\delta E_2 \\ i\delta E_1 + \alpha E_2 \\ \beta E_3 \end{pmatrix}.$$

On observe que E_1, E_2, B_3 et E_3, B_1, B_2 sont indépendants dans l'équation ci-dessus. Cette observation est également valable pour la deuxième équation de (1.2). Ainsi, grâce à la structure diagonale de $\mathbb{\epsilon}$, le système de Maxwell (1.2) peut être divisé en deux systèmes indépendants qui dissocient E_3, $\mathbf{B}_\perp = B_1\mathbf{e}_1 + B_2\mathbf{e}_2$ d'une part, et $\mathbf{E}_\perp = E_1\mathbf{e}_1 + E_2\mathbf{e}_2$, B_3 d'autre part. Au vu de l'équation précédente, nous définissons les deux opérateurs différentiels suivants :

$$\mathbf{curl}_\perp f = \begin{pmatrix} \partial_2 f \\ -\partial_1 f \end{pmatrix}, \qquad \mathrm{curl}_\perp (f_1\mathbf{e}_1 + f_2\mathbf{e}_2) = \partial_1 f_2 - \partial_2 f_1.$$

Le système pour le mode Ordinaire est donc le suivant :

$$\begin{cases} \mathbf{curl}_\perp E_3 = i\omega \mathbf{B}_\perp, \\ \mathrm{curl}_\perp \mathbf{B}_\perp = -\dfrac{i\omega\beta}{c^2} E_3, \end{cases} \tag{O-mode}$$

et le mode eXtraodinaire :

$$\begin{cases} \mathrm{curl}_\perp \mathbf{E}_\perp = i\omega B_3, \\ \mathbf{curl}_\perp B_3 = -\dfrac{i\omega}{c^2}\mathbb{\epsilon}_\perp \mathbf{E}_\perp, \end{cases} \qquad \text{avec} \quad \mathbb{\epsilon}_\perp = \begin{pmatrix} \alpha & -i\delta \\ i\delta & \alpha \end{pmatrix}. \tag{X-mode}$$

Pour la discussion qui suit, nous aurons besoin d'introduire quelques notations auxiliaires. Soit $\mathbf{x}_\perp = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\Delta_\perp v = \partial_{11}v + \partial_{22}v$, $\mathrm{div}_\perp \mathbf{v} = \partial_1 v_1 + \partial_2 v_2$ et $\nabla_\perp v = \partial_1 v\mathbf{e}_1 + \partial_2 v\mathbf{e}_2$. Dans ce cas, il est évident que $\mathbf{curl}_\perp v = -R_{\pi/2}\nabla_\perp v$ et $\mathrm{curl}_\perp \mathbf{v} = -\mathrm{div}_\perp R_{\pi/2}\mathbf{v}$, où $R_{\pi/2} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ est la matrice de rotation $\pi/2$ dans le plan orienté $(\mathbf{e}_1, \mathbf{e}_2)$. En particulier, $\mathrm{curl}_\perp \mathbf{curl}_\perp v = -\Delta_\perp v$.

Concentrons-nous maintenant sur les équations régissant les inconnues scalaires E_3 et B_3. L'EDP du second ordre dérivée du système pour le mode ordinaire est $-\Delta_\perp E_3 = \frac{\omega^2\beta}{c^2}E_3$. Dans le cas où le signe de β change continuellement de signe, cette équation rappelle une équation d'Airy, cf. [35].

D'autre part, l'EDP du second ordre dérivée du système (X-mode) est la suivante :

$$\mathrm{div}_\perp \left(R_{\pi/2}\mathbb{\epsilon}_\perp^{-1} R_{\pi/2} \nabla_\perp B_3 \right) = \frac{\omega^2}{c^2} B_3.$$

Nous supposerons que le tenseur $\mathbb{\epsilon}_\perp$ est inversible partout dans la région, plus précisément que $\alpha^2(\mathbf{x}_\perp) - \delta^2(\mathbf{x}_\perp) \neq 0$ pour tout $\mathbf{x}_\perp$, et donc que l'expression ci-dessus est bien définie. Définissons le tenseur 2×2 suivant :

$$\mathbb{\alpha} := c^2 R_{\pi/2}\mathbb{\epsilon}_\perp^{-1} R_{\pi/2} = \frac{c^2}{\delta^2 - \alpha^2}\begin{pmatrix} \alpha & i\delta \\ -i\delta & \alpha \end{pmatrix}.$$

Ensuite, étant donné que α et δ dépendent de la variable d'espace $\mathbf{x}_\perp$ seulement par l'intermédiaire de la densité du plasma $\mathcal{N}_e(\mathbf{x}_\perp)$, α et δ ont les mêmes courbes de niveau. De plus, en gardant à l'esprit (1.6), $\alpha = 1 - \delta/\delta^+$ où $\delta^+ = C_{\delta,\omega}/C_{\alpha,\omega}$.

Comme indiqué précédemment, nous supposons que le coefficient $\alpha(\mathbf{x}_\perp)$ s'annule sur une interface I. Compte tenu de la dernière remarque, le tenseur $\mathbb{\alpha}(\mathbf{x}_\perp)$ est constant sur I, et est égal à $\mathbb{\alpha} = i\mathbb{A}$, avec

$$\mathbb{A} = \frac{c^2}{{\delta^+}^2}\begin{pmatrix} 0 & -\delta^+ \\ \delta^+ & 0 \end{pmatrix},$$

qui est une matrice antisymétrique à valeurs réelles. Il s'ensuit que nous pouvons décomposer $\mathbb{a}(\mathbf{x}_\perp)$ comme

$$\mathbb{a}(\boldsymbol{x}_\perp) = \underbrace{-\alpha_0(\mathbf{x}_\perp)\mathbb{H}(\mathbf{x}_\perp)}_{\mathbb{a}_0(\mathbf{x}_\perp)} + i\mathbb{A}, \tag{1.7}$$

où $\alpha_0 = \frac{c^2\alpha}{\alpha^2-\delta^2}$, et $\mathbb{H}(\mathbf{x}_\perp)$ est une matrice hermitienne donnée par

$$\mathbb{H}(\mathbf{x}_\perp) = \begin{pmatrix} 1 & -i\left(\delta(\mathbf{x}_\perp) + \alpha(\mathbf{x}_\perp)/\delta^+\right) \\ i\left(\delta(\mathbf{x}_\perp) + \alpha(\mathbf{x}_\perp)/\delta^+\right) & 1 \end{pmatrix}.$$

Dans ce qui suit, nous supposerons que $\mathbb{H}(\mathbf{x}_\perp)$ est définie positive dans toute la région de calcul. Cela implique en particulier que le déterminant de $\mathbb{H}(\mathbf{x}_\perp)$ est positif pour tout $\mathbf{x}_\perp$, ce qui conduit à

$$\left|\delta(\mathbf{x}_\perp) + \alpha(\mathbf{x}_\perp)/\delta^+\right| < 1.$$

Puisque $\mathbb{A}$ est antisymétrique, $\operatorname{div}_\perp(\mathbb{A}\,\nabla_\perp B_3) = 0$, et l'EDP du second ordre régissant B_3 devient donc

$$-\operatorname{div}_\perp(\mathbb{a}_0\,\nabla_\perp B_3) - \omega^2 B_3 = 0. \tag{1.8}$$

Nous supposons que la densité des électrons $\mathcal{N}_e$ est $\mathscr{C}^2$-régulière, de sorte que α_0 et $\mathbb{H}$ soient également $\mathscr{C}^2$-réguliers. Étant donné $\mathbf{x}_\perp$ sur l'interface I, notons $\mathbf{n}(\mathbf{x}_\perp)$ la normale à l'interface au point $\mathbf{x}_\perp$. Alors, pour h un réel suffisamment petit, nous pouvons écrire le développement en série de α_0 :

$$\alpha_0(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) = \frac{\partial\alpha_0}{\partial\mathbf{n}}(\mathbf{x}_\perp)h + \frac{\partial^2\alpha_0}{\partial\mathbf{n}^2}(\mathbf{x}_\perp)h^2 + \mathcal{O}(h^3).$$

Par conséquent, nous supposons à partir de maintenant que $|\alpha_0(\mathbf{x}_\perp)|$ se comporte dans un voisinage de l'interface $I = \{\alpha_0(\mathbf{x}_\perp) = 0\}$ comme $\operatorname{dist}(\mathbf{x}_\perp, I)$ et ne dégénèrent pas au sens où il existe une constante $c > 0$ telle que $\left|\frac{\partial\alpha_0}{\partial\mathbf{n}}(\mathbf{x}_\perp)\right| > c$ pour tout $\mathbf{x}_\perp \in I$. Nous supposons également que l'interface I est une boucle C^1 (sans auto-intersection).

Ainsi, considérer le modèle dérivé du mode eXtraodinaire avec B_3 inconnu dans le voisinage de l'interface conduit à une EDP elliptique dégénérée. La communauté mathématique étudie les modèles dérivés du cadre des ondes dans le plasma froid depuis une décennie environ, cf. [25, 2, 26, 7, 42]. La résolution de cette équation sera l'objectif de la deuxième partie de cette thèse. Ce problème a déjà été étudié dans [49], où une méthode numérique basée sur une formulation variationnelle mixte a été proposée.

Remark 1.4.1. Remarquez que d'autres hypothèses peuvent être faites sur le comportement de α_0 : $\operatorname{dist}(\mathbf{x}_\perp, I)^2$, $\operatorname{dist}(\mathbf{x}_\perp, I)^3$, ou même une puissance fractionnaire. Cela signifierait que $\frac{\partial\alpha_0}{\partial\mathbf{n}}(\mathbf{x}_\perp)$ s'annule pour tout $\mathbf{x}_\perp$ sur l'interface. En particulier, cela conduirait à un type de singularité différent de celui étudié dans cette thèse. De plus, cela ne semble pas pertinent d'un point de vue physique puisque la densité du plasma est régulière en pratique, et nous excluons clairement la présence de chocs.

1.4.2 Plasma général

Le problème décrit ci-dessous n'a pas été étudié dans cette thèse. Néanmoins, nous pensons qu'il est intéressant de le formaliser, car il apparaît que de tels problèmes peuvent être liés à la littérature existante, ouvrant ainsi de nouvelles perspectives.

L'argument développé pour un plasma à une espèce ne peut pas être appliqué sans adaptation à un plasma réel, qui a en général 2 espèces au moins (les électrons et les ions). Dans ce cas, en utilisant (1.3), nous avons

$$\alpha(\boldsymbol{x}) = 1 - \sum_{s\in S} C_{s,\alpha,\omega}\mathcal{N}_s(\boldsymbol{x}), \qquad \delta(\boldsymbol{x}) = \sum_{s\in S} C_{s,\delta,\omega}\mathcal{N}_s(\boldsymbol{x}),$$

avec $C_{s,\alpha,\omega} > 0$ et $C_{s,\delta,\omega} \in \mathbb{R}$ pour toute espèce $s \in S$. On notera que $C_{s,\delta,\omega} > 0$ (respectivement $C_{s,\delta,\omega} < 0$) pour les espèces chargées positivement (resp. négativement), comme l'indiquent les équations (1.3).

Comme précédemment, nous supposons que le problème est indépendant de x_3, de sorte que le problème peut être séparé en modes ordinaires et extraordinaires, et que le signe de α change à travers une interface I. Nous supposons que les densités des espèces d'ions sont au moins $\mathscr{C}^2$, et la même hypothèse s'applique à δ et α. En outre, $|\alpha(\mathbf{x}_\perp)|$ se comporte dans un voisinage de l'interface $I = \{\alpha(\mathbf{x}_\perp) = 0\}$ comme $\mathrm{dist}(\mathbf{x}_\perp, I)$ et ne dégénèrent pas au sens où il existe une constante $c > 0$ telle que $\left|\frac{\partial\alpha}{\partial\mathbf{n}}(\mathbf{x}_\perp)\right| > c$ pour tout $\mathbf{x}_\perp \in I$.

Si nous reproduisons l'analyse précédente, le blocage provient du fait que $\alpha(\mathbf{x}_\perp)$ et $\delta(\mathbf{x}_\perp)$ ne partagent pas les mêmes courbes de niveau. Par conséquent, la matrice $\mathbb{\alpha}(\mathbf{x}_\perp)$ n'est plus constante sur l'interface I, et nous ne pouvons pas définir une matrice constante $\mathbb{A}$ telle que la décomposition (1.7) soit valide.

Nous pouvons néanmoins formaliser le problème de la manière suivante. La valeur de $\mathbb{\alpha}$ sur l'interface est $\mathbb{\alpha}(\mathbf{x}_\perp) = i\mathbb{A}(\mathbf{x}_\perp)$ avec

$$\mathbb{A}(\mathbf{x}_\perp) = \frac{c^2}{\delta(\mathbf{x}_\perp)^2}\begin{pmatrix} 0 & -\delta(\mathbf{x}_\perp) \\ \delta(\mathbf{x}_\perp) & 0 \end{pmatrix}, \text{ with } \mathbf{x}_\perp \in I.$$

Il est possible d'étendre la définition de cette matrice à l'ensemble du domaine. Pour tout $\mathbf{x}_\perp$ dans le domaine, il existe $\mathscr{P}(\mathbf{x}_\perp) \in I$, la projection de $\mathbf{x}_\perp$ sur l'interface I, et $s(\mathbf{x}_\perp) \in \mathbb{R}$ tel que $\mathbf{x}_\perp = \mathscr{P}(\mathbf{x}_\perp) + s(\mathbf{x}_\perp)\mathbf{n}(\mathscr{P}(\mathbf{x}_\perp))$, où $\mathbf{n}(\mathbf{y}_\perp)$ est le vecteur normal unitaire à l'interface au point $\mathbf{y}_\perp \in I$. Nous pouvons raisonnablement supposer que la dernière décomposition est unique dans le domaine qui nous intéresse. Par conséquent, nous étendons facilement $\mathbb{A}$ dans l'ensemble du domaine comme suit :

$$\mathbb{A}(\mathbf{x}_\perp) = \frac{c^2}{\delta(\mathscr{P}(\mathbf{x}_\perp))^2}\begin{pmatrix} 0 & -\delta(\mathscr{P}(\mathbf{x}_\perp)) \\ \delta(\mathscr{P}(\mathbf{x}_\perp)) & 0 \end{pmatrix}.$$

Ensuite, nous factorisons la matrice $\mathbb{\alpha} = -\alpha_0\mathbb{H} + i\mathbb{A}$ comme précédemment, avec $\alpha_0 = \frac{c^2\alpha}{\alpha^2-\delta^2}$ et

$$\mathbb{H}(\mathbf{x}_\perp) = \begin{pmatrix} 1 & i\tilde{\delta}(\mathbf{x}_\perp) \\ -i\tilde{\delta}(\mathbf{x}_\perp) & 1 \end{pmatrix}, \qquad \tilde{\delta}(\mathbf{x}_\perp) = \frac{\delta(\mathbf{x}_\perp)\left(\delta(\mathscr{P}(\mathbf{x}_\perp)) - \delta(\mathbf{x}_\perp)\right) + \alpha(\mathbf{x}_\perp)^2}{\delta(\mathscr{P}(\mathbf{x}_\perp))\alpha(\mathbf{x}_\perp)}.$$

En raison de l'hypothèse de régularité, $\tilde{\delta}$ ne dégénère pas dans le voisinage de l'interface. En effet, le développement de α et δ pour $h \in \mathbb{R}$ petit et $\mathbf{x}_\perp \in I$, c'est-à-dire,

$$\delta(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) = \delta(\mathbf{x}_\perp) + \frac{\partial\delta}{\partial\mathbf{n}}(\mathbf{x}_\perp)h + \mathscr{O}(h^2),$$
$$\alpha(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) = \frac{\partial\alpha}{\partial\mathbf{n}}(\mathbf{x}_\perp)h + \mathscr{O}(h^2),$$

donne

$$\tilde{\delta}(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) = \frac{\frac{\partial \delta}{\partial \mathbf{n}}(\mathbf{x}_\perp)}{\frac{\partial \alpha}{\partial \mathbf{n}}(\mathbf{x}_\perp)} + \mathcal{O}(h).$$

Par conséquent, la matrice $\mathbb{H}$ est hermitienne, continue, et nous supposons également qu'elle est définie positive dans la région de calcul, ce qui exige que $|\tilde{\delta}| < 1$. Ensuite, nous observons que nous pouvons également calculer la dérivée normale de la matrice $\mathbb{A}$, qui est $\frac{\partial}{\partial \mathbf{n}}\mathbb{A} = 0_{\mathbb{R}^{2\times 2}}$ par construction. Nous dirons donc que $\mathbb{A}$ est *transverse*. Enfin, l'équation du problème peut être résumée comme suit :

$$-\operatorname{div}_\perp (\mathbb{\alpha} \nabla_\perp B_3) - \omega^2 B_3 = 0, \tag{1.9}$$

avec $\mathbb{\alpha} = \alpha_0 \mathbb{H} + i\mathbb{A}$, et

- α_0 est une fonction continue, qui change de signe au travers d'une interface I, dont la dérivée normale ne s'annule sur I,
- $\mathbb{H}$ est une matrice hermitienne uniformément elliptique sur le domaine,
- $\mathbb{A}$ est une matrice antisymétrique, transverse à l'interface, dans le sens où $\left.\frac{\partial \mathbb{A}}{\partial \mathbf{n}}\right|_I = 0_{\mathbb{R}^{2\times 2}}$.

En fait, ce type d'opérateur a déjà été partiellement étudié, sans changement de signe, mais seulement avec la dégénérescence à la frontière dans la thèse de Baouendi. On peut aussi se référer à [4, 3].

1.5 Plan de la thèse

Ce travail est divisé en deux parties.

La première partie consiste en l'étude du modèle de plasma avec un fort champ magnétique en arrière-plan, ce qui correspond à un métamatériau hyperbolique. L'objectif est d'étendre les résultats de [22] au cas 3D et de dériver une condition de rayonnement. Le chapitre correspondant introduit une décomposition des champs électriques et magnétiques ressemblant à la décomposition habituelle TE et TM, puis il donne quelques résultats sur les deux problèmes résultants. Les résultats sont très partiels et constituent une ébauche sur le sujet.

La seconde partie consiste en l'étude de l'EDP dégénérée associée à l'équation (1.8). Le problème aux limites associé est bien posé dans un cadre variationnel « naturel ». Cependant, ce cadre n'inclut pas le comportement singulier présenté par les solutions physiques obtenues via le principe d'absorption limite, cf. [35, 14]. Remarquez que ce comportement singulier est important du point de vue physique puisqu'il induit le chauffage du plasma mentionné précédemment, voir aussi [27].

Le chapitre 4 introduit le problème dans un cadre simplifié et équivalent, et nous rappelons la formulation variationnelle utilisée dans [49] pour calculer les solutions singulières. Ensuite, le chapitre 5 étudie une sous-classe particulière de problèmes pour laquelle nous prouvons le principe d'absorption limite, et nous discutons de la régularité des solutions.

Le chapitre suivant 6 améliore le cadre fonctionnel de la formulation variationnelle proposée dans [49]. Nous prouvons la cohérence de la formulation variationnelle avec le principe d'absorption limite. Puis, nous établissons des résultats d'unicité et de stabilité de la solution de la version non régularisée du problème. Un des résultats clés de ce chapitre est la définition d'une notion faible de saut à travers l'interface à l'intérieur du domaine, qui permet de caractériser la décomposition de la solution d'absorption limite en une partie régulière et une partie singulière. Les résultats de ce chapitre peuvent être trouvés dans [23].

Enfin, le chapitre 7 propose deux formulations variationnelles alternatives. Nous comparons les performances numériques des différentes formulations variationnelles introduites dans cette deuxième partie de la thèse.

CHAPTER 2

Electromagnetic waves in plasma

Contents

2.1 Introduction on plasma

Plasma is the fourth state of the matter, and the more abundant form of matter in the universe. It is characterized by the presence of charged particles and ions, with proportions and densities which may vary in space and time. The temperature of a plasma is typically much higher than the ambient temperature, often exceeding several thousand Kelvin degrees. On Earth, plasmas are mainly used for industrial purpose. One of the most frequently cited example of industrial use is electric energy production via fusion nuclear reactors. Actually, this motivates large facets of academic and industrial research. Different kind of reactors are investigated nowadays, as Tokamaks [37], or Stellarators [39, Chapter 17].

There are several challenges to achieve a stable fusion reaction inside these reactors. We specify here three of them. The first one is the containment of the plasma inside the reactor. To do so, several devices are arranged such that poloidal and toroidal magnetic fields and a toroidal electric current are imposed [36], see Figure 2.1. It results in an imposed helical magnetic field. Nevertheless, this kind of installation is not sufficient to prevent instabilities of the plasma. In order to contain the plasma properly, measurements of the density of the plasma must be done. Because of the extreme temperature, intrusive measurement is impossible. Then, the second challenge consists in control the density of the plasma via reflectometry methods [43, 40, 30, 33]. It consists in sending electromagnetic waves with different frequencies and measuring the response. Finally, the last challenge is the plasma heating, and is done by sending electromagnetic waves at specific frequencies and directions depending on the characteristics of the plasma. Theoretically, there is currently three kinds of usable waves [39, Chapter 12]: ion cyclotron frequency waves, electron cyclotron frequency waves and lower-hybrid waves.

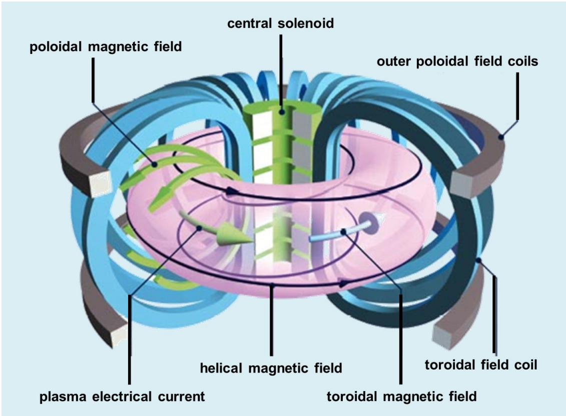


Figure 2.1: Representation of the magnetic fields and current inside a tokamak[1].

In order to deal with these challenges, a fine description of the electromagnetic fields inside the plasma are required. This report describes the work done on two very simplified configurations. Both configurations consider magnetized plasma with a constant imposed magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$. The first configuration results in a non-standard hyperbolic partial differential equation in space. The second configuration consists in the study of the lower-hybrid resonance, which leads to a sign-changing degenerate PDE.

2.2 Waves in plasma

The electromagnetic fields in a plasma are described by four vector-valued functions in space-time:

- the electric field $\boldsymbol{E}$,
- the electric displacement $\boldsymbol{D}$,
- the magnetic field $\boldsymbol{H}$,
- the magnetic induction or magnetizing field $\boldsymbol{B}$.

These four fields are linked together by the well-known Maxwell's equations:

$$\left|\begin{aligned} &\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}, \\ &\mathbf{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \\ &\operatorname{div} \boldsymbol{D} = \rho, \\ &\operatorname{div} \boldsymbol{B} = 0 \end{aligned}\right. \tag{2.1}$$

where $\boldsymbol{j}$ is the current density vector and ρ is the charge density. Additionally, it is completed by the two constitutive relations that holds at the microscopic scale:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}, \qquad \boldsymbol{B} = \mu_0 \boldsymbol{H},$$

[1]Source: https://www.iter.org/newsline/-/3037

where ε_0 is the vacuum permittivity and μ_0 is the vacuum permeability. On the other hand, the considered medium is a plasma, which is characterized by the presence of free electrically charged particles, such as electrons or ions. Consequently, a current density $\boldsymbol{j}$ is induced by the displacement of the charges inside the plasma. The presence of plasma current density makes a major difference with the classic medium because the plasma is made of free charges and these charges obviously move. More precisely, we consider a *cold magnetized collisionless* plasma.

The plasma can be described by two approach which are not equivalent: the fluid approach with the density of the particles and electrons and the kinetic approach with the distribution function on the particles. Because our plasma is considered *cold*, the fluid approach is relevant. The kinetic approach relies on the Boltzmann theory and will not be our subject of interest. The interested reader may refer to [55, Chapter 8, 56, Chapter 4].

Therefore, given the set of different ion species and electrons S, we can decompose the plasma current as

$$\boldsymbol{j} = \sum_{s \in S} \boldsymbol{j}_s = \sum_{s \in S} \mathcal{N}_s q_s \boldsymbol{v}_s$$

where, given an ion species $s \in S$, $\mathcal{N}_s$ is the ion density, i.e., the number of ions per volume unit, q_s is the ion charge and $\boldsymbol{v}_s$ is the velocity. The velocity and the electromagnetic fields are linked via the Navier-Stokes equation and the volume Lorentz force:

$$\mathcal{N}_s m_s \left(\frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s \cdot \nabla) \boldsymbol{v}_s \right) = \mathcal{N}_s q_s (\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}) - \operatorname{div} \mathbb{T},$$

where $\mathbb{T}$ is the fluid constraint tensor.

From this point, several simplification assumptions are made. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an orthonormal basis of $\mathbb{R}^3$, with (x_1, x_2, x_3) the normalized orthogonal coordinates. Since the plasma is *cold* and *collisionless*, we neglect the fluid constraint tensor $\mathbb{T}$. Next, we *linearize* the equation around the equilibrium $(\boldsymbol{v}_s, \boldsymbol{E}, \boldsymbol{B}) = (0, 0, \boldsymbol{B}_0)$ where $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_3$ is the *background magnetic field* imposed to the plasma. Therefore, expanding $\boldsymbol{v}_s$, $\boldsymbol{E}$, $\boldsymbol{B}$ at the first order and substituting these quantities into the Navier-Stokes equation yields

$$\frac{\partial \boldsymbol{v}_s}{\partial t} = \frac{q_s}{m_s} (\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}_0).$$

Finally, we assume that we are in *time-harmonic regime*, i.e., all the quantities $a(\boldsymbol{x}, t)$ which depend on time can be expanded as $a(\boldsymbol{x}, t) = \operatorname{Re}\left(\hat{a}(\boldsymbol{x}) e^{-i\omega t}\right)$. Then, the equations above becomes

$$-i\omega \hat{\boldsymbol{v}}_s = \frac{q_s}{m_s} \left(\hat{\boldsymbol{E}} + \hat{\boldsymbol{v}}_s \times (B_0 \boldsymbol{e}_3) \right).$$

The eigenvectors of the operator $\boldsymbol{e}_3 \times \cdot$ are $(\boldsymbol{e}_+, \boldsymbol{e}_-, \boldsymbol{e}_3)$ with $\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}} (\boldsymbol{e}_1 \mp i\boldsymbol{e}_2)$, and they constitute an orthonormal basis of the complex vector space $\mathbb{C}^3$. In this basis, we have

$$\hat{v}_{s,\pm} = \left(\frac{q_s}{m_s} \right) \frac{i}{\omega \mp \omega_c} \hat{E}_\pm, \qquad \hat{v}_{s,z} = \left(\frac{q_s}{m_s} \right) \frac{i}{\omega} \hat{E}_z,$$

with $\omega_{c,s} = \frac{q_s B_0}{m_s}$, the *cyclotron frequency* associated with the ion species s. Notice that $\hat{v}_{s,\pm} = \frac{1}{\sqrt{2}} \left(\hat{v}_{s,x} \pm i\hat{v}_{s,y} \right)$ and $\hat{E}_\pm = \frac{1}{\sqrt{2}} \left(\hat{E}_x \pm i\hat{E}_y \right)$. Then, the current density $\hat{\boldsymbol{j}}$ can be expressed in function of the electric field $\hat{\boldsymbol{E}}$ in the basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$:

$$\hat{\boldsymbol{j}}_s = \mathcal{N}_s q_s \hat{\boldsymbol{v}}_s = \sigma_s \hat{\boldsymbol{E}}, \quad \text{with} \quad \sigma_s = i\omega\varepsilon_0 \chi_s \quad \text{and} \quad \chi_s = \begin{pmatrix} -\frac{\omega_{p,s}^2}{\omega^2-\omega_{c,s}^2} & -i\frac{\omega_{c,s}\omega_{p,s}^2}{\omega(\omega^2-\omega_{c,s}^2)} & 0 \\ i\frac{\omega_{c,s}\omega_{p,s}^2}{\omega(\omega^2-\omega_{c,s}^2)} & -\frac{\omega_{p,s}^2}{\omega^2-\omega_{c,s}^2} & 0 \\ 0 & 0 & \frac{\omega_{p,s}^2}{\omega^2} \end{pmatrix},$$

where we assumed that the ion densities $\mathcal{N}_s$ do not vary in time. The matrices σ_s and χ_s are respectively called conductivity tensor and electric susceptibility tensor. Notice the introduction of the *plasma frequency* $\omega_{p,s} = \sqrt{\frac{\mathcal{N}_s q_s^2}{m_s \varepsilon_0}}$.

Remark 2.2.1. The approach can be generalized to any background magnetic field $\boldsymbol{B}_0(\boldsymbol{x})$. Actually, given a point $\boldsymbol{x} \in \mathbb{R}^3$, the electric susceptibility tensor $\chi_s(\boldsymbol{x})$ is always diagonal when it is expressed into the basis constituted by the eigenvectors of the operator $\boldsymbol{B}_0(\boldsymbol{x}) \times \cdot$. One can refer to [31, Chapter 2].

Finally, going back to the Maxwell's equations expressed in time-harmonic regime leads to

$$\left| \begin{aligned} &\textbf{curl}\, \hat{\boldsymbol{B}} = -\frac{i\omega}{c^2}\epsilon\hat{\boldsymbol{E}}, \\ &\textbf{curl}\, \hat{\boldsymbol{E}} = i\omega\hat{\boldsymbol{B}}, \end{aligned} \right. \tag{2.2}$$

where $c = (\mu_0\varepsilon_0)^{-1/2}$, and the well-known *cold plasma dielectric tensor* is given by

$$\begin{gathered} \epsilon = \mathbb{I}_3 + \sum_{s\in S} \chi_s = \begin{pmatrix} \alpha & -i\delta & 0 \\ i\delta & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \\ \alpha = 1 - \sum_{s\in S} \frac{\omega_{p,s}^2}{\omega^2 - \omega_{c,s}^2}, \quad \delta = \frac{1}{\omega}\sum_{s\in S} \frac{\omega_{c,s}\omega_{p,s}^2}{\omega^2 - \omega_{c,s}^2}, \quad \beta = 1 - \frac{\sum_{s\in S}\omega_{p,s}^2}{\omega^2}, \\ \omega_{c,s} = \frac{q_s B_0}{m_s}, \quad \omega_{p,s}^2 = \frac{\mathcal{N}_s q_s^2}{m_s\varepsilon_0}. \end{gathered} \tag{2.3}$$

Let us make few comments about the model above. Firstly, the cold plasma dielectric tensor varies within the space variable $\boldsymbol{x}$ and the frequency of the harmonic regime ω. Indeed, the plasma frequencies $\omega_{p,s}$ depend on the ion densities $\mathcal{N}_s$[2], and the dependence on the frequency ω indicates that the model is clearly dispersive. Secondly, ϵ is not necessarily positive for any frequency at any point of the space from the physics. This observation is the basis of this thesis and will be discussed in the two following paragraph. On the other hand, the case ϵ uniformly positive or uniformly negative corresponds to the classical Maxwell's equations which have already been studied from the mathematical point of view since the establishment of these equations.

The model above has been extensively studied in the Physics community, and we refer to the following monographs [55, 56, 39].

[2]Notice that $\omega_{p,s}$ does not depend on the frequency ω because $\mathcal{N}_s$ does not depend on time.

2.3 Plasma in a strong background magnetic field

As noticed the last paragraph, the dielectric tensor $\mathbb{\epsilon}$ is not necessarily positive. If we consider global plasma frequency $\omega_p^2 = \sum_{s\in S} \omega_{p,s}^2$, then we have $\beta = 1 - \frac{\omega_p^2}{\omega^2}$ negative whenever $0 < \omega < \omega_p$. On the other hand, we can clearly find frequencies $\omega_{p,s}$ and $\omega_{c,s}$ in such way that the 2×2 bloc

$$\begin{pmatrix} \alpha & -i\delta \\ i\delta & \alpha \end{pmatrix}$$

is positive definite, which is equivalent to $\alpha > |\delta|$. In order to simplify the analysis, we will assume that the background magnetic field $\boldsymbol{B}_0$ has a very large magnitude, in such way that the cyclotron frequencies $\omega_{c,s}$ are very large compared to the global plasma frequency ω_p. Then, in the view of the expressions (2.3) of α and δ, the dielectric tensor can be approximated for a given frequency $\omega > 0$ as

$$\mathbb{\epsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{pmatrix}. \tag{2.4}$$

We assume in the following that this approximation, which we denote *strong magnetic field limit*, is valid in the whole free space, and that ω_p does not vary in the whole space $\mathbb{R}^3$. Then, the objective of the first part of this thesis is to study the problem posed in *free space*:

$$\left| \begin{array}{l} \text{find } \hat{\boldsymbol{E}}, \widehat{\boldsymbol{B}} \text{ such that} \\ \boldsymbol{curl}\, \widehat{\boldsymbol{B}} + \frac{i\omega}{c^2} \mathbb{\epsilon} \hat{\boldsymbol{E}} = \hat{\boldsymbol{j}}, \\ \boldsymbol{curl}\, \hat{\boldsymbol{E}} - i\omega \widehat{\boldsymbol{B}} = \widehat{\boldsymbol{m}}, \end{array} \right. \tag{2.5}$$

where some source terms $\hat{\boldsymbol{j}}, \widehat{\boldsymbol{m}}$ have been added.

Whereas numerous works are devoted to isotropic models, i.e., models where both the dielectric permittivity and the magnetic permeability are scalars [10, 47, 11, 15, 16, 44, 8], few works are dedicated to anisotropic media, especially if the dielectric permittivity or magnetic permeability tensor is no longer sign-definite. Let us remark that there exist works dedicated to the study of Maxwell's equations with anisotropic elliptic dielectric permittivity and magnetic permeability tensors [19, 21, 20], which is unfortunately not our case. We study so-called hyperbolic metamaterials, see [53] and references therein. As far as we know, few mathematical analyses [12, 22] are devoted to the study of hyperbolic metamaterials. Notice that hyperbolic problems in the frequency domain may appear in fluid dynamics, see [28, 29]. Moreover, hyperbolic problems in bounded domains were first studied up to our knowledge in [38].

Our model has already been studied in the case of 2D-electromagnetic fields, i.e., the magnetic field and the electric fields does not depend on the y-variable in [22] in the harmonic regime, see [7, 6] for its time domain counterpart.

Remark 2.3.1. The system without source term can be rewritten in the time domain as:

$$\boldsymbol{curl}\, \boldsymbol{B} - \frac{1}{c^2} \partial_t \boldsymbol{E} = \mu_0 \boldsymbol{J}_p,$$
$$\boldsymbol{curl}\, \boldsymbol{E} + \partial_t \boldsymbol{B} = 0.$$

where $\partial_t \boldsymbol{J}_p = \omega_p^2 \varepsilon_0 (\boldsymbol{E} \cdot \boldsymbol{e}_3) \boldsymbol{e}_3$.

2.4 Plasma with a varying density

2.4.1 Single species plasma

When an electromagnetic wave is sent inside a plasma, it can transfer energy to the particles to produce plasma heating in a localized region. This phenomenon is related to the so-called resonant waves, and appears with the variation in space of the ion densities $\mathcal{N}_s(\boldsymbol{x})$. For simplicity, we assume that the plasma is constituted by only one ion species of density $\mathcal{N}_e(\boldsymbol{x})$. Therefore, the dielectric tensor

$$\varepsilon(\boldsymbol{x}) = \begin{pmatrix} \alpha(\boldsymbol{x}) & -i\delta(\boldsymbol{x}) & 0 \\ i\delta(\boldsymbol{x}) & \alpha(\boldsymbol{x}) & 0 \\ 0 & 0 & \beta(\boldsymbol{x}) \end{pmatrix},$$

also varies in space, where α, δ, β are given by (2.3), which can be written in our case

$$\alpha = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \delta = \frac{1}{\omega}\frac{\omega_c \omega_p^2}{\omega^2 - \omega_c^2}, \quad \beta = 1 - \frac{\omega_p^2}{\omega^2},$$
$$\omega_c = \frac{q_e B_0}{m_c}, \qquad \omega_p^2 = \frac{q_e^2}{m_e \varepsilon_0}\mathcal{N}_e.$$

We assume that the plasma density $\mathcal{N}_e$ varies in space so that we have

$$\alpha(\boldsymbol{x}) = 1 - C_{\alpha,\omega}\mathcal{N}_e(\boldsymbol{x}), \qquad \delta(\boldsymbol{x}) = C_{\delta,\omega}\mathcal{N}_e(\boldsymbol{x}), \qquad \beta(\boldsymbol{x}) = 1 - C_{\beta,\omega}\mathcal{N}_e(\boldsymbol{x}). \tag{2.6}$$

We consider a lower hybrid resonance in the plasma, see [55, Chapter 2-6] and recent works [35, 26, 13, 14, 27, 25, 48, 50, 49], which is characterized by the fact that $\alpha = 0$ on some curve inside the region. Like in the above cited works, we will be particularly interested in the cases when the density $\mathcal{N}_e(\boldsymbol{x})$ is s.t. the sign of α changes continuously between subregions separated by an interface.

With obvious notations, $\boldsymbol{E} = E_1\boldsymbol{e}_1 + E_2\boldsymbol{e}_2 + E_3\boldsymbol{e}_3$, etc. We will assume in this part that all quantities are independent of x_3, the variable corresponding to the direction of the exterior magnetic field, so that $E_1 = E_1(x_1, x_2)$, etc. Then, expanding the first equation of (2.2), one have

$$\begin{pmatrix} \partial_2 B_3 \\ -\partial_1 B_3 \\ \partial_1 B_2 - \partial_2 B_1 \end{pmatrix} = -\frac{i\omega}{c^2}\begin{pmatrix} \alpha E_1 - i\delta E_2 \\ i\delta E_1 + \alpha E_2 \\ \beta E_3 \end{pmatrix}.$$

Observe that E_1, E_2, B_3 and E_3, B_1, B_2 are independent in the equation above. This observation is also valid for the second equation of (2.2). Hence, thanks to the block diagonal structure of ε, the Maxwell system (2.2) can be split into two independent systems that dissociate E_3, $\mathbf{B}_\perp = B_1\mathbf{e}_1 + B_2\mathbf{e}_2$ on one hand, and $\mathbf{E}_\perp = E_1\mathbf{e}_1 + E_2\mathbf{e}_2$, B_3 on the other. In the view of the previous equation we define the two following differential operator

$$\mathbf{curl}_\perp f = \begin{pmatrix} \partial_2 f \\ -\partial_1 f \end{pmatrix}, \qquad \mathrm{curl}_\perp (f_1\mathbf{e}_1 + f_2\mathbf{e}_2) = \partial_1 f_2 - \partial_2 f_1.$$

Then, the system for the Ordinary mode is:

$$\begin{cases} \mathbf{curl}_\perp E_3 = i\omega \mathbf{B}_\perp, \\ \operatorname{curl}_\perp \mathbf{B}_\perp = -\dfrac{i\omega\beta}{c^2} E_3, \end{cases} \tag{O-mode}$$

and the eXtraodinary mode:

$$\begin{cases} \operatorname{curl}_\perp \mathbf{E}_\perp = i\omega B_3, \\ \mathbf{curl}_\perp B_3 = -\dfrac{i\omega}{c^2}\mathbb{\epsilon}_\perp \mathbf{E}_\perp, \end{cases} \quad \text{where} \quad \mathbb{\epsilon}_\perp = \begin{pmatrix} \alpha & -i\delta \\ i\delta & \alpha \end{pmatrix}. \tag{X-mode}$$

For the discussion that follows, we will need to introduce auxiliary notation. Let $\mathbf{x}_\perp = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\Delta_\perp v = \partial_{11}v + \partial_{22}v$, $\operatorname{div}_\perp \mathbf{v} = \partial_1 v_1 + \partial_2 v_2$ and $\nabla_\perp v = \partial_1 v\mathbf{e}_1 + \partial_2 v\mathbf{e}_2$. In this case we evidently have that $\mathbf{curl}_\perp v = -R_{\pi/2}\nabla_\perp v$ and $\operatorname{curl}_\perp \mathbf{v} = -\operatorname{div}_\perp R_{\pi/2}\mathbf{v}$, where $R_{\pi/2} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the $\pi/2$ rotation matrix in the oriented plane $(\mathbf{e}_1, \mathbf{e}_2)$. In particular, $\operatorname{curl}_\perp \mathbf{curl}_\perp v = -\Delta_\perp v$.

Let us now focus on the equations governing the scalar unknowns E_3 and B_3. The second-order PDE derived from the system for the Ordinary mode is $-\Delta_\perp E_3 = \frac{\omega^2\beta}{c^2}E_3$. In the case when the sign of β changes continuously, this equation is reminiscent of an Airy equation, cf. [35].

On the other hand, the second-order PDE derived from the (X-mode) is

$$\operatorname{div}_\perp \left(R_{\pi/2}\mathbb{\epsilon}_\perp^{-1}R_{\pi/2}\nabla_\perp B_3\right) = \frac{\omega^2}{c^2}B_3.$$

We will assume that the tensor $\mathbb{\epsilon}_\perp$ is invertible everywhere in the region, more precisely, that $\alpha^2(\mathbf{x}_\perp) - \delta^2(\mathbf{x}_\perp) \neq 0$ for all $\mathbf{x}_\perp$, and thus the above expression is well-defined. Let us define the two-by-two tensor

$$\mathbb{\alpha} := c^2 R_{\pi/2}\mathbb{\epsilon}_\perp^{-1}R_{\pi/2} = \frac{c^2}{\delta^2 - \alpha^2}\begin{pmatrix} \alpha & i\delta \\ -i\delta & \alpha \end{pmatrix}.$$

Then, since it holds that α and δ depend on the space variable $\mathbf{x}_\perp$ only via the density of the plasma $\mathcal{N}_e(\mathbf{x}_\perp)$, α and δ have the same level curves. Moreover, with (2.6) in mind, $\alpha = 1 - \delta/\delta^+$ where $\delta^+ = C_{\delta,\omega}/C_{\alpha,\omega}$.

As discussed before, we assume that the coefficient $\alpha(\mathbf{x}_\perp)$ vanishes on some interface I. In view of the last remark, the tensor $\mathbb{\alpha}(\mathbf{x}_\perp)$ is constant on I, and is equal to $\mathbb{\alpha} = i\mathbb{A}$, with

$$\mathbb{A} = \frac{c^2}{{\delta^+}^2}\begin{pmatrix} 0 & -\delta^+ \\ \delta^+ & 0 \end{pmatrix},$$

which is a real-valued skew-symmetric matrix. From this it follows that we can decompose $\mathbb{\alpha}(\mathbf{x}_\perp)$ as

$$\mathbb{\alpha}(\mathbf{x}_\perp) = \underbrace{-\alpha_0(\mathbf{x}_\perp)\mathbb{H}(\mathbf{x}_\perp)}_{\mathbb{\alpha}_0(\mathbf{x}_\perp)} + i\mathbb{A}, \tag{2.7}$$

where $\alpha_0 = \frac{c^2\alpha}{\alpha^2 - \delta^2}$, and $\mathbb{H}(\mathbf{x}_\perp)$ is a Hermitian matrix given by

$$\mathbb{H}(\mathbf{x}_\perp) = \begin{pmatrix} 1 & -i\left(\delta(\mathbf{x}_\perp) + \alpha(\mathbf{x}_\perp)/\delta^+\right) \\ i\left(\delta(\mathbf{x}_\perp) + \alpha(\mathbf{x}_\perp)/\delta^+\right) & 1 \end{pmatrix}.$$

In what follows, we will assume that $\mathbb{H}(\mathbf{x}_\perp)$ is positive definite in the whole computational region. This requires in particular that the determinant of $\mathbb{H}(\mathbf{x}_\perp)$ is positive for all $\mathbf{x}_\perp$, which leads to

$$\left|\delta(\mathbf{x}_\perp) + \alpha(\mathbf{x}_\perp)/\delta^+\right| < 1.$$

Since $\mathbb{A}$ is skew-symmetric, $\mathrm{div}_\perp(\mathbb{A}\,\nabla_\perp B_3) = 0$, so the second-order PDE governing B_3 becomes

$$-\,\mathrm{div}_\perp\left(\alpha_0\,\nabla_\perp B_3\right) - \omega^2 B_3 = 0. \tag{2.8}$$

We suppose that the electron density $\mathcal{N}_e$ is $\mathscr{C}^2$-regular, so that α_0 and $\mathbb{H}$ are also $\mathscr{C}^2$-regular. Given $\mathbf{x}_\perp$ on the interface I, let $\mathbf{n}(\mathbf{x}_\perp)$ be the normal to the interface at the point $\mathbf{x}_\perp$. Then, for h a real small enough, we can write the series expansion of α_0:

$$\alpha_0(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) = \frac{\partial\alpha_0}{\partial\mathbf{n}}(\mathbf{x}_\perp)h + \frac{\partial^2\alpha_0}{\partial\mathbf{n}^2}(\mathbf{x}_\perp)h^2 + \mathcal{O}(h^3).$$

Therefore, we assume from now on that $|\alpha_0(\mathbf{x}_\perp)|$ behaves in a neighborhood of the interface $I = \{\alpha_0(\mathbf{x}_\perp) = 0\}$ like $\mathrm{dist}(\mathbf{x}_\perp, I)$ and does not degenerate in the sense that there is a constant $c > 0$ such that $\left|\frac{\partial\alpha_0}{\partial\mathbf{n}}(\mathbf{x}_\perp)\right| > c$ for all $\mathbf{x}_\perp \in I$. We also assume that the interface I is a C^1-loop (without self-intersections).

Since we are interested in the behavior of the physical solution in a neighborhood of the interface, we do not consider the problem in the whole space $\mathbb{R}^3$, but rather inside a bounded domain D which contains the interface. Then, several boundary conditions exist on ∂D: Dirichlet conditions, Neumann conditions, Robin conditions, and absorbing conditions. By simplicity, we will mostly use absorbing conditions

$$\alpha_0\nabla_\perp B_3\cdot\mathbf{n} + i\lambda B_3 = f, \qquad \text{on } \partial D,$$

where f represents some source term. Notice that absorbing conditions may be used to model emitting or receiving antennas.

Considering the model derived from the eXtraodinary mode with unknown B_3 in the neighborhood of the interface leads to a degenerate elliptic PDE. The Mathematics community studies the models derived from the framework of the waves in the cold plasma from a decade [25, 2, 26, 7, 42]. How to solve this equation will be the goal of the second part of this thesis. This problem has already been investigated in [49], where a numerical method based on a mixed variational formulation was proposed.

Remark 2.4.1. Notice that other assumptions could be made on the behavior of α_0: $\mathrm{dist}(\mathbf{x}_\perp, I)^2$, $\mathrm{dist}(\mathbf{x}_\perp, I)^3$, or even a fractional power. It would mean that $\frac{\partial\alpha_0}{\partial\mathbf{n}}(\mathbf{x}_\perp)$ vanishes for all $\mathbf{x}_\perp$ on the interface In particular, this would lead to a different type of singularity than the one studied in this thesis. Moreover, it does not seem relevant from the physical point of view since the density of the plasma is smooth in practice, and we clearly exclude the presence of shock.

2.4.2 General plasma

The problem described below has not been studied in this thesis. Nevertheless, we think that it is interesting to formalize it because it appears that such problems may be connected to some existing literature, opening new perspectives.

The argument developed for single species plasma cannot be applied without adaptation for a real plasma, which as in general 2 species at least (the electrons and the ions). In that case, using (2.3), we have

$$\alpha(\boldsymbol{x}) = 1 - \sum_{s\in S} C_{s,\alpha,\omega}\mathcal{N}_s(\boldsymbol{x}), \qquad \delta(\boldsymbol{x}) = \sum_{s\in S} C_{s,\delta,\omega}\mathcal{N}_s(\boldsymbol{x}),$$

with $C_{s,\alpha,\omega} > 0$ and $C_{s,\delta,\omega} \in \mathbb{R}$ for all species $s \in S$. Notice that $C_{s,\delta,\omega} > 0$ (respectively $C_{s,\delta,\omega} < 0$) for positively (resp. negatively) charged species, as indicated by equations (2.3).

As before, we assume that the problem is independent of x_3, so that the problem can be separated into ordinary and extraordinary modes, and the sign of α changes through an interface I. We assume that the densities of the ion species are at least $\mathscr{C}^2$-regular, and the same assumption applies to δ and α. Moreover, $|\alpha(\mathbf{x}_\perp)|$ behaves in a neighborhood of the interface $I = \{\alpha(\mathbf{x}_\perp) = 0\}$ like $\operatorname{dist}(\mathbf{x}_\perp, I)$ and does not degenerate in the sense that there is a constant $c > 0$ such that $\left|\frac{\partial\alpha}{\partial\mathbf{n}}(\mathbf{x}_\perp)\right| > c$ for all $\mathbf{x}_\perp \in I$.

If we replicate the previous analysis, the bottleneck arises from the fact that $\alpha(\mathbf{x}_\perp)$ and $\delta(\mathbf{x}_\perp)$ do not share the same level curves. Therefore, the matrix $\mathbb{a}(\mathbf{x}_\perp)$ is no longer constant on the interface I, and we cannot define a constant matrix $\mathbb{A}$ such that the decomposition (2.7) holds.

Nonetheless, we can still formalize the problem in the following way. The value of $\mathbb{a}$ on the interface is $\mathbb{a}(\mathbf{x}_\perp) = i\mathbb{A}(\mathbf{x}_\perp)$ with

$$\mathbb{A}(\mathbf{x}_\perp) = \frac{c^2}{\delta(\mathbf{x}_\perp)^2}\begin{pmatrix} 0 & -\delta(\mathbf{x}_\perp) \\ \delta(\mathbf{x}_\perp) & 0 \end{pmatrix}, \text{ with } \mathbf{x}_\perp \in I.$$

It is possible to extend the definition of this matrix to the whole domain. For all $\mathbf{x}_\perp$ in the domain, there is $\mathscr{P}(\mathbf{x}_\perp) \in I$, the projection of $\mathbf{x}_\perp$ on the interface I, and $s(\mathbf{x}_\perp) \in \mathbb{R}$ such that $\mathbf{x}_\perp = \mathscr{P}(\mathbf{x}_\perp) + s(\mathbf{x}_\perp)\mathbf{n}(\mathscr{P}(\mathbf{x}_\perp))$, where $\mathbf{n}(\mathbf{y}_\perp)$ is the unit normal vector to the interface at the point $\mathbf{y}_\perp \in I$. We can reasonably assume that the last decomposition is unique in the domain of interest. Therefore, we easily extend $\mathbb{A}$ in the whole domain as

$$\mathbb{A}(\mathbf{x}_\perp) = \frac{c^2}{\delta(\mathscr{P}(\mathbf{x}_\perp))^2}\begin{pmatrix} 0 & -\delta(\mathscr{P}(\mathbf{x}_\perp)) \\ \delta(\mathscr{P}(\mathbf{x}_\perp)) & 0 \end{pmatrix}.$$

Then, we factorize the matrix $\mathbb{a} = -\alpha_0\mathbb{H} + i\mathbb{A}$ as before, with $\alpha_0 = \frac{c^2\alpha}{\alpha^2-\delta^2}$ and

$$\mathbb{H}(\mathbf{x}_\perp) = \begin{pmatrix} 1 & i\tilde{\delta}(\mathbf{x}_\perp) \\ -i\tilde{\delta}(\mathbf{x}_\perp) & 1 \end{pmatrix}, \qquad \tilde{\delta}(\mathbf{x}_\perp) = \frac{\delta(\mathbf{x}_\perp)\left(\delta(\mathscr{P}(\mathbf{x}_\perp)) - \delta(\mathbf{x}_\perp)\right) + \alpha(\mathbf{x}_\perp)^2}{\delta(\mathscr{P}(\mathbf{x}_\perp))\alpha(\mathbf{x}_\perp)}.$$

Due to the regularity assumption, $\tilde{\delta}$ does not degenerate in the neighborhood the interface. Indeed, expanding α and δ for small $h \in \mathbb{R}$ and $\mathbf{x}_\perp \in I$, that is,

$$\begin{aligned}
\delta(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) &= \delta(\mathbf{x}_\perp) + \frac{\partial\delta}{\partial\mathbf{n}}(\mathbf{x}_\perp)h + \mathscr{O}(h^2), \\
\alpha(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) &= \frac{\partial\alpha}{\partial\mathbf{n}}(\mathbf{x}_\perp)h + \mathscr{O}(h^2),
\end{aligned}$$

yields

$$\tilde{\delta}(\mathbf{x}_\perp + h\mathbf{n}(\mathbf{x}_\perp)) = \frac{\frac{\partial\delta}{\partial\mathbf{n}}(\mathbf{x}_\perp)}{\frac{\partial\alpha}{\partial\mathbf{n}}(\mathbf{x}_\perp)} + \mathscr{O}(h).$$

Therefore, the matrix $\mathbb{H}$ is Hermitian, continuous, and we also assume that it is positive definite in the computational region, which requires that $|\tilde{\delta}| < 1$. Next, observe that we can also compute the normal derivative of the matrix $\mathbb{A}$, which is $\frac{\partial}{\partial \mathbf{n}}\mathbb{A} = 0_{\mathbb{R}^{2\times 2}}$ by construction. Therefore, we will say that $\mathbb{A}$ is *transverse*. Finally, the equation of the problem can be summarized as

$$- \operatorname{div}_\perp (\mathbb{\alpha} \nabla_\perp B_3) - \omega^2 B_3 = 0, \tag{2.9}$$

with $\mathbb{\alpha} = \alpha_0 \mathbb{H} + i\mathbb{A}$, where

- α_0 is a continuous sign-changing function, whose normal derivative does not vanish on the locus of the sign-change,
- $\mathbb{H}$ is a hermitian matrix uniformly elliptic on the domain,
- $\mathbb{A}$ is a skew-symmetric matrix, transverse to the interface, in the sense that $\left.\frac{\partial \mathbb{A}}{\partial \mathbf{n}}\right|_I = 0_{\mathbb{R}^{2\times 2}}$.

Actually, this type of operator have already been partially studied, without the sign change but only with the degeneracy at the boundary in the thesis of Baouendi [5]. One may refer also to [4, 3].

2.5 Outline

This work is divided in two parts.

The first part consists in the study of the model of plasma in a strong background magnetic field, which corresponds to a hyperbolic metamaterial. The objective is to extend the results of [22] to the 3D-case and to derive a radiation condition. The corresponding chapter introduces a splitting of the electric and magnetic fields resembling the usual TE and TM decomposition, then, it gives some results on the two resulting problems. The results are in a very partial state, and constitute a rough draft on the subject.

The second part consists in the study of the degenerate PDE associated to the equation (2.8) augmented by absorbing boundary conditions. The associated boundary-value problem is well-posed within a "natural" variational framework. However, this framework does not include the singular behavior presented by the physical solutions obtained via the limiting absorption principle, cf. [35, 14]. Notice that this singular behavior is important from the physical point of view since it induces the plasma heating mentioned before, see also [27].

Chapter 4 introduces the problem in a simplified and equivalent setting, and we recall the variational formulation used in [49] to compute the singular solutions. Then, Chapter 5 studies a particular subclass of problem for which we prove the limiting absorption principle, and we discuss the regularity of the solutions.

Next, Chapter 6 improves the functional framework of the variational formulation proposed in [49]. We prove the consistency of the variational formulation with the limiting absorption principle. Then, we establish uniqueness, and stability results of the solution of the non-regularized version of the problem. One of the key results of this chapter is the definition of a notion of weak jump through the interface inside the domain, which allows to characterize the decomposition of the limiting absorption solution into a regular and a singular parts. The results of this Chapter can be found in [23].

Finally, Chapter 7 proposes two alternative variational formulations. We compare numerical performance of the different variational formulations introduced in this second part of the thesis.

PART I

Hyperbolic problem in free space

CHAPTER 3

Hyperbolic Maxwell problem in free space

Contents

Given an exterior magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$ with $B_0 \to +\infty$, the harmonic Maxwell's equations in a cold plasma are

$$\left| \begin{array}{l} \boldsymbol{curl}\, \boldsymbol{B} + \dfrac{i\omega}{c^2}\varepsilon \boldsymbol{E} = 0, \\ \boldsymbol{curl}\, \boldsymbol{E} - i\omega \boldsymbol{B} = 0, \end{array} \right.$$

where $\boldsymbol{E}$, $\boldsymbol{B}$ are perturbations of the electromagnetic fields at equilibrium, and the dielectric tensor ε becomes a symmetric dielectric tensor given by

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta(\omega) \end{pmatrix}, \quad \text{with } \beta(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \tag{3.1}$$

We immediately notice that $\mathbb{\varepsilon}$ is not necessarily a sign-definite matrix (i.e., a matrix with non-vanishing eigenvalues of the same sign) because $1 - \frac{\omega_p^2}{\omega^2} < 0$ for $0 < \omega < \omega_p$. Moreover, $\mathbb{\varepsilon}$ is not elliptic, even for non-standard definition of elliptic tensor [19].

Finally, up to a renormalization[1] of ω, ω_p, $\boldsymbol{E}$ and $\boldsymbol{B}$, we assume that the speed of light is $c = 1$. Introducing source terms $\boldsymbol{j}, \boldsymbol{m}$, we rewrite the problem to solve as

$$\left|\begin{array}{l} \text{find } \boldsymbol{E}, \boldsymbol{B} \text{ such that} \\ \boldsymbol{curl}\, \boldsymbol{B} + i\omega\mathbb{\varepsilon}\boldsymbol{E} = \boldsymbol{j}, \\ \boldsymbol{curl}\, \boldsymbol{E} - i\omega\boldsymbol{B} = \boldsymbol{m}. \end{array}\right. \tag{3.2}$$

This problem has already been investigated in [22]. With the assumption of 2D-fields, the transverse electric and the transverse magnetic are decoupled. Then, it has been showed in [22] that the transverse magnetic problem solves a hyperbolic equation in free space, whereas the transverse electric problem solves an elliptic equation problem. The goal of this chapter is to extend the results to the case of 3D-fields and to provides insights into a Silver-Müller condition, which is the equivalent of the radiation condition for Maxwell's equation.

The first difficulty is the splitting of the problem into two sub-problems, each of which captures either the elliptic or hyperbolic behavior of the problem. This is the subject of the first section, which will supply sub-problems : the transverse electric problem and the transverse magnetic problem. These problems share analogous properties with the classic ones. Then, the second and third sections are devoted to the transverse electric and magnetic problems. The transverse magnetic problem will have an important step in the resolution of a 3D hyperbolic scalar problem in free space.

3.1 Problem splitting

3.1.1 Problem with absorption

We begin the study of the system with a simple case to solve: this is the case where $\omega \in \mathbb{C} \setminus \mathbb{R}$, so that $\omega^2 \notin \mathbb{R}^+$.

We suppose that $\boldsymbol{j}, \boldsymbol{m}, \boldsymbol{E}, \boldsymbol{B}$ are in $\boldsymbol{L}^2(\mathbb{R}^3) := \left(L^2(\mathbb{R}^3)\right)^3$. Subsequently, this requires that $\boldsymbol{curl}\, \boldsymbol{E}$ and $\boldsymbol{curl}\, \boldsymbol{B}$ belong also to $\boldsymbol{L}^2(\mathbb{R}^3)$. Thus, it is natural to first look for solutions in

$$\boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) := \left\{\boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3) ; \boldsymbol{curl}\, \boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3)\right\}.$$

Thus, the inhomogeneous problem is written as:

$$\left|\begin{array}{l} \text{find } (\boldsymbol{E}, \boldsymbol{B}) \in \left(\boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3)\right)^2 \text{ such that} \\ \boldsymbol{curl}\, \boldsymbol{B} + i\omega\mathbb{\varepsilon}\boldsymbol{E} = \boldsymbol{j}, \\ \boldsymbol{curl}\, \boldsymbol{E} - i\omega\boldsymbol{B} = \boldsymbol{m}, \end{array}\right. \tag{3.3}$$

with $\mathbb{\varepsilon}$ given by (3.1) and $\boldsymbol{j}, \boldsymbol{m} \in \boldsymbol{L}^2(\mathbb{R}^3)$. We can eliminate $\boldsymbol{B}$ from the above and rewrite the problem as a second order system:

$$\left|\begin{array}{l} \text{find } \boldsymbol{E} \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \text{ such that} \\ \boldsymbol{curl}\,\boldsymbol{curl}\, \boldsymbol{E} - \omega^2\mathbb{\varepsilon}\boldsymbol{E} = \boldsymbol{F}, \end{array}\right. \tag{3.4}$$

[1] The renormalization is $\frac{\omega}{c} = \tilde{\omega}$, $\frac{\omega_p}{c} = \tilde{\omega}_p$ and $c\boldsymbol{B} = \tilde{\boldsymbol{B}}$.

where $\boldsymbol{F} = \boldsymbol{curl}\,\boldsymbol{m} + i\omega\boldsymbol{j} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)'$.

Remark 3.1.1. Given $\boldsymbol{m} \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\boldsymbol{curl}\,\boldsymbol{m}$ is well-defined in the sense of the distributions $\mathscr{D}'(\mathbb{R}^3)$ and can easily be extended by continuity in $\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)'$:

$$\langle \boldsymbol{curl}\,\boldsymbol{m}, \boldsymbol{g}\rangle_{\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)',\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)} = \int_{\mathbb{R}^3} \boldsymbol{m} \cdot \overline{\boldsymbol{curl}\,\boldsymbol{g}}\, \mathrm{d}\boldsymbol{x}.$$

Notice that the duality bracket is linear with respect to the first variable and antilinear with respect to the second one.

Consider the following sesquilinear form, associated with the equation (3.4), defined on $\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3) \times \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$ by:

$$a_{\boldsymbol{E}}(\boldsymbol{f}, \boldsymbol{g}) := \int_{\mathbb{R}^3} \left(\boldsymbol{curl}\,\boldsymbol{f} \cdot \overline{\boldsymbol{curl}\,\boldsymbol{g}} - \omega^2 \boldsymbol{f} \cdot \overline{\boldsymbol{g}} + \omega_p^2 f_z \overline{g_z}\right) \mathrm{d}\boldsymbol{x}.$$

The index $\boldsymbol{E}$ stands for electric. Then, the variational form of the (3.4) problem is:

$$\left| \begin{array}{l} \text{find } \boldsymbol{E} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3) \text{ such that} \\ a_{\boldsymbol{E}}(\boldsymbol{E}, \boldsymbol{g}) = \langle \boldsymbol{F}, \boldsymbol{g}\rangle_{\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)}, \qquad \forall \boldsymbol{g} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3). \end{array} \right. \tag{3.5}$$

with $\boldsymbol{F} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)'$. The following lemma shows the equivalence between the problems (3.3) and (3.5).

Lemma 3.1.2. *$\boldsymbol{E}$ is a solution of (3.5) with $\boldsymbol{F} = \boldsymbol{curl}\,\boldsymbol{m} + i\omega\boldsymbol{j}$, if and only if $(\boldsymbol{E}, \boldsymbol{B}) \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)^2$ is solution of (3.3).*

Proof. Let $(\boldsymbol{E}, \boldsymbol{B})$ be solutions of (3.3). Then, $\boldsymbol{E}$ solves (3.4) in $\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)'$. Testing the equation against $\boldsymbol{g} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$ and integrating by parts gives the result.

On the other hand, given $\boldsymbol{E} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$ be a solution of (3.5), $\boldsymbol{B} = \frac{1}{i\omega}(\boldsymbol{curl}\,\boldsymbol{E} - \boldsymbol{m}) \in \boldsymbol{L}^2(\mathbb{R}^3)$. Then, $(\boldsymbol{E}, \boldsymbol{B})$ solves (3.3) in $(\mathscr{D}'(\mathbb{R}^3))^3$. Finally, because $\boldsymbol{L}^2(\mathbb{R}^3) \hookrightarrow (\mathscr{D}'(\mathbb{R}^3))^3$, the equalities hold in $\boldsymbol{L}^2(\mathbb{R}^3)$ and $\boldsymbol{B} \in \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$. ☐

Lemma 3.1.3. *If $\omega \in \mathbb{C} \setminus \mathbb{R}$, then the problem (3.5) admits a unique solution and there exists $C > 0$ depending on ω such that*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)} \leq C_\omega \|\boldsymbol{F}\|_{\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)'}.$$

Proof. The proof is based on the Lax-Milgram theorem. In fact, $\boldsymbol{F}$ is by definition a continuous anti-linear form on $\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$ and $a_{\boldsymbol{E}}$ is a continuous sesquilinear form on $\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3) \times \boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$. Then, it remains to check. The following identity allows us to check the coercivity of $a_{\boldsymbol{E}}$ on $\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)$, which concludes the proof:

$$\mathrm{Im}\left(a_{\boldsymbol{E}}(\boldsymbol{f}, \omega\boldsymbol{f})\right) = -\mathrm{Im}(\omega)\left(\|\boldsymbol{curl}\,\boldsymbol{f}\|^2_{L^2(\mathbb{R}^3)} + |\omega|^2 \|\boldsymbol{f}\|^2_{L^2(\mathbb{R}^3)} + \omega_p^2 \|f_z\|^2_{L^2(\mathbb{R}^3)}\right).$$

☐

The previous lemmas finally allow us to conclude in the case where $\omega \in \mathbb{C} \setminus \mathbb{R}$:

Theorem 3.1.4. *If $\omega \in \mathbb{C} \setminus \mathbb{R}$, the problem (3.3) is well-posed, i.e., there is a unique solution $(\boldsymbol{E}, \boldsymbol{B}) \in (\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3))^2$ and there exists $C_\omega > 0$, which depends on ω, such that*

$$\|\boldsymbol{E}\|^2_{\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)} + \|\boldsymbol{B}\|^2_{\boldsymbol{H}(\boldsymbol{curl};\mathbb{R}^3)} \leq C_\omega \left(\|\boldsymbol{j}\|^2_{L^2(\mathbb{R}^3)} + \|\boldsymbol{m}\|^2_{L^2(\mathbb{R}^3)}\right).$$

3.1.2 Plane wave analysis

We have seen in the previous section that the system (3.3) is well-defined in the case where $\omega \in \mathbb{C} \setminus \mathbb{R}$. However, similarly to the Helmholtz equation, this approach is not valid for $\omega \in \mathbb{R}$. Then, seeking for plane waves solutions shows that two kind of plane waves can appear. This leads to split the problem (3.3) into two sub-problems, one carrying the *hyperbolic* behavior of the system and the other carrying the *elliptic* behavior of the system.

This idea comes from [6], where the time dependence, however, was considered. The study starts with the analysis of the plane waves of the problem. We seek solutions of (3.2) in the form $\boldsymbol{E}(\boldsymbol{x}) = \hat{\boldsymbol{E}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ with $\boldsymbol{k} \in \mathbb{R}^3$. Then, given a plane wave $\hat{\boldsymbol{E}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, it solves

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = 0, \tag{3.6}$$

if and only if

$$-\boldsymbol{k} \times \left(\boldsymbol{k} \times \hat{\boldsymbol{E}}\right) - \omega^2 \varepsilon \hat{\boldsymbol{E}} = -\boldsymbol{k} \times \left(\boldsymbol{k} \times \hat{\boldsymbol{E}}\right) + \omega_p^2 \left(\hat{\boldsymbol{E}} \cdot \boldsymbol{e}_z\right) \boldsymbol{e}_z - \omega^2 \hat{\boldsymbol{E}} = 0.$$

Let $\mathbb{A}(\boldsymbol{k})$ be the matrix such that $\mathbb{A}(\boldsymbol{k})\hat{\boldsymbol{E}} = -\boldsymbol{k} \times \left(\boldsymbol{k} \times \hat{\boldsymbol{E}}\right) + \omega_p^2 \left(\hat{\boldsymbol{E}} \cdot \boldsymbol{e}_z\right) \boldsymbol{e}_z$. Then, the previous equality can be rewritten as the eigenvalue problem

$$\left[\mathbb{A}(\boldsymbol{k}) - \omega^2 \mathbb{I}_3\right] \hat{\boldsymbol{E}} = 0.$$

More precisely, we want $\boldsymbol{k} \in \mathbb{R}^3$, $\hat{\boldsymbol{E}} \in \mathbb{R}^3$ such that ω^2 is an eigenvalue of $\mathbb{A}(\boldsymbol{k})$ associated to the eigenvector $\hat{\boldsymbol{E}}$. This allows us to define the dispersion relation of the system:

$$F_\omega(\boldsymbol{k}) := \det\left(\mathbb{A}(\boldsymbol{k}) - \omega^2 \mathbb{I}_3\right),$$

Then, we seek $\boldsymbol{k} \in \mathbb{R}^3$ such that $F_\omega(\boldsymbol{k}) = 0$, and, for such $\boldsymbol{k}$, we determine the eigenspaces of $\mathbb{A}(\boldsymbol{k})$ associated to ω^2. Notice that if $F_\omega(\boldsymbol{k}) \neq 0$, then $\hat{\boldsymbol{E}} = 0$. By definition, $F_\omega(\boldsymbol{k})$ is a third order polynomial in ω^2. Let us define the following vectors

$$\boldsymbol{k}_\parallel := \begin{pmatrix} k_x \\ k_y \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{k}_\perp := \begin{pmatrix} -k_y \\ k_x \\ 0 \end{pmatrix}.$$

Lemma 3.1.5 ([6]). *The dispersion function is written:*

$$\begin{aligned} F_\omega(\boldsymbol{k}) &= \left(\omega^2 - \omega_\perp(\boldsymbol{k})^2\right)\left(\omega^2 - \omega_\parallel^+(\boldsymbol{k})^2\right)\left(\omega^2 - \omega_\parallel^-(\boldsymbol{k})^2\right) \\ &= \left(\omega^2 - \omega_p^2\right)\left(\omega^2 - |\boldsymbol{k}|^2\right)\left(\omega^2 - \beta(\omega)^{-1}|\boldsymbol{k}_\parallel|^2 - k_z^2\right), \end{aligned} \tag{3.7}$$

where $\omega_\perp(\boldsymbol{k})^2 = |\boldsymbol{k}|^2$, *and* $\omega_\parallel^\pm(\boldsymbol{k})^2 = \frac{1}{2}\left(\omega_p^2 + |\boldsymbol{k}|^2 \pm \sqrt{\Delta(\boldsymbol{k})}\right)$ *with* $\Delta(\boldsymbol{k}) = \left(\omega_p^2 + |\boldsymbol{k}|^2\right)^2 - 4k_z^2\omega_p^2 \leq 0$. *Then* F_ω *vanishes if:*

1. $\omega_\perp(\boldsymbol{k})^2 = \omega^2$ *and the associated eigenspace is* $\Lambda_\perp(\boldsymbol{k}) := \operatorname{span}\left(\boldsymbol{k}_\parallel, \boldsymbol{e}_z\right)^\perp$.

2. $\omega_\parallel^\pm(\boldsymbol{k})^2 = \omega^2$ *and the associated eigenspaces are subset of* $\Lambda_\parallel(\boldsymbol{k}) := \operatorname{span}\left(\boldsymbol{k}_\parallel, \boldsymbol{e}_z\right)$.

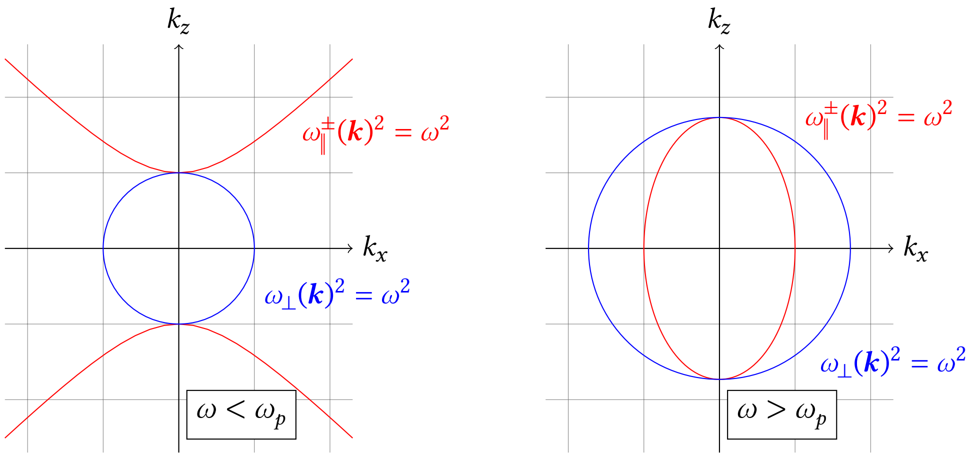


Figure 3.1: Dispersion curves in the plane $\{k_y = 0\}$ with $\omega_p^2 = 2$ and $\omega^2 = 1$ on the left plot and $\omega_p^2 = 2$ and $\omega^2 = 3$ on the right plot.

Remark 3.1.6. The second item can be rewritten so that $\boldsymbol{k}$ and ω are related through the following quadratic equation:

$$\frac{|\boldsymbol{k}_\parallel|^2}{\omega^2 - \omega_p^2} + \frac{k_z^2}{\omega^2} = 1. \tag{3.8}$$

Therefore, the associated curve is an ellipse if $\omega > \omega_p$ whereas the curve becomes a hyperbola if $\omega < \omega_p$, see figure 3.1.

Proof. See Appendix A.1. □

Let $\boldsymbol{E} \in \left(\mathcal{S}'\left(\mathbb{R}^3\right)\right)^3$ be a solution of the system (3.6). If we consider its Fourier transform $\hat{\boldsymbol{E}}(\boldsymbol{k}) := \mathcal{F}[\boldsymbol{E}](\boldsymbol{k})$, then according to plane wave analysis, the solution sought has two components $\hat{\boldsymbol{E}}(\boldsymbol{k}) = \hat{\boldsymbol{E}}_\perp(\boldsymbol{k}) + \hat{\boldsymbol{E}}_\parallel(\boldsymbol{k})$. The support of $\hat{\boldsymbol{E}}_\perp(\boldsymbol{k})$ is a subset of $\{\omega^2 = |\boldsymbol{k}|^2\}$, and we have in particular $\boldsymbol{k}_\parallel \cdot \hat{\boldsymbol{E}}_\perp = 0$, $\boldsymbol{e}_z \cdot \hat{\boldsymbol{E}}_\perp = 0$. On the other hand, the support of $\hat{\boldsymbol{E}}_\parallel(\boldsymbol{k})$ is a subset of the hyperbola defined by equation (3.8), and $\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{E}}_\parallel = 0$. Taking the inverse Fourier transform of $\hat{\boldsymbol{E}}_\perp(\boldsymbol{k})$ and $\hat{\boldsymbol{E}}_\parallel(\boldsymbol{k})$, we can split $\boldsymbol{E} = \boldsymbol{E}_\perp + \boldsymbol{E}_\parallel$ such that

$$\partial_x E_{\perp,x}(\boldsymbol{x}) + \partial_y E_{\perp,y}(\boldsymbol{x}) = 0, \quad E_{\perp,z}(\boldsymbol{x}) = 0, \quad \text{and} \quad \partial_x E_{\parallel,y}(\boldsymbol{k}) - \partial_y E_{\parallel,x}(\boldsymbol{k}) = 0,$$

where the identities hold in $\left(\mathcal{S}'\left(\mathbb{R}^3\right)\right)^3$.

This discussion justifies the separation of the system into two subsystems: a first part related to $\boldsymbol{E}_\perp$, and a second part related to $\boldsymbol{E}_\parallel$. Then, the next section is devoted to the splitting of vector fields.

3.1.3 Anisotropic Helmholtz decomposition

To decompose the system (3.3) into two subsystems, we are going to base ourselves on a lemma of anisotropic decomposition of the vector fields of $\mathbb{R}^3$ in $\mathbb{C}^2$, taken from [6]. We introduce the two following differential operators

$$\operatorname{curl}_\perp \boldsymbol{F} := \partial_x F_y - \partial_y F_x, \qquad \operatorname{div}_\perp \boldsymbol{F} := \partial_x F_x + \partial_y F_y,$$

and the following functional spaces

$$H(\text{curl}_\perp 0;\mathbb{R}^3) := \left\{F \in L^2\left(\mathbb{R}^3\right) \, : \, \text{curl}_\perp F = 0\right\}, \tag{3.9}$$

$$H(\text{div}_\perp 0;\mathbb{R}^3) := \left\{F \in L^2\left(\mathbb{R}^3\right) \, : \, \text{div}_\perp F = 0, F_z = 0\right\}. \tag{3.10}$$

Lemma 3.1.7 (Anisotropic Helmholtz decomposition, [6]). *The following decomposition holds :* $L^2\left(\mathbb{R}^3\right) = H(\text{curl}_\perp 0;\mathbb{R}^3) \overset{\perp}{\oplus} H(\text{div}_\perp 0;\mathbb{R}^3)$.

Proof. Let $F \in L^2\left(\mathbb{R}^3\right)$ and its Fourier transform $\hat{F} := \mathscr{F}(F)$. For any $k \in \mathbb{R}^3$ such that $k_x^2 + k_y^2 \neq 0$, we consider the following two orthogonal vector subspaces:

$$\Lambda_\parallel(k) = \left\{v \in \mathbb{R}^3 \, : \, v \cdot k_\perp = 0\right\}, \qquad \Lambda_\perp(k) = \left\{v \in \mathbb{R}^3 \, : \, v \cdot k_\parallel = 0, v \cdot e_z = 0\right\},$$

and the associated orthogonal projectors $P_\parallel(k)$ and $P_\perp(k)$. Then, it is natural to set $\hat{F}_\parallel(k) = P_\parallel(k)\hat{F}(k)$ and $\hat{F}_\perp(k) = P_\perp(k)\hat{F}(k)$. Since the set $\left\{k \in \mathbb{R}^3 \, : \, k_x = k_y = 0\right\}$ is negligible with respect to Lebesgue measure, and $\left\|P_\parallel(k)\right\|_\infty = \|P_\perp(k)\|_\infty = 1$, $\hat{F}_\parallel(k)$, $\hat{F}_\perp(k)$ are functions of $L^2\left(\mathbb{R}^3\right)$. Next, applying inverse Fourier transform, one obtains $F_\parallel, F_\perp \in L^2\left(\mathbb{R}^3\right)$ such that $F = F_\parallel + F_\perp$. Moreover, due to the definition of $\Lambda_\parallel(k)$ and $\Lambda_\perp(k)$, we clearly have $F_\parallel \in H(\text{curl}_\perp 0;\mathbb{R}^3)$ and $F_\perp \in H(\text{div}_\perp 0;\mathbb{R}^3)$, and this decomposition is unique because $\mathbb{R}^3 = \Lambda_\parallel(k) \overset{\perp}{\oplus} \Lambda_\perp(k)$. Finally, the two spaces are orthogonal thanks to the Plancherel identity:

$$\left(F_\parallel, F_\perp\right)_{L^2(\mathbb{R}^3)} = \left(\hat{F}_\parallel, \hat{F}_\perp\right)_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \hat{F}_\parallel(k) \cdot \overline{\hat{F}_\perp(k)} \, dk = 0.$$

□

Given $F = F_\parallel + F_\perp \in L^2\left(\mathbb{R}^3\right)$, $F_\parallel \in H(\mathbf{curl}_\perp 0;\mathbb{R}^3)$ will be denoted as the longitudinal component, and $F_\perp \in H(\text{div}_\perp 0;\mathbb{R}^3)$ the transverse component. Also, notice that $F_\perp$ and e_z are orthogonal. Then, in the view of Maxwell's system, it is natural to split fields of $H(\boldsymbol{curl};\mathbb{R}^3)$. The following lemma is about the regularity of the components of such vector fields.

Lemma 3.1.8. *Let be* $F \in H(\boldsymbol{curl};\mathbb{R}^3)$. *We have* $F_\parallel, F_\perp \in H(\boldsymbol{curl};\mathbb{R}^3)$. *Moreover,* $\boldsymbol{curl}\, F_\parallel \in H(\text{div}_\perp 0;\mathbb{R}^3)$ *and* $\boldsymbol{curl}\, F_\perp \in H(\text{curl}_\perp 0;\mathbb{R}^3)$.

Proof. We have by direct computation

$$\left(k \times F_\parallel\right) \cdot (k \times F_\perp) = |k|^2 F_\parallel \cdot F_\perp - \left(k \cdot F_\parallel\right)(k \cdot F_\perp) = 0.$$

Therefore, using Plancherel theorem, we have

$$\begin{aligned}\|\boldsymbol{curl}\, F\|^2_{L^2(\mathbb{R}^3)} &= \left\|k \times \hat{F}\right\|^2_{L^2(\mathbb{R}^3)} = \left\|k \times \hat{F}_\parallel + k \times \hat{F}_\perp\right\|^2_{L^2(\mathbb{R}^3)} \\ &= \left\|k \times \hat{F}_\parallel\right\|^2_{L^2(\mathbb{R}^3)} + \left\|k \times \hat{F}_\perp\right\|^2_{L^2(\mathbb{R}^3)} = \left\|\boldsymbol{curl}\, F_\parallel\right\|^2_{L^2(\mathbb{R}^3)} + \|\boldsymbol{curl}\, F_\perp\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

which shows that $F_\parallel, F_\perp \in H(\boldsymbol{curl};\mathbb{R}^3)$. Next, we have

$$k_\parallel \cdot \left(k \times \hat{F}_\parallel\right) = -k_z k_\perp \cdot F_\parallel = 0, \qquad e_z \cdot \left(k \times \hat{F}_\parallel\right) = k_\perp \cdot F_\parallel = 0,$$

$$k_\perp \cdot \left(k \times \hat{F}_\perp\right) = \left(-|k_\parallel|^2 e_z + k_z k_\parallel\right) \cdot F_\perp = 0,$$

so that $k \times \hat{F}_\parallel \in \Lambda_\perp(k)$, $k \times \hat{F}_\perp \in \Lambda_\parallel(k)$ which concludes the proof. □

The anisotropic Helmholtz decomposition allows decomposing vector fields of $\boldsymbol{L}^2\left(\mathbb{R}^3\right)$. In particular, it makes possible to demonstrate the equivalence between the resolution of the system (3.3) and the two subsystems resulting from the Helmholtz decomposition, in the case where $\omega \in \mathbb{C} \setminus \mathbb{R}$.

However, in the case where $\omega \in \mathbb{R}$, this decomposition is not necessarily valid because, a priori, the solution of (3.3) does not belong to $\mathbf{L}^2(\mathbb{R}^3)$, but rather to a larger weighted L^2 space, see e.g., [46, §2.6.5]. Thus, it would be desirable to find an anisotropic Helmholtz decomposition in a more general space than $\boldsymbol{L}^2\left(\mathbb{R}^3\right)$. The key tool of the proof of the previous lemma is the Fourier transform, which is an isomorphism of $L^2\left(\mathbb{R}^3\right)$. Since the Fourier transform is also an isomorphism of $\mathcal{S}'\left(\mathbb{R}^3\right)$, it is natural to look for an extension of the decomposition for distributions which belong to $\left(\mathcal{S}'\left(\mathbb{R}^3\right)\right)^3$.

However, the decomposition is not valid for every distribution. Let us analyze which distributions are problematic. Consider for example the constant distribution $\boldsymbol{e}_x$. Obviously,

$$\operatorname{div}_\perp \boldsymbol{e}_x = \operatorname{curl}_\perp \boldsymbol{e}_x = 0,$$

so that it has an infinite number of decomposition like in Lemma 3.1.7. Following the proof of Lemma 3.1.7 with $\boldsymbol{F} = \boldsymbol{e}_x$, we observe that the distributions $P_\parallel \hat{\boldsymbol{F}}$, $P_\perp \hat{\boldsymbol{F}}$ do not have sense. Indeed, given a test function $\boldsymbol{\varphi} \in \left(\mathcal{S}\left(\mathbb{R}^3\right)\right)^3$, we would have

$$\left\langle P_\parallel \hat{\boldsymbol{F}}, \boldsymbol{\varphi} \right\rangle_{(\mathcal{S}(\mathbb{R}^3))^3} = \left\langle (2\pi)^{3/2} \delta_0 \boldsymbol{e}_x, P_\parallel \boldsymbol{\varphi} \right\rangle_{(\mathcal{S}(\mathbb{R}^3))^3},$$

which has no sense since $\left(P_\parallel \boldsymbol{\varphi}\right)(0)$ is not defined. Similarly, it is easy to see that polynomial distributions $\sum_{j\in\{x,y,z\}} p_j(\boldsymbol{x})\boldsymbol{e}_j$, where p_j is a polynomial, can have several possible splitting as longitudinal and transversal component, according to the terminology introduced above, and all these distributions are singular at the point $\boldsymbol{x} = 0$.

Then, it suffices to consider distributions for which the multiplication with the projectors $P_\parallel$ and $P_\perp$ is valid. These observations lead us to define the following admissible class of distributions.

Assumption 3.1.9. *Given a distribution* $\boldsymbol{F} \in \mathcal{S}'\left(\mathbb{R}^3\right)^3$, *there is* $\varepsilon > 0$ *and* $\hat{\boldsymbol{F}}_{reg} \in L^1(B_\varepsilon)^3$ *such that* $\mathcal{F}(\boldsymbol{F}) = \hat{\boldsymbol{F}}_{reg}$ *in* $\mathcal{D}'(B_\varepsilon)^3$.

This assumption is sufficient to ensure the uniqueness of the decomposition, since it naturally excludes polynomial distributions.

Lemma 3.1.10. *Let be* $\boldsymbol{F} \in \left(\mathcal{S}'\left(\mathbb{R}^3\right)\right)^3$ *satisfying assumption 3.1.9. Then there is* $\boldsymbol{F}_\perp, \boldsymbol{F}_\parallel \in \left(\mathcal{S}'\left(\mathbb{R}^3\right)\right)^3$ *satisfying assumption 3.1.9 such that* $\boldsymbol{F} = \boldsymbol{F}_\perp + \boldsymbol{F}_\parallel$, $\operatorname{div}_\perp \boldsymbol{F}_\perp = 0$, $(\boldsymbol{F}_\perp)_z = 0$ *and* $\operatorname{curl}_\perp \boldsymbol{F}_\parallel = 0$. *Moreover,* $\boldsymbol{F}_\parallel$ *and* $\boldsymbol{F}_\perp$ *are uniquely defined.*

Proof. Thanks to the assumption 3.1.9, $P_\parallel \mathcal{F}(\boldsymbol{F}_\parallel)$, $P_\parallel \mathcal{F}(\boldsymbol{F}_\perp)$ are well-defined, and so that $\boldsymbol{F}_\parallel$ and $\boldsymbol{F}_\perp$. It only remains to prove the uniqueness of the decomposition. Let $\boldsymbol{F}_\parallel$, $\boldsymbol{F}_\perp$ satisfying assumption 3.1.9 be such that $\boldsymbol{F}_\parallel + \boldsymbol{F}_\perp = 0$. Therefore, we have $\mathcal{F}(\boldsymbol{F}_\parallel) = P_\parallel \mathcal{F}(\boldsymbol{F}_\parallel) = -P_\parallel \mathcal{F}(\boldsymbol{F}_\perp) = 0$ which ends the proof. □

Then, we will use the results of this section to decompose the electric and magnetic fields in the next one.

Remark 3.1.11. Notice that the assumption (3.1.9) is not restrictive in our case. For example, when $\omega \in \mathbb{R}$, a source term $\boldsymbol{j}$ in (3.3) will generally have a compact support, which implies that $\mathcal{F}(\boldsymbol{j})$ are analytic and assumption 3.1.9 is obviously verified.

3.1.4 Reduced Maxwell problems

We will now apply the anisotropic Helmholtz decomposition to the system

$$\left|\begin{aligned} & i\omega \varepsilon \boldsymbol{E} + \boldsymbol{curl}\,\boldsymbol{B} = \boldsymbol{j}, \\ & -i\omega \boldsymbol{B} + \boldsymbol{curl}\,\boldsymbol{E} = \boldsymbol{m}, \end{aligned}\right. \qquad \text{with } \varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}, \tag{3.11}$$

for $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $\boldsymbol{j}, \boldsymbol{m} \in \boldsymbol{L}^2(\mathbb{R}^3)$ first, and then for $\omega \in \mathbb{R} \setminus \{\pm\omega_p, 0\}$ and $\boldsymbol{E}, \boldsymbol{B} \in \left(\mathcal{S}'(\mathbb{R}^3)\right)^3$.

For $\omega \in \mathbb{C} \setminus \mathbb{R}$, according to Theorem 3.1.4, the solutions of this system are such that $\boldsymbol{E}, \boldsymbol{B} \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3)$. Then, using Lemma 3.1.7, we split

$$\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp, \qquad \boldsymbol{B} = \boldsymbol{B}_\parallel + \boldsymbol{B}_\perp, \qquad \boldsymbol{m} = \boldsymbol{m}_\parallel + \boldsymbol{m}_\perp, \qquad \boldsymbol{j} = \boldsymbol{j}_\parallel + \boldsymbol{j}_\perp,$$

where

- $\boldsymbol{E}_\parallel, \boldsymbol{B}_\parallel \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \cap \boldsymbol{H}(\mathrm{curl}_\perp 0; \mathbb{R}^3)$ and $\boldsymbol{m}_\parallel, \boldsymbol{j}_\parallel \in \boldsymbol{H}(\mathrm{curl}_\perp 0; \mathbb{R}^3)$,
- $\boldsymbol{E}_\perp, \boldsymbol{B}_\perp \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \cap \boldsymbol{H}(\mathrm{div}_\perp 0; \mathbb{R}^3)$ and $\boldsymbol{m}_\perp, \boldsymbol{j}_\perp \in \boldsymbol{H}(\mathrm{div}_\perp 0; \mathbb{R}^3)$.

Notice that we also have $\varepsilon \boldsymbol{E}_\parallel \in \boldsymbol{H}(\mathrm{curl}_\perp 0; \mathbb{R}^3)$ and $\varepsilon \boldsymbol{E}_\perp = \boldsymbol{E}_\perp \in \boldsymbol{H}(\mathrm{div}_\perp 0; \mathbb{R}^3)$, as ε only affects the third component. We also have that $\mathrm{curl}_\perp \varepsilon \boldsymbol{F} = \mathrm{curl}_\perp \boldsymbol{F}$ and $\mathrm{div}_\perp \varepsilon \boldsymbol{F} = \mathrm{div}_\perp \boldsymbol{F}$.

Therefore, using the fact that the spaces $\boldsymbol{H}(\mathrm{curl}_\perp 0; \mathbb{R}^3)$ and $\boldsymbol{H}(\mathrm{div}_\perp 0; \mathbb{R}^3)$ are in direct sum and Lemma 3.1.8, the previous system naturally decomposes into the two following problems:

$$\left|\begin{aligned} & \text{find } \boldsymbol{E}_\parallel \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \cap \boldsymbol{H}(\mathrm{curl}_\perp 0; \mathbb{R}^3), \\ & \boldsymbol{B}_\perp \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \cap \boldsymbol{H}(\mathrm{div}_\perp 0; \mathbb{R}^3) \text{ such that} \\ & i\omega \varepsilon \boldsymbol{E}_\parallel + \boldsymbol{curl}\,\boldsymbol{B}_\perp = \boldsymbol{j}_\parallel, \\ & -i\omega \boldsymbol{B}_\perp + \boldsymbol{curl}\,\boldsymbol{E}_\parallel = \boldsymbol{m}_\perp, \end{aligned}\right. \tag{3.12}$$

and

$$\left|\begin{aligned} & \text{find } \boldsymbol{E}_\perp \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \cap \boldsymbol{H}(\mathrm{div}_\perp 0; \mathbb{R}^3), \\ & \boldsymbol{B}_\parallel \in \boldsymbol{H}(\boldsymbol{curl}; \mathbb{R}^3) \cap \boldsymbol{H}(\mathrm{curl}_\perp 0; \mathbb{R}^3) \text{ such that} \\ & i\omega \boldsymbol{E}_\perp + \boldsymbol{curl}\,\boldsymbol{B}_\parallel = \boldsymbol{j}_\perp, \\ & -i\omega \boldsymbol{B}_\parallel + \boldsymbol{curl}\,\boldsymbol{E}_\perp = \boldsymbol{m}_\parallel. \end{aligned}\right. \tag{3.13}$$

Notice that $\varepsilon \boldsymbol{E}_\perp = \boldsymbol{E}_\perp$, so that ε disappears in the second problem.

In the view of the plane wave analysis, the first problem (3.12) can be described as *hyperbolic*, whereas the second problem as *elliptic*. Provided that $\omega \in \mathbb{C} \setminus \mathbb{R}$, and in the view of Lemma 3.1.8, solving the initial problem (3.3) is equivalent to solving both (3.12) and (3.13). On the other hand, the two problems taken separately are also well-posed for $\omega \in \mathbb{C} \setminus \mathbb{R}$: the well-posedness of (3.12) is reminiscent of Theorem 3.1.4, and the well-posedness of (3.13) comes from the classic Maxwell's equations theory.

Moreover, the third component of $\boldsymbol{B}_\perp$ and $\boldsymbol{E}_\perp$ vanishes, so that they are orthogonal with the background magnetic field $\boldsymbol{B}_0$. Therefore, the system (3.12) (resp., (3.13)) can also be called as the *transverse magnetic problem* or TM problem (resp., *transverse electric problem* or TE problem).

For $\omega \in \mathbb{R}$, as for the Helmholtz equation, the solutions are not expected to belong to $\boldsymbol{L}^2(\mathbb{R}^3)$ but rather in a weighted L^2 space. The following proposition shows the equivalence between the original system (3.11) and the split systems in the sense of the distribution.

Lemma 3.1.12. *Let be $\omega \in \mathbb{R}\backslash\{-\omega_p, 0, \omega_p\}$ and $\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{j}, \boldsymbol{m} \in \left(\mathscr{S}'\left(\mathbb{R}^3\right)\right)^3$ satisfying the system (3.11). If $\boldsymbol{j}, \boldsymbol{m}$ follow assumption 3.1.9, then there exist unique $\boldsymbol{E}_\parallel$, $\boldsymbol{E}_\perp$, $\boldsymbol{B}_\parallel$, $\boldsymbol{B}_\perp$ such that $\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp$, $\boldsymbol{B} = \boldsymbol{B}_\parallel + \boldsymbol{B}_\perp$ as in lemma 3.1.10, and*

$$\left| \begin{aligned} i\omega\varepsilon\boldsymbol{E}_\parallel + \boldsymbol{curl}\,\boldsymbol{B}_\perp &= \boldsymbol{j}_\parallel, \\ -i\omega\boldsymbol{B}_\perp + \boldsymbol{curl}\,\boldsymbol{E}_\parallel &= \boldsymbol{m}_\perp, \end{aligned} \right. \qquad \left| \begin{aligned} i\omega\boldsymbol{E}_\perp + \boldsymbol{curl}\,\boldsymbol{B}_\parallel &= \boldsymbol{j}_\perp, \\ -i\omega\boldsymbol{B}_\parallel + \boldsymbol{curl}\,\boldsymbol{E}_\perp &= \boldsymbol{m}_\parallel. \end{aligned} \right.$$

Notice that, given $\boldsymbol{E}_\parallel, \boldsymbol{E}_\perp, \boldsymbol{B}_\parallel, \boldsymbol{B}_\perp$ solutions of the two systems above, we can always reconstruct $\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp$ and $\boldsymbol{B} = \boldsymbol{B}_\parallel + \boldsymbol{B}_\perp$ solutions of the original problem (3.2). Therefore, it is sufficient that $\boldsymbol{j}, \boldsymbol{m} \in \left(\mathscr{S}'\left(\mathbb{R}^3\right)\right)^3$ satisfy Assumption 3.1.9, to have the equivalence between the system 3.11 and the two sub-systems.

Proof. The key point is that if $\boldsymbol{j}$ and $\boldsymbol{m}$ follow assumption 3.1.9, then $\boldsymbol{E}$ and $\boldsymbol{B}$ follow it too. We have

$$\begin{aligned} \boldsymbol{curl}\,\boldsymbol{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} &= \boldsymbol{curl}\,\boldsymbol{m} + i\omega\boldsymbol{j}, \\ \boldsymbol{curl}\,\varepsilon^{-1}\,\boldsymbol{curl}\,\boldsymbol{B} - \omega^2\boldsymbol{B} &= \boldsymbol{curl}\,\varepsilon^{-1}\boldsymbol{j} - i\omega\boldsymbol{m}. \end{aligned}$$

Taking the Fourier transform of the previous equations, $\hat{\boldsymbol{E}} = \mathscr{F}(\boldsymbol{E})$ (respectively $\boldsymbol{B}, \boldsymbol{j}, \boldsymbol{m}$), we obtain the two systems

$$\mathbb{A}_E\hat{\boldsymbol{E}} = i\boldsymbol{k}\times\hat{\boldsymbol{m}} + i\omega\hat{\boldsymbol{j}}, \qquad \mathbb{A}_B\hat{\boldsymbol{B}} = i\boldsymbol{k}\times\varepsilon^{-1}\hat{\boldsymbol{j}} - i\omega\hat{\boldsymbol{m}},$$

where $\mathbb{A}_E\hat{\boldsymbol{E}} = -\boldsymbol{k}\times\left(\boldsymbol{k}\times\hat{\boldsymbol{E}}\right) - \omega^2\varepsilon\hat{\boldsymbol{E}}$ and $\mathbb{A}_B\hat{\boldsymbol{B}} = -\boldsymbol{k}\times\varepsilon^{-1}\left(\boldsymbol{k}\times\hat{\boldsymbol{B}}\right) - \omega^2\hat{\boldsymbol{B}}$. The matrices $\mathbb{A}_E(\boldsymbol{k}), \mathbb{A}_B(\boldsymbol{k})$ are smooth and there is $\varepsilon > 0$ such that there are invertible for every $\boldsymbol{k} \in B_\varepsilon$. Therefore, we have in $\mathscr{D}'(B_\varepsilon)^3$

$$\hat{\boldsymbol{E}} = {\mathbb{A}_E}^{-1}\left(i\boldsymbol{k}\times\hat{\boldsymbol{m}} + i\omega\hat{\boldsymbol{j}}\right), \qquad \hat{\boldsymbol{B}} = {\mathbb{A}_B}^{-1}\left(i\boldsymbol{k}\times\varepsilon^{-1}\hat{\boldsymbol{j}} - i\omega\hat{\boldsymbol{m}}\right).$$

Finally, as ${\mathbb{A}_E}^{-1}, {\mathbb{A}_B}^{-1}$ are smooth in B_ε, and $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{j}}$ satisfy Assumption 3.1.9, then $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ also satisfy this assumption. Applying lemma 3.1.10 concludes the proof. □

The two sub-problems defined above involve both the Sobolev space $\boldsymbol{H}\left(\mathrm{div}_\perp\, 0; \mathbb{R}^3\right)$, or at least $\boldsymbol{E}_\perp, \boldsymbol{B}_\perp$ have a vanishing third component. Therefore, in the two subsystems (3.12) and (3.13), the number of unknowns involved is not 6 like the usual Maxwell's equations, but only 5. This justifies, in our case, the use of the term *reduced*.

Then, we will define below differential operators adapted to this *reduced* setting. The vector fields with three components will be written with an ***italic bold font***, the vector fields with two components with a **roman bold font** and a tilde ~, and the scalar field with a medium weight font. We introduce the following differential operators, provided $\boldsymbol{F} = (F_x, F_y, F_z)^\top$, $\tilde{\mathbf{F}} = (F_x, F_y)^\top$

and a scalar function f:

$$\widetilde{\operatorname{curl}}\, \boldsymbol{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \end{pmatrix}, \qquad \operatorname{curl}_\perp \boldsymbol{F} = \partial_x F_y - \partial_y F_x, \qquad \operatorname{div}_\perp \boldsymbol{F} = \partial_x F_x + \partial_y F_y,$$

$$\boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \begin{pmatrix} -\partial_z F_y \\ \partial_z F_x \\ \partial_x F_y - \partial_y F_x \end{pmatrix}, \qquad \operatorname{curl}_\perp \tilde{\mathrm{F}} = \partial_x F_y - \partial_y F_x, \qquad \operatorname{div}_\perp \tilde{\mathrm{F}} = \partial_x F_x + \partial_y F_y,$$

$$\boldsymbol{curl}_\perp f = \begin{pmatrix} \partial_y f \\ -\partial_x f \\ 0 \end{pmatrix}, \qquad \tilde{\nabla} f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}, \qquad \Delta_\perp f = \partial_x^2 f + \partial_y^2 f.$$

Notice that the font of the differential operator has been chosen according to its output.

The usual $\boldsymbol{curl}$-operator has the following relation $\boldsymbol{curl}\,\boldsymbol{curl}\,\boldsymbol{F} = \nabla \operatorname{div} \boldsymbol{F} - \boldsymbol{\Delta F}$. Similarly, we have numerous relations involving the operators above, and can be found in Appendix A.2. But two noticeable identities are the following:

$$\widetilde{\operatorname{curl}}\, \boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \tilde{\nabla} \operatorname{div}_\perp \tilde{\mathrm{F}} - \boldsymbol{\Delta} \tilde{\mathrm{F}} \tag{3.14}$$

$$\widetilde{\operatorname{curl}}\, \varepsilon^{-1}\, \boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \beta^{-1} \tilde{\nabla} \operatorname{div}_\perp \tilde{\mathrm{F}} - \boldsymbol{\Delta}_\beta \tilde{\mathrm{F}}, \tag{3.15}$$

where $\boldsymbol{\Delta}$ is the usual Laplacian operator on each coordinate of $\tilde{\mathrm{F}}$, and $\boldsymbol{\Delta}_\beta$ is the following scaled operator

$$\Delta_\beta := \beta^{-1}(\partial_x^2 + \partial_y^2) + \partial_z^2 \tag{3.16}$$

applied on each coordinate of $\tilde{\mathrm{F}}$. Furthermore, the relations like $\operatorname{div} \boldsymbol{curl}\, \boldsymbol{f} = 0$ are false in general with the operators defined above, but the identity $\operatorname{div} \boldsymbol{curl}_\perp \tilde{\mathrm{F}} = 0$ still holds.

Remark 3.1.13. Given the usual $\boldsymbol{curl}$-operator and $\varepsilon = \operatorname{diag}(1, 1, \beta)$, we have the following identity for $\boldsymbol{F} = (F_x, F_y, F_z)^\top$:

$$\boldsymbol{curl}\, \varepsilon^{-1}\, \boldsymbol{curl}\, \boldsymbol{F} = \begin{pmatrix} \partial_x \operatorname{div}\left(\beta^{-1} \varepsilon \boldsymbol{F}\right) - \Delta_\beta F_x \\ \partial_x \operatorname{div}\left(\beta^{-1} \varepsilon \boldsymbol{F}\right) - \Delta_\beta F_y \\ \partial_x \operatorname{div} \boldsymbol{F} - \Delta F_z \end{pmatrix}.$$

The anisotropic Helmholtz decomposition 3.1.7 then separates the first two components of the third in the above identity. It results (3.15) and (3.14).

We have $\boldsymbol{L}^2(\mathbb{R}^3) = L^2(\mathbb{R}^3)^3$ and $\tilde{\mathrm{L}}^2(\mathbb{R}^3) = L^2(\mathbb{R}^3)^2$. Then, the operators above induce the following Sobolev spaces

$$\boldsymbol{H}\left(\widetilde{\operatorname{curl}}; \mathbb{R}^3\right) := \left\{\boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3) \,:\, \widetilde{\operatorname{curl}}\, \boldsymbol{F} \in \tilde{\mathrm{L}}^2(\mathbb{R}^3)\right\}, \tag{3.17}$$

$$\tilde{\mathrm{H}}\left(\boldsymbol{curl}_\perp; \mathbb{R}^3\right) := \left\{\tilde{\mathrm{F}} \in \tilde{\mathrm{L}}^2(\mathbb{R}^3) \,:\, \boldsymbol{curl}_\perp \tilde{\mathrm{F}} \in \boldsymbol{L}^2(\mathbb{R}^3)\right\}, \tag{3.18}$$

$$\tilde{\mathrm{H}}\left(\operatorname{div}_\perp 0; \mathbb{R}^3\right) := \left\{\tilde{\mathrm{F}} \in \tilde{\mathrm{L}}^2(\mathbb{R}^3) \,:\, \operatorname{div}_\perp \tilde{\mathrm{F}} = 0\right\}. \tag{3.19}$$

With these operators and spaces defined above, we can rewrite the subsystems (3.12) and (3.13).

The transverse electric subsystem becomes

$$\left|\begin{array}{l} \text{find } \tilde{\mathrm{E}}_\perp \in \tilde{\mathrm{H}}\left(\boldsymbol{curl}_\perp;\mathbb{R}^3)\right) \cap \tilde{\mathrm{H}}\left(\operatorname{div}_\perp 0;\mathbb{R}^3\right), \\ \boldsymbol{B}_\parallel \in \boldsymbol{H}\left(\widetilde{\mathbf{curl}};\mathbb{R}^3\right) \cap \boldsymbol{H}\left(\operatorname{curl}_\perp 0;\mathbb{R}^3\right) \text{ such that} \\ i\omega\tilde{\mathrm{E}}_\perp + \widetilde{\mathbf{curl}}\,\boldsymbol{B}_\parallel = \tilde{\mathrm{j}}_\perp, \\ -i\omega\boldsymbol{B}_\parallel + \boldsymbol{curl}_\perp\,\tilde{\mathrm{E}}_\perp = \boldsymbol{m}_\parallel, \end{array}\right. \tag{3.20}$$

where $\boldsymbol{m}_\parallel \in \boldsymbol{H}\left(\operatorname{curl}_\perp 0;\mathbb{R}^3\right)$ and $\tilde{\mathrm{j}}_\perp \in \tilde{\mathrm{H}}\left(\operatorname{div}_\perp 0;\mathbb{R}^3\right)$. The corresponding one unknown equations are

$$\begin{aligned} \widetilde{\mathbf{curl}}\,\boldsymbol{curl}_\perp\,\tilde{\mathrm{E}}_\perp - \omega^2\tilde{\mathrm{E}}_\perp &= \widetilde{\mathbf{curl}}\,\boldsymbol{m}_\parallel + i\omega\tilde{\mathrm{j}}_\perp, \\ \boldsymbol{curl}_\perp\,\widetilde{\mathbf{curl}}\,\boldsymbol{B}_\parallel - \omega^2\boldsymbol{B}_\parallel &= \boldsymbol{curl}_\perp\,\tilde{\mathrm{j}}_\perp - i\omega\boldsymbol{m}_\parallel. \end{aligned} \tag{3.21}$$

The transverse magnetic subsystem becomes

$$\left|\begin{array}{l} \text{find } \boldsymbol{E}_\parallel \in \boldsymbol{H}\left(\widetilde{\mathbf{curl}};\mathbb{R}^3\right) \cap \boldsymbol{H}\left(\operatorname{curl}_\perp 0;\mathbb{R}^3\right), \\ \tilde{\mathrm{B}}_\perp \in \tilde{\mathrm{H}}\left(\boldsymbol{curl}_\perp;\mathbb{R}^3)\right) \cap \tilde{\mathrm{H}}\left(\operatorname{div}_\perp 0;\mathbb{R}^3\right) \text{ such that} \\ i\omega\varepsilon\boldsymbol{E}_\parallel + \boldsymbol{curl}_\perp\,\tilde{\mathrm{B}}_\perp = \boldsymbol{j}_\parallel, \\ -i\omega\tilde{\mathrm{B}}_\perp + \widetilde{\mathbf{curl}}\,\boldsymbol{E}_\parallel = \tilde{\mathrm{m}}_\perp, \end{array}\right. \tag{3.22}$$

where $\boldsymbol{j}_\parallel \in \boldsymbol{H}\left(\operatorname{curl}_\perp 0;\mathbb{R}^3\right)$ and $\tilde{\mathrm{m}}_\perp \in \tilde{\mathrm{H}}\left(\operatorname{div}_\perp 0;\mathbb{R}^3\right)$. The corresponding one unknown equations are

$$\begin{aligned} \boldsymbol{curl}_\perp\,\widetilde{\mathbf{curl}}\,\boldsymbol{E}_\parallel - \omega^2\varepsilon\boldsymbol{E}_\parallel &= \boldsymbol{curl}_\perp\,\tilde{\mathrm{m}}_\perp + i\omega\boldsymbol{j}_\parallel, \\ \widetilde{\mathbf{curl}}\,\varepsilon^{-1}\,\boldsymbol{curl}_\perp\,\tilde{\mathrm{B}}_\perp - \omega^2\tilde{\mathrm{B}}_\perp &= \widetilde{\mathbf{curl}}\,\varepsilon^{-1}\boldsymbol{j}_\parallel - i\omega\tilde{\mathrm{m}}_\perp. \end{aligned} \tag{3.23}$$

Now that the original system has been split into a reduced transverse electric problem, and a reduced transverse magnetic problem, the two next section are devoted to their analysis.

Remark 3.1.14. The search for plane wave solutions $\tilde{\mathrm{E}}_\perp(\boldsymbol{x}) = \hat{\tilde{\mathrm{E}}}_\perp e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $\tilde{\mathrm{B}}_\perp(\boldsymbol{x}) = \hat{\tilde{\mathrm{B}}}_\perp e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ leads to define the following dispersion function

$$\begin{aligned} F_\omega^{TE}(\boldsymbol{k}) &:= \det\left(\mathbb{A}^{TE}(\boldsymbol{k}) - \omega^2\mathbb{I}_2\right), \\ F_\omega^{TM}(\boldsymbol{k}) &:= \det\left(\mathbb{A}^{TM}(\boldsymbol{k}) - \omega^2\mathbb{I}_2\right), \end{aligned}$$

where $\mathbb{A}^{TE}(\boldsymbol{k}) = |\boldsymbol{k}|^2\mathbb{I}_2 - \mathrm{k}_\parallel\mathrm{k}_\parallel^\top$, $\mathbb{A}^{TM}(\boldsymbol{k}) = \left(\beta^{-1}|\mathrm{k}_\parallel|^2 + k_z^2\right)\mathbb{I}_2 - \beta^{-1}\mathrm{k}_\parallel\mathrm{k}_\parallel^\top$ and $\mathrm{k}_\parallel = (k_x, k_y)^\top$. The eigenvalue-eigenvector pairs of $\mathbb{A}^{TE}(\boldsymbol{k})$ are $\left(|\boldsymbol{k}|^2, \mathrm{k}_\perp\right)$, with $\mathrm{k}_\perp = (-k_y, k_x)^\top$, and $\left(k_z^2, \mathrm{k}_\parallel\right)$, and the eigenvalue-eigenvector pairs of $\mathbb{A}^{TM}(\boldsymbol{k})$ are $\left(\beta^{-1}|\mathrm{k}_\parallel|^2 + k_z^2, \mathrm{k}_\perp\right)$ and $\left(k_z^2, \mathrm{k}_\parallel\right)$. Therefore,

$$\begin{aligned} F_\omega^{TE}(\boldsymbol{k}) &= \left(k_z^2 - \omega^2\right)\left(|\boldsymbol{k}|^2 - \omega^2\right), \\ F_\omega^{TM}(\boldsymbol{k}) &= \left(k_z^2 - \omega^2\right)\left(\beta^{-1}|\mathrm{k}_\parallel|^2 + k_z^2 - \omega^2\right), \end{aligned}$$

Then, the transverse electric problem (3.20) captures the elliptic part whereas the transverse magnetic problem (3.22) captures the hyperbolic part of the solution of the initial problem (3.3). On the other hand, we observe that considering the problem with 5 unknowns introduce the term $k_z^2 - \omega^2$ in the dispersion relation, which was not present in the original problem, see Lemma 3.1.5. In particular, this term will introduce unwanted terms in the fundamental solutions of the sub-problems.

3.2 Transverse electric problem

This section is devoted to the study of the reduced transverse electric problem (3.20). Firstly, we establish the existence of a solution by computing a fundamental solution. Then, its uniqueness is discussed via the Silver-Müller condition.

3.2.1 Existence of solutions

The principle of the computation of the fundamental solution is the same as for the usual Maxwell's system. However, contrary to usual Maxwell's system, the reduced TE problem is not symmetric. Therefore, the fundamental solution to the TE problem (3.20) is a pair of distributions $(\tilde{\mathbb{E}}_\perp, \mathbb{B}_\parallel) \in \tilde{\boldsymbol{\mathcal{S}}}' \times \boldsymbol{\mathcal{S}}'$ where $\tilde{\boldsymbol{\mathcal{S}}}' = \mathcal{S}'(\mathbb{R}^3)^{2\times 2} \times \mathcal{S}'(\mathbb{R}^3)^{2\times 3}$ and $\boldsymbol{\mathcal{S}}' = \mathcal{S}'(\mathbb{R}^3)^{3\times 2} \times \mathcal{S}'(\mathbb{R}^3)^{3\times 3}$ such that

$$\left| \begin{aligned} & i\omega\tilde{\mathbb{E}}_\perp + \widetilde{\mathbf{curl}}\, \mathbb{B}_\parallel = \tilde{\mathbb{j}}_\perp, \\ & -i\omega\mathbb{B}_\parallel + \boldsymbol{curl}_\perp\, \tilde{\mathbb{E}}_\perp = \mathbb{m}_\parallel, \end{aligned} \right. \tag{3.24}$$

where $\tilde{\mathbb{j}}_\perp = (\delta_0 \mathbb{I}_2, 0_{2\times 3})$, $\mathbb{m}_\parallel = (0_{3\times 2}, \delta_0\mathbb{I}_3)$ and δ_0 is the Dirac mass. The differential operators must be taken column-wise.

Let us explain briefly the notation above by the following simple application. We decompose $\tilde{\mathbb{E}}_\perp = \left(\tilde{\mathbb{E}}^j_\perp, \tilde{\mathbb{E}}^m_\perp\right)$ with $\tilde{\mathbb{E}}^j_\perp \in \mathcal{S}'(\mathbb{R}^3)^{2\times 2}$, $\tilde{\mathbb{E}}^m_\perp \in \mathcal{S}'(\mathbb{R}^3)^{2\times 3}$, and $\mathbb{B}_\parallel = \left(\mathbb{B}^j_\parallel, \mathbb{B}^m_\parallel\right)$ with $\mathbb{B}^j_\parallel \in \mathcal{S}'(\mathbb{R}^3)^{3\times 2}$, $\mathbb{B}^m_\parallel \in \mathcal{S}'(\mathbb{R}^3)^{3\times 3}$. Then, given the following source term $\tilde{\mathbf{j}}_\perp \in \mathscr{C}^\infty_0(\mathbb{R}^3)^2$ and $\boldsymbol{m}_\parallel \in \mathscr{C}^\infty_0(\mathbb{R}^3)^3$ for the reduced TE problem, we can define the following solutions

$$\boldsymbol{E}_\perp = \tilde{\mathbb{E}}_\perp * \begin{pmatrix} \tilde{\mathbf{j}}_\perp \\ \boldsymbol{m}_\parallel \end{pmatrix} = \tilde{\mathbb{E}}^j_\perp * \tilde{\mathbf{j}}_\perp + \tilde{\mathbb{E}}^m_\perp * \boldsymbol{m}_\parallel, \qquad \text{and} \qquad \boldsymbol{B}_\parallel = \mathbb{B}_\parallel * \begin{pmatrix} \tilde{\mathbf{j}}_\perp \\ \boldsymbol{m}_\parallel \end{pmatrix} = \mathbb{B}^j_\parallel * \tilde{\mathbf{j}}_\perp + \mathbb{B}^m_\parallel * \boldsymbol{m}_\parallel,$$

where $*$ is the matrix convolution operator.

Recall the fundamental solution of the Helmholtz equation for $\omega \in \mathbb{C}\setminus\mathbb{R}$:

$$\mathscr{G}_\omega(\boldsymbol{x}) = \frac{e^{\gamma i\omega|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}, \qquad \text{with } \gamma = \operatorname{sign}\operatorname{Im}\omega. \tag{3.25}$$

Proposition 3.2.1. *For $\omega \in \mathbb{C}\setminus\mathbb{R}$, the fundamental solution of the TE problem (3.24) is decomposed into two parts*

$$\tilde{\mathbb{E}}_\perp = \tilde{\mathbb{E}}^{reg}_\perp + \tilde{\mathbb{E}}^{sing}_\perp, \qquad \mathbb{B}_\parallel = \mathbb{B}^{reg}_\parallel + \mathbb{B}^{sing}_\parallel,$$

with

$$\tilde{\mathbb{E}}^{reg}_\perp = \left(i\omega\mathscr{G}_\omega\mathbb{I}_2,\ \widetilde{\mathbf{curl}}\,(\mathscr{G}_\omega\mathbb{I}_3)\right), \qquad \mathbb{B}^{reg}_\parallel = \left(\boldsymbol{curl}_\perp(\mathscr{G}_\omega\mathbb{I}_2),\ -i\omega\mathscr{G}_\omega\mathbb{I}_3 + \frac{1}{i\omega}\operatorname{Hess}\mathscr{G}_\omega\right)$$

and

$$\begin{aligned} \tilde{\mathbb{E}}^{sing}_\perp &= \left(i\omega\tilde{\nabla}\operatorname{div}_\perp\left(\mathscr{G}_\omega * \underline{\mathscr{G}^{1D}_\omega}\mathbb{I}_2\right),\ -\tilde{\nabla}\operatorname{curl}_\perp\partial_z\left(\mathscr{G}_\omega * \underline{\mathscr{G}^{1D}_\omega}\mathbb{I}_3\right)\right) \\ \mathbb{B}^{sing}_\parallel &= \left(\boldsymbol{curl}_\perp\operatorname{div}_\perp\partial_z\left(\mathscr{G}_\omega * \underline{\mathscr{G}^{1D}_\omega}\mathbb{I}_2\right),\ -i\omega\,\boldsymbol{curl}_\perp\operatorname{curl}_\perp\mathscr{G}_\omega * \underline{\mathscr{G}^{1D}_\omega}\mathbb{I}_3\right), \end{aligned}$$

where $\underline{\mathscr{G}^{1D}_\omega} = \delta_0(x,y)\otimes\mathscr{G}^{1D}_\omega(z)$ *and* $\mathscr{G}^{1D}_\omega(z) = -\gamma\frac{e^{\gamma i\omega|x|}}{2i\omega}$, $\gamma = \operatorname{sign}\operatorname{Im}\omega$.

Proof. In the view of identity (3.14) and of the first one-unknown equation (3.21),

$$-\Delta\tilde{\mathbb{E}}_\perp + \tilde{\nabla}\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp - \omega^2\tilde{\mathbb{E}}_\perp = \widetilde{\mathbf{curl}}\,\mathbb{m}_\parallel + i\omega\tilde{\mathbb{j}}_\perp, \tag{3.26}$$

the first step of the proof consists in the computation of $\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp$. The application of identities (A.9) and (A.8) to (3.24) gives

$$\begin{aligned}\operatorname{div}_\perp\left(\widetilde{\mathbf{curl}}\,\mathbb{B}_\parallel + i\omega\tilde{\mathbb{E}}_\perp\right) &= -\partial_z\operatorname{curl}_\perp\mathbb{B}_\parallel + i\omega\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp = \operatorname{div}_\perp\tilde{\mathbb{j}}_\perp,\\ \operatorname{curl}_\perp\left(\boldsymbol{curl}_\perp\tilde{\mathbb{E}}_\perp - i\omega\mathbb{B}_\parallel\right) &= \partial_z\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp - i\omega\operatorname{curl}_\perp\mathbb{B}_\parallel = \operatorname{curl}_\perp\mathbb{m}_\parallel.\end{aligned}$$

Therefore, $\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp$ solves the following Helmholtz equation

$$-\partial_z^2\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp - \omega^2\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp = i\omega\operatorname{div}_\perp\tilde{\mathbb{j}}_\perp - \partial_z\operatorname{curl}_\perp\mathbb{m}_\parallel.$$

Notice that it is the 1D Helmholtz equation settled in $\mathbb{R}^3$. Therefore, for $\omega\in\mathbb{C}\setminus\mathbb{R}$, the unique distribution $\underline{\mathscr{G}_\omega^{1D}}\in\mathscr{S}'\left(\mathbb{R}^3\right)$ solving $-\partial_z^2\underline{\mathscr{G}_\omega^{1D}} - \omega^2\underline{\mathscr{G}_\omega^{1D}} = \delta_0(\boldsymbol{x})$ is $\underline{\mathscr{G}_\omega^{1D}} = \delta_0(x,y)\otimes\mathscr{G}_\omega^{1D}(z)$ where $\mathscr{G}_\omega^{1D}$ is the fundamental solution of the 1D Helmholtz equation, see Lemma A.3.1. Then, we have

$$\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp = \underline{\mathscr{G}_\omega^{1D}} * \left(i\omega\operatorname{div}_\perp\tilde{\mathbb{j}}_\perp - \partial_z\operatorname{curl}_\perp\mathbb{m}_\parallel\right) = \left(i\omega\operatorname{div}_\perp\left(\underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right),\ -\partial_z\operatorname{curl}_\perp\left(\underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_3\right)\right),$$

where $*$ must be understood as the convolution of a scalar function with a matrix. Next, going back to (3.26), $\tilde{\mathbb{E}}_\perp$ solves a vector 3D Helmholtz equation. Then, given $\mathscr{G}_\omega$ its fundamental solution, we have that

$$\begin{aligned}\tilde{\mathbb{E}}_\perp &= \mathscr{G}_\omega * \left(\widetilde{\mathbf{curl}}\,\mathbb{m}_\parallel + i\omega\tilde{\mathbb{j}}_\perp - \tilde{\nabla}\operatorname{div}_\perp\tilde{\mathbb{E}}_\perp\right)\\ &= \left(i\omega\mathscr{G}_\omega\mathbb{I}_2 + i\omega\tilde{\nabla}\operatorname{div}_\perp\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right),\ \widetilde{\mathbf{curl}}\left(\mathscr{G}_\omega\mathbb{I}_3\right) - \tilde{\nabla}\operatorname{curl}_\perp\partial_z\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_3\right)\right).\end{aligned}$$

Notice that the convolution product $\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}$ is well-defined in the sense of distribution. Indeed, the mapping $(x,y)\mapsto\mathscr{G}_\omega(x,y,\cdot)$ belongs to $L^1\left(\mathbb{R}^2, L^1(\mathbb{R})\right)$ and $\mathscr{G}_\omega^{1D}\in L^1(\mathbb{R})$, so that the convolution $\mathscr{G}_\omega(x,y,\cdot) * \mathscr{G}_\omega^{1D}$ exists for almost every $(x,y)\in\mathbb{R}^2$.

As for $\mathbb{B}_\parallel$, we use (3.24) which yields

$$\begin{aligned}\mathbb{B}_\parallel &= \frac{1}{i\omega}\left(\boldsymbol{curl}_\perp\tilde{\mathbb{E}}_\perp - \mathbb{m}_\parallel\right)\\ &= \Bigg(\boldsymbol{curl}_\perp\left(\mathscr{G}_\omega\mathbb{I}_2\right) + \boldsymbol{curl}_\perp\tilde{\nabla}\operatorname{div}_\perp\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right),\\ &\qquad \frac{1}{i\omega}\boldsymbol{curl}_\perp\widetilde{\mathbf{curl}}\left(\mathscr{G}_\omega\mathbb{I}_3\right) - \frac{1}{i\omega}\delta_0\mathbb{I}_3 - \frac{1}{i\omega}\boldsymbol{curl}_\perp\tilde{\nabla}\operatorname{curl}_\perp\partial_z\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_3\right)\Bigg).\end{aligned}$$

Notice that $\boldsymbol{curl}_\perp\tilde{\nabla} f = -\partial_z\boldsymbol{curl}_\perp f$. Using the last identity and (A.4), the fact $\mathscr{G}_\omega$ and $\underline{\mathscr{G}_\omega^{1}}$ are the fundamental solutions of the 3D and 1D Helmholtz equation in $\mathbb{R}^3$, we have the following expressions

$$\begin{aligned}\mathbb{B}_\parallel &= \Bigg(\boldsymbol{curl}_\perp\left(\mathscr{G}_\omega\mathbb{I}_2\right) - \boldsymbol{curl}_\perp\operatorname{div}_\perp\partial_z\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right),\\ &\qquad -i\omega\mathscr{G}_\omega\mathbb{I}_3 + \frac{1}{i\omega}\operatorname{Hess}\mathscr{G}_\omega - \frac{1}{i\omega}\boldsymbol{curl}_\perp\operatorname{curl}_\perp\left(\mathscr{G}_\omega\mathbb{I}_3 - \partial_z^2\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_3\right)\Bigg).\\ &= \Bigg(\boldsymbol{curl}_\perp\left(\mathscr{G}_\omega\mathbb{I}_2\right) + \boldsymbol{curl}_\perp\operatorname{div}_\perp\partial_z\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right),\\ &\qquad -i\omega\mathscr{G}_\omega\mathbb{I}_3 + \frac{1}{i\omega}\operatorname{Hess}\mathscr{G}_\omega - i\omega\,\boldsymbol{curl}_\perp\operatorname{curl}_\perp\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_3\Bigg).\end{aligned}$$

□

Remark 3.2.2. With the Fourier transform, we have that $\mathscr{F}\mathscr{G}_\omega(\boldsymbol{k}) = (|\boldsymbol{k}|^2 - \omega^2)^{-1}$ and $\mathscr{F}\underline{\mathscr{G}_\omega^{1D}}(\boldsymbol{k}) = (k_z^2 - \omega^2)^{-1}$. Then, we have

$$\mathscr{F}\left[\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\right](\boldsymbol{k}) = \mathscr{F}\mathscr{G}_\omega(\boldsymbol{k}) \times \mathscr{F}\underline{\mathscr{G}_\omega^{1D}}(\boldsymbol{k}) = \frac{1}{(|\boldsymbol{k}|^2 - \omega^2)(k_z^2 - \omega^2)}.$$

Therefore, the separation of the fundamental solution into regular and singular parts is justified by the remark 3.1.14.

As in the classic fundamental solution of the Maxwell's equation, the Hessian of $\mathscr{G}_\omega$ appears in $\mathbb{B}_\parallel^{reg}$. Therefore, although the convolution $\operatorname{Hess}\mathscr{G}_\omega * \boldsymbol{m}_\parallel$ may be well-defined in the sense of the distributions, it may not represent a function. To address this issue, we make an additional assumption on the regularity of $\boldsymbol{m}_\parallel$.

Proposition 3.2.3. *For $\omega \in \mathbb{C} \setminus \mathbb{R}$, given $\boldsymbol{m}_\parallel \in \boldsymbol{H}(\operatorname{curl}_\perp 0; \mathbb{R}^3) \cap \boldsymbol{H}(\operatorname{div}; \mathbb{R}^3)$, $\tilde{\mathbf{j}}_\perp \in \tilde{\mathbf{H}}(\operatorname{div}_\perp 0; \mathbb{R}^3)$, the unique solution of the TE problem (3.20) is*

$$\tilde{\mathrm{E}}_\perp = \tilde{\mathbb{E}}_\perp^{reg} * \begin{pmatrix} \tilde{\mathbf{j}}_\perp \\ \boldsymbol{m}_\parallel \end{pmatrix} = \mathscr{G}_\omega * \left(i\omega\tilde{\mathbf{j}}_\perp + \widetilde{\operatorname{curl}}\,\boldsymbol{m}_\parallel\right),$$

$$\mathrm{B}_\parallel = \mathbb{B}_\parallel^{reg} * \begin{pmatrix} \tilde{\mathbf{j}}_\perp \\ \boldsymbol{m}_\parallel \end{pmatrix} = \mathscr{G}_\omega * \left(\boldsymbol{curl}_\perp\,\tilde{\mathbf{j}}_\perp - i\omega\boldsymbol{m}_\parallel\right) + \frac{1}{i\omega}\nabla\mathscr{G}_\omega * \operatorname{div}\boldsymbol{m}_\parallel.$$

Proof. It is easy to check that $\tilde{\mathbb{E}}_\perp^{sing} * \left(\tilde{\mathbf{j}}_\perp, \boldsymbol{m}_\parallel\right)^\top$ and $\tilde{\mathbb{B}}_\parallel^{sing} * \left(\tilde{\mathbf{j}}_\perp, \boldsymbol{m}_\parallel\right)^\top$ vanish under the above constraints on $\boldsymbol{m}_\parallel$ and $\tilde{\mathbf{j}}_\perp$. For example,

$$\tilde{\nabla}\operatorname{div}_\perp\left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right) * \tilde{\mathbf{j}}_\perp = \left(\mathscr{G}_\omega * \underline{\mathscr{G}_\omega^{1D}}\right) * \tilde{\nabla}\operatorname{div}_\perp\tilde{\mathbf{j}}_\perp = 0.$$

The regularity comes from the fact that $\mathscr{G}_\omega, \partial_i\mathscr{G}_\omega \in L^1\left(\mathbb{R}^3\right)$ for $i \in \{x, y, z\}$, and that the source terms are square-integrable. ☐

Remark 3.2.4. The complete fundamental solution to the usual Maxwell's equations can be written in a similar manner as $\underline{\mathbb{E}} = (\mathbb{E}, \mathbb{B})$ and $\underline{\mathbb{B}} = (\mathbb{B}, -\mathbb{E})$, where $\mathbb{E}, \mathbb{B} \in \mathscr{S}'\left(\mathbb{R}^3\right)^{3\times 3}$ with

$$\mathbb{E} = i\omega\mathscr{G}_\omega\mathbb{I}_3 - \frac{1}{i\omega}\operatorname{Hess}\mathscr{G}_\omega, \qquad \mathbb{B} = \boldsymbol{curl}\,\mathscr{G}_\omega\mathbb{I}_3.$$

Then, given $\boldsymbol{j}_\perp$ and $\boldsymbol{m}_\parallel$ as in Proposition 3.2.3, one can verify that the convolutions of $\underline{\mathbb{E}}$, $\underline{\mathbb{B}}$ with the source terms coincide with the expressions given in Proposition 3.2.3.

Let us now prove the existence of solution for $\omega \in \mathbb{R} \setminus \{\pm\omega_p, 0\}$. First, recall the definition of the outgoing fundamental solution.

Lemma 3.2.5. *For all $\omega \in \mathbb{R} \setminus \{0\}$, we have $\lim_{\nu \to 0+} \mathscr{G}_{\omega+i\nu} = \mathscr{G}_\omega^+$ in $W_{loc}^{1,1}\left(\mathbb{R}^3\right)$ where $G_\omega^+(\boldsymbol{x}) = \frac{e^{i\omega|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}$.*

This allows us to state the following existence theorem.

Theorem 3.2.6 (Existence of classic solutions). *Let $\omega \in \mathbb{R} \setminus \{0\}$, $\boldsymbol{m}_\parallel \in \boldsymbol{H}(\operatorname{curl}_\perp 0; \mathbb{R}^3) \cap \left(\mathscr{C}_0^2\left(\mathbb{R}^3\right)\right)^3$, and $\tilde{\mathbf{j}}_\perp \in \tilde{\mathbf{H}}(\operatorname{div}_\perp 0; \mathbb{R}^3) \cap \left(\mathscr{C}_0^2\left(\mathbb{R}^3\right)\right)^2$. Then,*

$$\tilde{\mathrm{E}}_\perp^+ = \mathscr{G}_\omega^+ * \left(i\omega\tilde{\mathbf{j}}_\perp + \widetilde{\operatorname{curl}}\,\boldsymbol{m}_\parallel\right), \qquad \boldsymbol{B}_\parallel^+ = \mathscr{G}_\omega^+ * \left(\boldsymbol{curl}_\perp\,\tilde{\mathbf{j}}_\perp - i\omega\boldsymbol{m}_\parallel\right) + \frac{1}{i\omega}\nabla\mathscr{G}_\omega^+ * \operatorname{div}\boldsymbol{m}_\parallel.$$

are such that $\tilde{\mathrm{E}}_\perp^+ \in \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^2$, $\boldsymbol{B}_\parallel^+ \in \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^3$, and satisfy (3.20) in a strong sense.

3.2.2 Silver-Müller condition

Classically, in the case of unbounded problems, the uniqueness is ensured by a *radiation condition*, e.g., the Sommerfeld condition for the Helmholtz problem, or the Silver-Müller condition for the harmonic Maxwell's problem. In fact, without the radiation condition, it is easy to construct infinitely many solutions of the TE problem (3.20) by considering convex combination of $\left(\tilde{\mathrm{E}}_\perp^+, \boldsymbol{B}_\parallel^+\right)$ given in Theorem 3.2.6 and

$$\tilde{\mathrm{E}}_\perp^- = \mathscr{G}_\omega^- * \left(i\omega\tilde{\mathrm{j}}_\perp + \widetilde{\mathbf{curl}}\,\boldsymbol{m}_\parallel\right), \qquad \boldsymbol{B}_\parallel^- = \mathscr{G}_\omega^- * \left(\boldsymbol{curl}_\perp\,\tilde{\mathrm{j}}_\perp - i\omega\boldsymbol{m}_\parallel\right) + \frac{1}{i\omega}\nabla\mathscr{G}_\omega^- * \operatorname{div}\boldsymbol{m}_\parallel,$$

where $\mathscr{G}_\omega^- := \lim_{\nu\to0+}\mathscr{G}_{\omega-i\nu}$ and $\mathscr{G}_{\omega-i\nu}$ given by (3.25). With a normalized speed of light $c = 1$, recall that the weak form of the usual Silver-Müller conditions reads

$$\int_{S_R} |\boldsymbol{E} - \boldsymbol{B}\times\boldsymbol{n}|^2\,\mathrm{d}s \xrightarrow[R\to+\infty]{} 0, \tag{3.27}$$

$$\int_{S_R} |\boldsymbol{B} + \boldsymbol{E}\times\boldsymbol{n}|^2\,\mathrm{d}s \xrightarrow[R\to+\infty]{} 0, \tag{3.28}$$

where S_R is the sphere of radius R, refer to [24, §6.2, 17, §1.2] for details. These two conditions are equivalent and both select the outgoing solution. Then, according to the Remark 3.2.4, the outgoing solution $\left(\boldsymbol{E}_\perp^+, \boldsymbol{B}_\parallel^+\right)$ satisfies the conditions above, where $\boldsymbol{E}_\perp^+ = \left(\tilde{\mathrm{E}}_\perp^+, 0\right)^\top$.

Theorem 3.2.7. *Let $\omega \in \mathbb{R}\setminus\{0\}$ and $\left(\tilde{\mathrm{E}}_\perp, \boldsymbol{B}_\parallel\right) \in \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^2 \times \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^3$ be a solution of the homogeneous reduced TE problem* (3.20)*. If $\left(\boldsymbol{E}_\perp, \boldsymbol{B}_\parallel\right)$ satisfies either* (3.27) *or* (3.28)*, then it vanishes.*

3.3 Transverse magnetic problem

This section is devoted to the study of the reduced transverse magnetic problem (3.22). Because this problem is less “classic” than the previous one, we first study its scalar counterpart, which has the same importance for the reduced TM problem as the Helmholtz equation for the usual Maxwell's problem. Then, we study the existence and uniqueness of the solution for the TM problem.

3.3.1 Associated scalar equation

The scaled operator appears naturally if we use the identity (3.15) with the one unknown equation (3.23) involving $\tilde{\mathrm{B}}_\perp$. The resolution reduced TM problem is clearly linked to the resolution of the scaled Helmholtz equation:

$$\left|\begin{array}{l}\text{find } u \text{ such that}\\ -\beta(\omega)^{-1}\left(\partial_x^2 + \partial_y^2\right)u - \partial_z^2 u - \omega^2 u = f, \quad \text{in } \mathscr{D}'\left(\mathbb{R}^3\right),\end{array}\right. \qquad \text{with } \beta(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \tag{3.29}$$

The spaces to which u and f belong will be specified later. Obviously, $\beta(\omega) > 0$ for $\omega \in (\omega_p, +\infty)$ and $\beta(\omega) < 0$ for $\omega \in (0, \omega_p)$. Therefore, the equation is elliptic for $\omega \in (\omega_p, +\infty)$ and is hyperbolic for $\omega \in (0, \omega_p)$. We are interested in the latter case.

Let us make some opening remarks on this equation. The existence of a distribution $\mathscr{G} \in \mathscr{S}'\left(\mathbb{R}^3\right)$ which solves the equation above with a Dirac mass as a source term has already been treated in the classic literature, see e.g., [34, Theorem 6.2.3, p. 141]. However, the above equation is mostly encountered in the study of wave equations in time regime, i.e., when $\beta < 0$ does not depend on ω, and the variable z is replaced by $t \in \mathbb{R}$:

$$-\beta^{-1}\left(\partial_x^2 + \partial_y^2\right)u - \partial_t^2 u - \omega^2 u = f(x, y, t).$$

In this case, having causal solutions is natural and desirable. This implies that the support of a solution u should be a subset of the cone

- $\left\{(x, y, t) \,:\, |\beta|^{-1/2}\, t > (x^2 + y^2)^{1/2}\right\}$ for a forward solution, i.e., vanishing for $t < 0$,
- $\left\{(x, y, t) \,:\, |\beta|^{-1/2}\, t < -(x^2 + y^2)^{1/2}\right\}$ for a backward solution, i.e., vanishing for $t > 0$.

In both cases, a solution should vanish on the "non-causal" cone

$$\left\{(x, y, t) \,:\, |\beta|^{-1/2}\, |t| < (x^2 + y^2)^{1/2}\right\}.$$

In our case, the "causality" is a priori not required in the sense that the support of the fundamental solution of (3.29) is not necessarily a subset of the cones

$$\left\{(x, y, z) \in \mathbb{R}^3 \,:\, |z| > |\beta(\omega)|^{1/2}(x^2 + y^2)^{1/2}\right\}.$$

As a matter of fact, we seek the solution u to be the limiting absorption solution of $u_{\omega+i\nu}$ when $\nu > 0$ goes to 0.

Finally, we denote the equation (3.29) as the *scaled Helmholtz equation* even if $\beta(\omega)$ may be negative. This equation must be distinguished with the *usual scaled Helmholtz equation*

$$-\beta^{-1}\left(\partial_x^2 + \partial_y^2\right)u - \partial_z^2 u - \omega^2 u = f, \tag{3.30}$$

with $\beta > 0$. Some results on the usual scaled Helmholtz equation can be found in Appendix A.3.

This problem has been extensively studied in the 2D case in [22] for $(0, \omega_p)$. Therefore, the analysis presented in this section follows the same steps: the problem with absorption is studied first. Then, an existence theorem via the computation of a fundamental solution is stated. Finally, the section is concluded by a uniqueness condition

3.3.1.1 Problem with absorption

First consider the following problem, with $\operatorname{Im}\omega \neq 0$:

$$\left|\begin{array}{l} \text{find } u_\omega \in H^1(\Omega) \text{ such that} \\ -\beta(\omega)^{-1}\left(\partial_x^2 + \partial_y^2\right)u_\omega - \partial_z^2 u_\omega - \omega^2 u_\omega = f, \end{array}\right. \tag{3.31}$$

where $f \in L^2(\Omega)$. Then, we have the following well-posedness result.

Proposition 3.3.1. *Given $\omega \in \mathbb{C} \setminus \mathbb{R}$, the problem (3.31) is well-posed for all $f \in L^2(\Omega)$, and there is $C(\omega) > 0$ such that*

$$\|u_\omega\|_{H^1(\mathbb{R}^3)} \leq \frac{|\omega|}{C(\omega)}\, \|f\|_{L^2(\mathbb{R}^3)}\,.$$

Proof. Notice that we can express the equation of the problem (3.31) in the divergence equation

$$-\operatorname{div}\left(\beta(\omega)^{-1}\mathbb{e}\nabla u\right)-\omega^2 u=f,\qquad \mathbb{e}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\beta(\omega)\end{pmatrix}. \tag{3.32}$$

Then, the problem (3.31) is equivalent to

$$\left|\begin{array}{l}\text{Find } u_\omega\in H^1(\Omega) \text{ such that}\\ a_\omega(u_\omega,v)=\ell(v),\qquad \text{for all } v\in H^1(\Omega),\end{array}\right.$$

where

$$a_\omega(u,v)=\int_{\mathbb{R}^3}\left(\beta^{-1}\mathbb{e}\nabla u\cdot\overline{\nabla v}-\omega^2 u\bar{v}\right)\mathrm{d}\boldsymbol{x},\qquad\text{and}\qquad \ell(v)=\int_{\mathbb{R}^3} f\,\bar{v}\,\mathrm{d}\boldsymbol{x}.$$

Obviously, a_ω and ℓ are continuous with respect to the $H^1\left(\mathbb{R}^3\right)$-norm. A quick computation yields that $\operatorname{Im}\left(\overline{\omega}\beta(\omega)^{-1}\right)$ and $\operatorname{Im}\overline{\omega}$ have the same sign. Therefore, we have for $u\in H^1\left(\mathbb{R}^3\right)$

$$\operatorname{Im} a_\omega(u,\omega u)=\operatorname{Im}\left(\overline{\omega}\beta(\omega)^{-1}\right)\left(\|\partial_x u\|^2_{L^2(\mathbb{R}^3)}+\left\|\partial_y u\right\|^2_{L^2(\mathbb{R}^3)}\right)+\operatorname{Im}(\overline{\omega})\,\|\partial_z u\|^2_{L^2(\mathbb{R}^3)}+\operatorname{Im}(\overline{\omega})|\omega|^2\,\|u\|^2_{L^2(\mathbb{R}^3)}.$$

Consequently, there is a constant $C(\omega)>0$ such that for all $u\in H^1\left(\mathbb{R}^3\right)$

$$|\operatorname{Im} a_\omega\left(u,\omega u\right)|\geq C(\omega)\,\|u\|^2_{H^1(\mathbb{R}^3)}.$$

The Lax-Milgram theorem allows us to conclude about the well-posedness. ☐

A flaw in previous proposition is that it cannot be applied for $\omega\in\mathbb{R}$. Indeed, the constant $C(\omega)$ goes to zero when ω approaches the real axis. Therefore, in order to show the existence of a limiting absorption solution, another characterization of the solution is needed. In this view, the following proposition gives the fundamental solution $\mathscr{G}_\omega^\beta\in\mathcal{S}'\left(\mathbb{R}^3\right)$ of the problem:

$$-\beta(\omega)^{-1}\left(\partial_x^2+\partial_y^2\right)u-\partial_z^2 u-\omega^2 u=\delta_0, \tag{3.33}$$

for $\omega\in\mathbb{C}\setminus\mathbb{R}$. From this point forward, we employ the principal determination of the complex square root, with the branch cut along $(-\infty,0]$. Notice that $\operatorname{Re}\sqrt{z}>0$ for all $z\in\mathbb{C}\setminus(-\infty,0]$.

Proposition 3.3.2. *The unique solution of* (3.33) *for* $\omega\in\mathbb{C}\setminus\mathbb{R}$ *is*

$$\mathscr{G}_\omega^\beta(x,y,z)=\beta(\omega)\frac{\exp\left(\gamma i\omega\sqrt{\beta(\omega)(x^2+y^2)+z^2}\right)}{4\pi\sqrt{\beta(\omega)(x^2+y^2)+z^2}},\qquad \gamma=\operatorname{sign}(\operatorname{Im}\omega). \tag{3.34}$$

Moreover, we have $\mathscr{G}_\omega^\beta\in L^1\left(\mathbb{R}^3\right)$.

This fundamental solution must be compared with the fundamental solution of the scaled Helmholtz equation (3.30), see Lemma A.3.3. However, the main difference is that $\beta(\omega)$ is not real in general so that Lemma A.3.3 cannot be applied directly. Therefore, the proof below develops a stronger argument using the analyticity of the problem.

Remark 3.3.3. The existence of two different fundamental solutions would contradict Proposition 3.3.1. Indeed, given $f\in L^2\left(\mathbb{R}^3\right)$, $\mathscr{G}_\omega^\beta * f$ is the solution of (3.31).

Before starting the proof, let us fix the convention used with the Fourier transform. Recall that the unitary Fourier transform in $\mathbb{R}^3$ is given by

$$\mathscr{F}[u](\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} u(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{d}\boldsymbol{x}.$$

Still, in the view of the scaled Helmholtz equation 3.31, it is appropriate to consider the partial unitary Fourier transform along the x, y-directions:

$$\mathscr{F}_{x,y}[u](k_x, k_y, z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x, y, z) e^{-i\left(xk_x + yk_y\right)} \mathrm{d}x\mathrm{d}y.$$

Proof of proposition 3.3.2. The proof follows [22], and is separated in two steps. The first step consists in showing that there is a unique fundamental solution $\mathscr{G}_\omega^\beta$ for all $\omega \in \mathbb{C} \setminus \mathbb{R}$ and that the mapping $\omega \mapsto \mathscr{G}_\omega^\beta$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. Next, we compute an explicit expression for $\mathscr{G}_\omega^\beta$ on an appropriate subset of $\mathbb{C} \setminus \mathbb{R}$. The analytic continuation theorem will allow us to conclude since this expression will also be analytic on $\mathbb{C} \setminus \mathbb{R}$.

First, we apply the partial Fourier transform to (3.33) along the x, y-directions, which results in a 1D Helmholtz equation:

$$-\partial_z^2 \mathscr{F}_{x,y} \mathscr{G}_\omega^\beta - \left(\omega^2 - \beta(\omega)^{-1}(|k_x|^2 + |k_y|^2)\right) \mathscr{F}_{x,y} \mathscr{G}_\omega^\beta = \delta_0(z).$$

Let

$$\sigma_\omega(k_x, k_y) := \sqrt{\beta(\omega)^{-1}\left(k_x^2 + k_y^2\right) - \omega^2}.$$

It is defined unambiguously for k_x, k_y, ω such that $\beta(\omega)^{-1}\left(k_x^2 + k_y^2\right) - \omega^2 \in \mathbb{C} \setminus (-\infty, 0)$. A quick computation gives

$$\operatorname{Im}\left(\beta(\omega)^{-1}\right) = -\operatorname{Im}(\omega^2) \frac{\omega_p^2}{|\omega^2 - \omega_p^2|^2},$$

so that

$$\operatorname{Im}\left(\beta(\omega)^{-1}\left(k_x^2 + k_y^2\right) - \omega^2\right) = -\operatorname{Im}(\omega^2)\left(1 + \frac{|\boldsymbol{k}_\parallel|^2 \omega_p^2}{|\omega^2 - \omega_p^2|^2}\right) = -2\operatorname{Re}(\omega)\operatorname{Im}(\omega)\left(1 + \frac{|\boldsymbol{k}_\parallel|^2 \omega_p^2}{|\omega^2 - \omega_p^2|^2}\right),$$

where $|\boldsymbol{k}_\parallel|^2 = k_x^2 + k_y^2$. Notice that the last expression vanishes only if $\omega \in \mathbb{R} \cup i\mathbb{R}$. Of course, we exclude the case $\omega \in \mathbb{R}$, and for $\omega \in i\mathbb{R} \setminus \{0\}$, we have $\beta(\omega)^{-1} \in (0, +\infty)$, and similarly for $\beta(\omega)^{-1}\left(k_x^2 + k_y^2\right) - \omega^2$. Therefore, σ_ω is well-defined for all $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $(k_x, k_y) \in \mathbb{R}^2$, and the 1D Helmholtz equation becomes

$$-\partial_z^2 \mathscr{F}_{x,y} \mathscr{G}_\omega^\beta - (i\sigma_\omega)^2 \mathscr{F}_{x,y} \mathscr{G}_\omega^\beta = \delta_0(z).$$

By definition of the complex square root, $\operatorname{Im}(i\sigma_\omega) = \operatorname{Re}\sigma_\omega > 0$. Therefore, according to lemma A.3.1, we have for all $\omega \in \mathbb{C} \setminus \mathbb{R}$

$$\mathscr{F}_{x,y} \mathscr{G}_\omega^\beta(k_x, k_y, z) = \frac{e^{-\sigma_\omega(k_x,k_y)|z|}}{2\sigma_\omega(k_x, k_y)}, \qquad \forall (k_x, k_y, z) \in \mathbb{R}^3. \tag{3.35}$$

On one hand, the mapping $(k_x, k_y) \mapsto \sigma_\omega(k_x, k_y)$ cannot vanish for $\omega \in \mathbb{C} \setminus \mathbb{R}$. On the other hand, $|\sigma_\omega(k_x, k_y)| \gtrsim (k_x^2 + k_y^2)^{1/2}$ and $\operatorname{Re}\sigma_\omega(k_x, k_y) \gtrsim (k_x^2 + k_y^2)^{1/2}$ for large $(k_x^2 + k_y^2)^{1/2}$. Thus, $(k_x, k_y) \mapsto \mathscr{F}_{x,y} \mathscr{G}_\omega^\beta(k_x, k_y, z) \in L^1\left(\mathbb{R}^2\right)$ for all $z \in \mathbb{R}^*$ and

$$\mathscr{G}_\omega^\beta(x, y, z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{-\sigma_\omega(k_x,k_y)|z|}}{2\sigma_\omega(k_x, k_y)} e^{i\left(k_x x + k_y y\right)} \mathrm{d}x\mathrm{d}y. \tag{3.36}$$

The mapping $\omega \mapsto \sigma_\omega(k_x, k_y)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. Therefore, classic integration theorems (see, e.g., [57, Theorem I.7, p. 308]) imply that $\omega \mapsto \mathcal{G}^\beta_\omega(x, y, z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$ for almost every $(x, y, z) \in \mathbb{R}^3$.

Next, we remark that $\beta(\omega) \in (0, +\infty)$ for $\omega \in i\mathbb{R}$. As a consequence, applying Lemma A.3.3 to any $\omega \in i\mathbb{R}$ with $\tilde{\beta} = \beta(\omega)$ yields

$$\mathcal{G}^\beta_\omega(x, y, z) = \beta(\omega) \frac{\exp\left(\gamma i\omega\sqrt{\beta(\omega)(x^2+y^2)+z^2}\right)}{4\pi\sqrt{\beta(\omega)(x^2+y^2)+z^2}}, \qquad \gamma = \operatorname{sign}(\operatorname{Im}\omega).$$

Due to the analyticity of the mapping $\omega \mapsto \beta(\omega)$ on $\mathbb{C}\setminus\{0\}$ and the fact that $\beta(\omega) \in (-\infty, 0]$ only if $\omega \in \mathbb{R}$, the mapping of ω to the latter expression is also analytic on $\mathbb{C} \setminus \mathbb{R}$ for all $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$. The proof is concluded, as announced, with the application of the analytic continuation theorem.

Finally, given $\omega \in \mathbb{C} \setminus \mathbb{R}$, one may check that $\mathcal{G}^\beta_\omega$ decreases exponentially at infinity. Indeed,

$$\left|\mathcal{G}^\beta_\omega(\boldsymbol{x})\right| = \frac{|\beta(\omega)| \exp\left(-\gamma \operatorname{Im}\left(\omega\sqrt{\beta(\omega)(x^2+y^2)+z^2}\right)\right)}{4\pi\left|\beta(\omega)(x^2+y^2)+z^2\right|^{1/2}}, \qquad \gamma = \operatorname{sign}(\operatorname{Im}\omega),$$

and one may check that $\gamma \operatorname{Im}\left(\omega\sqrt{\beta(\omega)(x^2+y^2)+z^2}\right)$ is positive and tends to infinity as $|\boldsymbol{x}|$ increases. On the other hand, it is also easy to check that $\mathcal{G}^\beta_\omega \in L^1_{loc}(B_1(0))$ with the spherical coordinates. The two previous arguments yields $\mathcal{G}^\beta_\omega \in L^1\left(\mathbb{R}^3\right)$. □

Remark 3.3.4. The same kind of argument can be applied for the complex scaled Helmholtz equation, by first showing that $\beta \mapsto \mathcal{G}^\beta_\omega$ is analytic via the integral representation, and then by continuing analytically (A.13).

As a direct consequence of the proposition, the convolution of $\mathcal{G}^\beta_\omega$ with any $L^2\left(\mathbb{R}^3\right)$-function is valid. Therefore, the unique solution exhibited in Proposition 3.3.1 can be represented as the following lemma.

Corollary 3.3.5. *Given $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $f \in L^2\left(\mathbb{R}^3\right)$, the unique solution of (3.31) is $u_\omega = \mathcal{G}^\beta_\omega * f$.*

3.3.1.2 Limiting absorption solution

Now that the fundamental solution of (3.33) is known for $\omega \in \mathbb{C} \setminus \mathbb{R}$, the following proposition describes the behavior of $\mathcal{G}^\beta_\omega$ when $\operatorname{Im}\omega$ tends to 0. Due to the branch cut of the square root, the signs of $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$ play an important role. For this reason, we restrict ourselves to the case $\operatorname{Re}\omega > 0$. Let us define for all $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$

$$d_\beta(\boldsymbol{x}) := \left|\beta(\omega)(x^2+y^2)+z^2\right|^{1/2}. \tag{3.37}$$

Remark 3.3.6. This function must be compared to the "elliptic" distance $\sqrt{|\beta|(x^2+y^2)+z^2}$ to the origin which is equivalent to the euclidean distance ; we use the term "elliptic" because the level set are ellipse. Obviously, d_β is not a distance. Nevertheless, it can be viewed as it in the hyperbolic system of coordinates ; notice that the level sets of d_β are hyperbola. Recall that the hyperbolic system of coordinates writes

$$\sqrt{x^2+y^2} = \frac{\rho\sinh\theta}{|\beta|^{1/2}}, \qquad z = \rho\cosh\theta$$

with $\rho > 0$ and $\theta \in (0, +\infty)$, so that $d_\beta(\boldsymbol{x}) = \rho$. However, major difficulties appear with the "hyperbolic" distance. Indeed, the hyperbolic system of coordinates is bijective from the cone $\left\{z > \sqrt{|\beta|(x^2+y^2)}\right\}$ to $\mathbb{R}^+ \times \mathbb{R}^2$. In particular, the "hyperbolic" distance vanishes on the boundaries of the cone and not only at the origin of the system of coordinates.

Proposition 3.3.7. *Let $\omega \in (0, +\infty) \setminus \{\omega_p\}$. Then,*

$$\mathscr{G}^\beta_{\omega,\pm}(\boldsymbol{x}) := \lim_{\nu \to 0+} \mathscr{G}^\beta_{\omega \pm i\nu}(\boldsymbol{x})$$

exists for almost every $\boldsymbol{x} \in \mathbb{R}^3$. Moreover, the convergence holds in $L^1_{loc}\left(\mathbb{R}^3\right)$ and there is the two following cases:

- *if $\omega > \omega_p$, then*

$$\mathscr{G}^\beta_{\omega,\pm}(\boldsymbol{x}) = \beta(\omega) \frac{e^{\pm i\omega d_\beta(\boldsymbol{x})}}{4\pi d_\beta(\boldsymbol{x})},$$

- *if $\omega \in (0, \omega_p)$, then*

$$\mathscr{G}^\beta_{\omega,\pm}(\boldsymbol{x}) = \begin{cases} \beta(\omega) \dfrac{e^{\pm i\omega d_\beta(\boldsymbol{x})}}{4\pi d_\beta(\boldsymbol{x})} & \text{if } \boldsymbol{x} \in C^\beta_p := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ z^2 > |\beta(\omega)|(x^2+y^2)\right\}, \\ \pm\beta(\omega) \dfrac{e^{-\omega d_\beta(\boldsymbol{x})}}{4i\pi d_\beta(\boldsymbol{x})} & \text{if } \boldsymbol{x} \in C^\beta_e := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ z^2 < |\beta(\omega)|(x^2+y^2)\right\}. \end{cases} \tag{3.38}$$

Before proving this proposition, let us make some remarks. It is clear that the appearance of different cases is due to the presence of the complex square root. In the first case, $\omega > \omega_p$ and $\beta(\omega) > 0$, so that we retrieve the classic outgoing (respectively, ingoing) fundamental solution $\mathscr{G}^\beta_{\omega,+}$ (resp. $\mathscr{G}^\beta_{\omega,-}$) of the scaled Helmholtz solution. In the second case, we observe first that the solutions are not "causal" since they obviously do not vanish in the "non-causal" cone C^β_e. The outgoing and ingoing solutions decrease exponentially in C^β_e (where "e" stands for evanescent) as $d_\beta(\boldsymbol{x})$ increases, while a propagative behavior is observed in C^β_p (where "p" stands for propagative), see Figure 3.2.

Proof. Until the end of the proof, we use $\rho = \sqrt{x^2+y^2}$ and $d^{\pm\nu}_\beta(\boldsymbol{x}) = \sqrt{\beta(\omega \pm i\nu)\rho^2 + z^2}$ for $\omega \in (0, +\infty) \setminus \{\omega_p\}$ and $\nu > 0$. The proof is divided in three parts. First, the pointwise limit is computed. Next, we prove that $\mathscr{G}^\beta_{\omega,\pm} \in L^1_{loc}\left(\mathbb{R}^3\right)$. Finally, the convergence in $L^1_{loc}\left(\mathbb{R}^3\right)$ is established.

Step 1 : pointwise limit. If $\beta(\omega)\rho^2 + z^2 > 0$, then

$$d^{\pm\nu}_\beta(\boldsymbol{x}) \xrightarrow[\nu \to 0+]{} d_\beta(\boldsymbol{x}).$$

This occurs when $\omega > \omega_p$, or $\omega \in (0, \omega_p)$ and $\boldsymbol{x} \in C^\beta_p$. Next if $\beta(\omega)\rho^2 + z^2 < 0$, i.e., $\omega \in (0, \omega_p)$ and $\boldsymbol{x} \in C^\beta_e$, then the limit depends on the sign of

$$\operatorname{Im}\left(\beta(\omega \pm i\nu)\rho^2 + z^2\right) = \pm \frac{2\rho^2\omega_p^2\omega\nu}{(\omega^2+\nu^2)^2}. \tag{3.39}$$

Then, using the definition of the complex square root,

$$d^{\pm\nu}_\beta(\boldsymbol{x}) \xrightarrow[\nu \to 0+]{} \pm i d_\beta(\boldsymbol{x}).$$

The incorporation of the above limits in (3.34) gives the pointwise limit almost everywhere.

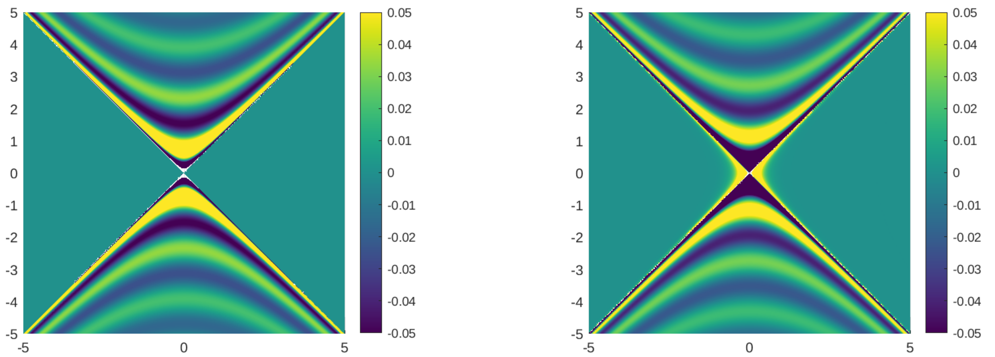


Figure 3.2: The real part (on the left) and the imaginary part (on the right) of the fundamental solution $\mathscr{G}^{\beta}_{\omega,+}$ in the plane $\{y = 0\}$. The abscissa corresponds to the variable x and the ordinate to the variable z. We have chosen $\omega = 4$ and $\omega_p = 4\sqrt{2}$, such that $\beta = -1$. Notice that the fundamental solution is propagative in the upper and lower cones C^{β}_{p}, and is evanescent in the rest of the domain C^{β}_{e}. Moreover, because the fundamental solution is singular, the colors near the boundaries of C^{β}_{p} and C^{β}_{e} become saturated.

Step 2 : $\mathscr{G}^{\beta}_{\omega,\pm} \in L^1_{loc}(\mathbb{R}^3)$. Next, the end of proof focuses on the case $\omega \in (0, \omega_p)$, where $\beta(\omega) = 1 - \frac{\omega_p^2}{\omega^2} < 0$, since the case $\omega > \omega_p$ is analogous to the scaled Helmholtz equation. Let $R > 0$ and the cylinder $B_R = \{(x, y, z), \max\left(|z|, |\beta|^{1/2}\rho\right) \leq R\}$. We define $K_{p,R} = C^{\beta}_{p} \cap B_R$ and $K_{e,R} = C^{\beta}_{e} \cap B_R$, see Figure 3.3. Then, we have

$$\int_{K_{p,R}} \left|\mathscr{G}^{\beta}_{\omega,\pm}(\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x} = \frac{|\beta|}{4\pi} \int_{\varphi=0}^{2\pi} \int_{z=-R}^{R} \int_{\rho=0}^{|z|/|\beta|^{1/2}} \frac{\rho \,\mathrm{d}\rho}{\sqrt{z^2 - |\beta|\rho^2}} \mathrm{d}z = \frac{R^2}{2},$$

and

$$\int_{K_{e,R}} \left|\mathscr{G}^{\beta}_{\omega,\pm}(\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x} \leq \frac{|\beta|}{4\pi} \int_{K_{e,R}} \frac{\mathrm{d}\boldsymbol{x}}{d_{\beta}(\boldsymbol{x})} = |\beta| \int_{z=0}^{R} \int_{\rho=z}^{R/|\beta|^{1/2}} \frac{\rho \,\mathrm{d}\rho}{\sqrt{|\beta|\rho^2 - z^2}} \mathrm{d}z = \frac{\pi R^2}{4}.$$

This shows that $\mathscr{G}^{\beta}_{\omega,\pm} \in L^1_{loc}(\mathbb{R}^3)$.

Step 3 : convergence in $L^1_{loc}(\mathbb{R}^3)$. We focus on the proof for $\mathscr{G}^{\beta}_{\omega,+}$ since the proofs for $\mathscr{G}^{\beta}_{\omega,+}$ and $\mathscr{G}^{\beta}_{\omega,-}$ are identical. It suffices to show the convergence on $K_{p,R}$ and $K_{e,R}$ for any $R > 0$. First, for all $\boldsymbol{x} \in K_{p,R}$, we have

$$\begin{aligned}
&\mathscr{G}^{\beta}_{\omega+i\nu}(\boldsymbol{x}) - \mathscr{G}^{\beta}_{\omega,+}(\boldsymbol{x}) \\
&\quad = \beta(\omega + i\nu) \frac{e^{i\omega d^{\nu}_{\beta}(\boldsymbol{x})}}{4\pi d^{\nu}_{\beta}(\boldsymbol{x})} - \beta(\omega) \frac{e^{i\omega d_{\beta}(\boldsymbol{x})}}{4\pi d_{\beta}(\boldsymbol{x})} \\
&\quad = \underbrace{\frac{\beta(\omega + i\nu) e^{i\omega d^{\nu}_{\beta}(\boldsymbol{x})}}{4\pi}}_{(A)} \left(\frac{1}{d^{\nu}_{\beta}(\boldsymbol{x})} - \frac{1}{d_{\beta}(\boldsymbol{x})} \right) + \underbrace{\frac{\beta(\omega + i\nu) e^{i\omega d^{\nu}_{\beta}(\boldsymbol{x})} - \beta(\omega) e^{i\omega d_{\beta}(\boldsymbol{x})}}{4\pi d_{\beta}(\boldsymbol{x})}}_{(B)}.
\end{aligned}$$

The term denoted as (B) is easily bounded by $1/d_{\beta}(\boldsymbol{x}) \in L^1_{loc}(\mathbb{R}^3)$, because the denominator is uniformly bounded on $K_{p,R}$. In the same way, the term in front of (A) is also uniformly bounded on $K_{p,R}$. Therefore, it remains to show that

$$\frac{1}{d_\beta^\nu(\boldsymbol{x})} - \frac{1}{d_\beta(\boldsymbol{x})} \xrightarrow[\nu\to 0+]{L^1(K_{p,R})} 0.$$

An easy computation shows that

$$\frac{1}{d_\beta^\nu(\boldsymbol{x})} - \frac{1}{d_\beta(\boldsymbol{x})} = -\frac{1}{d_\beta(\boldsymbol{x})}\left(1 - \frac{1}{\sqrt{\frac{\beta(\omega+i\nu)\rho^2+z^2}{\beta(\omega)\rho^2+z^2}}}\right) = -\frac{1}{d_\beta(\boldsymbol{x})}\left(1 - \frac{1}{\sqrt{1+\frac{(\beta(\omega+i\nu)-\beta(\omega))\rho^2}{\beta(\omega)\rho^2+z^2}}}\right) \tag{3.40}$$

Again, $1/d_\beta(\boldsymbol{x}) \in L^1_{loc}(\mathbb{R}^3)$, so it suffices to bound the last part in L^∞-norm uniformly with respect to ν small enough. However, it is not obvious because $\beta(\omega)\rho^2 + z^2$ goes to 0 near the boundary of C_p^β, so that the ratio

$$\frac{(\beta(\omega+i\nu)-\beta(\omega))\rho^2}{\beta(\omega)\rho^2+z^2}$$

is not uniformly bounded. As a consequence, following the approach in [22, Appendix D], we shall decompose $K_{p,R} = K_{p,R}^{reg,\nu} \cup K_{p,R}^{sing,\nu}$, with the definitions of $K_{p,R}^{reg,\nu}$ and $K_{p,R}^{sing,\nu}$ provided below.

Since $s \in \mathbb{C} \setminus (-\infty,-1] \mapsto 1 - \frac{1}{\sqrt{1+s}}$ is analytic in a neighborhood of 0, there are two positive constants C_1, C_2 such that

$$|s| \le C_1 \implies \left|1 - \frac{1}{\sqrt{1+s}}\right| \le C_2|s|. \tag{3.41}$$

In the same way, for $\nu > 0$ small enough, there is $C_\beta > 0$ such that $|\beta(\omega+i\nu) - \beta(\omega)| \le C_\beta \nu$. Then consider the following sets

$$K_{p,R}^{reg,\nu} := \left\{(x,y,z) \in \mathbb{R}^3 \,:\, \left(|\beta| + \frac{C_\beta \nu}{C_1}\right)\rho^2 \le z^2 \le R\right\}, \tag{3.42}$$

$$K_{p,R}^{sing,\nu} := \left\{(x,y,z) \in \mathbb{R}^3 \,:\, \begin{vmatrix} |\beta|\rho^2 \le & z^2 \le R, \\ & z^2 < \left(|\beta| + \frac{C_\beta \nu}{C_1}\right)\rho^2 \end{vmatrix}\right\}, \tag{3.43}$$

see Figure 3.3. Notice that the set $K_{p,R}^{reg,\nu}$ is designed in a such way that, for all $\boldsymbol{x} \in K_{p,R}^{reg,\nu}$ and for $\nu > 0$ small enough,

$$\left|\frac{(\beta(\omega+i\nu)-\beta(\omega))\rho^2}{\beta(\omega)\rho^2+z^2}\right| \le (C_\beta \nu)\left(\frac{C_1}{C_\beta \nu}\right) = C_1. \tag{3.44}$$

Moreover, as ν goes to 0, $\mathbb{1}_{K_{p,R}^{reg,\nu}}$ converges almost everywhere to $\mathbb{1}_{K_{p,R}}$. Then, for any $\boldsymbol{x} \in K_{p,R}^{reg,\nu}$, the combination of (3.40), (3.41) and (3.44) gives

$$\left|\frac{1}{d_\beta^\nu(\boldsymbol{x})} - \frac{1}{d_\beta(\boldsymbol{x})}\right| \le \frac{C_1 C_2}{d_\beta(\boldsymbol{x})}.$$

As a consequence, since the last inequality stands for $\nu > 0$ small enough and $\left(\mathscr{G}^\beta_{\omega+i\nu} - \mathscr{G}^\beta_{\omega,+}\right)\mathbb{1}_{K_{p,R}^{reg,\nu}}$ converges to 0 almost everywhere, then by Lebesgue's dominated convergence theorem, we have that

$$\mathscr{G}^\beta_{\omega+i\nu}\mathbb{1}_{K_{p,R}^{reg,\nu}} \xrightarrow[\nu\to 0+]{L^1(K_{p,R})} \mathscr{G}^\beta_{\omega,+}. \tag{3.45}$$

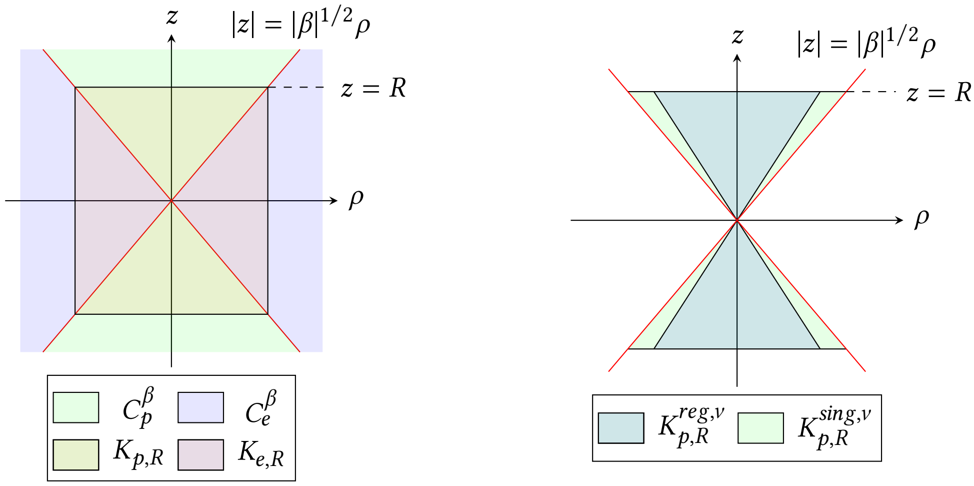


Figure 3.3: The domains $C_p^\beta, C_e^\beta, K_{p,R}, K_{e,R}$ on the left and $K_{p,R}^{reg,\nu}, K_{p,R}^{sing,\nu}$ on the right.

The final step consists in demonstrating that $\mathscr{G}_{\omega+i\nu}^\beta \mathbb{1}_{K_{p,R}^{sing,\nu}}$ converges to 0 in L^1-norm. Using (3.39), we have

$$\left|\frac{1}{d_\beta^\nu(\boldsymbol{x})}\right| \leq \frac{1}{\left|\mathrm{Im}\left(\beta(\omega \pm i\nu)\rho^2 + z^2\right)\right|^{1/2}} \lesssim \frac{1}{\rho\sqrt{\nu}}.$$

On the other hand, thanks to the definition (3.43) of $K_{p,R}^{sing,\nu}$, it follows that $\rho \gtrsim \frac{d_\beta(\boldsymbol{x})}{\sqrt{\nu}}$ for all $\boldsymbol{x} \in K_{p,R}^{sing,\nu}$. This leads to the inequality

$$\int \left|\mathscr{G}_{\omega+i\nu}^\beta\right| \mathbb{1}_{K_{p,R}^{sing,\nu}} \lesssim \int \left|\frac{1}{d_\beta}\right| \mathbb{1}_{K_{p,R}^{sing,\nu}} \xrightarrow[\nu \to 0+]{} 0. \tag{3.46}$$

Finally, (3.45) and (3.46) ensures that $\mathscr{G}_{\omega+i\nu}^\beta$ converges to $\mathscr{G}_{\omega,+}^\beta$ in $L^1(K_{p,R})$. A similar argument gives the same conclusion in $L^1(K_{e,R})$, and the proof is concluded. □

Let us conclude on the fundamental solution by the following lemma.

Lemma 3.3.8. *For $\omega \in (0, \omega_p)$, we have $\nabla \mathscr{G}_{\omega,\pm}^\beta \in L_{loc}^1\left(C_p^\beta \cup C_e^\beta\right)^3$, and*

$$\nabla \mathscr{G}_{\omega,\pm}^\beta(\boldsymbol{x}) = \begin{cases} \left(\pm i\omega - \dfrac{1}{d_\beta(\boldsymbol{x})}\right) \mathscr{G}_{\omega,\pm}^\beta(\boldsymbol{x}) \dfrac{\beta(\omega)\varepsilon^{-1}\boldsymbol{x}}{d_\beta(\boldsymbol{x})}, & \text{if } \boldsymbol{x} \in C_p^\beta, \\ \left(\omega - \dfrac{1}{d_\beta(\boldsymbol{x})}\right) \mathscr{G}_{\omega,\pm}^\beta(\boldsymbol{x}) \dfrac{\beta(\omega)\varepsilon^{-1}\boldsymbol{x}}{d_\beta(\boldsymbol{x})}, & \text{if } \boldsymbol{x} \in C_e^\beta. \end{cases} \tag{3.47}$$

On the other hand, $\nabla \mathscr{G}_{\omega,\pm}^\beta \notin L_{loc}^1\left(\mathbb{R}^3\right)^3$.

Proof. The computation of the gradient is obvious, and clearly shows that $\nabla \mathscr{G}_{\omega,\pm}^\beta \in L_{loc}^1\left(C_p^\beta \cup C_e^\beta\right)^3$. On the other hand, the computation of the L^1-norm on $K_{p,R}$ like in the second step of the proof of Proposition 3.3.7 leads to estimate the integral

$$\int_{\rho=0}^{|z|} \frac{\rho}{z^2 - \rho^2} \mathrm{d}\rho = \infty.$$

□

Now that we have a suitable fundamental solution for $\omega \in (0, \omega_p)$, let us state the following results about the existence of strong and weak solutions. Since the proofs follows exactly [22], we only transpose the result in the 3D case.

Proposition 3.3.9. *Let $\omega \in (0, \omega_p)$, and $f \in \mathscr{C}_0^2(\mathbb{R}^3)$. Then, $u_\omega^+ := \mathscr{G}_{\omega,+}^\beta * f \in \mathscr{C}^2(\mathbb{R}^3)$ and is a strong solution of* (3.29).

This result can be extended by density to functions in some weighted Sobolev spaces. Let us define the following spaces with their norms:

$$L^2_{\perp,-}(\mathbb{R}^3) := \left\{u \in L^2_{loc}(\mathbb{R}^3) \ : \ \|u\|_{\perp,-} < \infty\right\}$$
$$H^1_{\perp,+}(\mathbb{R}^3) := \left\{u \in L^2_{loc}(\mathbb{R}^3) \ : \ \nabla u \in \left(L^2_{loc}(\mathbb{R}^3)\right)^3, \|u\|_{H^1_{\perp,+}} < \infty\right\},$$
$$\|u\|^2_{\perp,-} = \int_{\mathbb{R}^3} |u|^2(1+z^2)\mathrm{d}\boldsymbol{x}, \qquad \|u\|^2_{H^1_{\perp,+}} = \int_{\mathbb{R}^3} \left(|u|^2 + |\nabla u|^2\right)\frac{1}{1+z^2}\mathrm{d}\boldsymbol{x}.$$

Notice that $L^2_{\perp,-}(\mathbb{R}^3) = L^2_w(\mathbb{R}; L^2(\mathbb{R}^2))$ where L^2_w is the Lebesgue space weighted by $w(\boldsymbol{x}) = 1+z^2$. Then, we have the following result.

Proposition 3.3.10. *Given $f \in L^2_{\perp,-}(\mathbb{R}^3)$, the function $u_\omega^+ = \mathscr{G}_{\omega,+}^\beta * f \in H^1_{\perp,+}(\mathbb{R}^3)$ is well-defined. Moreover, u_ω^+ solves the problem* (3.29).

The optimal result can be found in [22]. The main idea of the proof consist in expressing the norms involved with the partial Fourier transform along x, y-directions via the Plancherel Theorem:

$$\|u\|^2_{\perp,-} = \int_{\mathbb{R}^3} \left|\mathscr{F}_{x,y}[u](k_x,k_y,z)\right|^2 (1+z^2)\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}z,$$
$$\|u\|^2_{H^1_{\perp,+}} = \int_{\mathbb{R}^3} \left(\left(1+k_x^2+k_y^2\right)\left|\mathscr{F}_{x,y}[u](k_x,k_y,z)\right|^2 + \left|\partial_z\mathscr{F}_{x,y}[u](k_x,k_y,z)\right|^2\right)\frac{1}{1+z^2}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}z.$$

Then, defining $\mathscr{F}_{x,y}\left[\mathscr{G}_{\omega,+}^\beta\right]$ as the limit of (3.35) when $\operatorname{Im}\omega \to 0+$, we have

$$\mathscr{F}_{x,y}\left[u_\omega^+\right] = \mathscr{F}_{x,y}\left[\mathscr{G}_{\omega,+}^\beta\right] *_z \mathscr{F}_{x,y}[f],$$

and the estimation of u_ω^+ with the norms as above is easy.

3.3.1.3 Radiation condition

Fourier's radiation condition It has been seen in § 3.3.1.1 that the application of the partial Fourier transform in the x, y-directions leads to the following 1D Helmholtz equation:

$$-\partial_z^2\mathscr{F}_{x,y}[u] - \left(\omega^2 - \beta(\omega)^{-1}|\mathbf{k}_\parallel|^2\right)\mathscr{F}_{x,y}[u] = \mathscr{F}_{x,y}[f], \qquad \text{a.e. } \boldsymbol{k}_\parallel = (k_x,k_y) \in \mathbb{R}^2. \tag{3.48}$$

Recall that $\beta(\omega) < 0$ for $\omega \in (0, \omega_p)$. Given $\boldsymbol{k}_\parallel \in \mathbb{R}^2$, if $\mathscr{F}_{x,y}[u]$ satisfies the following outgoing radiation condition

$$\left|\partial_{|z|}\mathscr{F}_{x,y}[u] - i\sqrt{\omega^2 - \beta(\omega)^{-1}|\mathbf{k}_\parallel|^2}\mathscr{F}_{x,y}[u]\right| \xrightarrow[|z|\to+\infty]{} 0, \qquad \text{a.e. } \boldsymbol{k}_\parallel \in \mathbb{R}^2, \tag{3.49}$$

then the solution $\mathscr{F}_{x,y}[u]$ of (3.48) is unique. On the other hand, the $\mathscr{F}_{x,y}[u]$ must exist if we want a such condition. Then, a sufficient condition to guarantee this consists in imposing some regularity on the following mapping:

$$\left[(x,y) \in \mathbb{R}^2 \mapsto u(x,y,z)\right] \in L^2(\mathbb{R}^2), \quad \text{a.e. } z \in \mathbb{R}. \tag{3.50}$$

Definition 3.3.11 (Outgoing Fourier Sommerfeld condition, [22]). A function $u \in H^1_{loc}\left(\mathbb{R}^3\right)$ satisfies the outgoing Fourier radiation condition if it satisfies (3.49) and (3.50).

Let us make few comments on this condition. Although $\mathscr{G}^\beta_{\omega,+}(\cdot,\cdot,z) \notin L^2\left(\mathbb{R}^2\right)$ because of it singular behavior near the boundary of the cones C^β_p and C^β_e, it is compatible with the radiation condition. Using (3.35) and letting $\operatorname{Im}\omega$ tend to 0+, its partial Fourier transform in $\mathcal{S}'\left(\mathbb{R}^2\right)$ reads

$$\mathscr{F}_{x,y}\left[\mathscr{G}^\beta_{\omega,+}\right](k_x,k_y,z) = -\frac{e^{i\sqrt{\omega^2-\beta(\omega)(k_x^2+k_y^2)}|z|}}{2i\sqrt{\omega^2-\beta(\omega)(k_x^2+k_y^2)}}.$$

Hence, it trivially[2] satisfies (3.49). Together with Proposition 3.3.10, this leads to the following theorem.

Theorem 3.3.12. *Let $\omega \in (0,\omega_p)$. For all $f \in L^2_{\perp,-}\left(\mathbb{R}^3\right)$, the function $u^+_\omega = \mathscr{G}^\beta_{\omega,+} * f \in H^1_{\perp,+}\left(\mathbb{R}^3\right)$ is the unique solution of the problem (3.29) which satisfies the Outgoing Fourier Sommerfeld condition (3.3.11).*

Towards an alternative radiation condition The usual radiation condition (in a strong form) for the Helmholtz equation is reads

$$r\left|\partial_r u - i\omega u\right| \xrightarrow[r\to+\infty]{} 0$$

where $r = |\boldsymbol{x}|$. This condition ensures the uniqueness of the solution, via the Rellich's Lemma, see e.g., [45, Lemma 9.8]. Obviously, the fundamental solution (A.12) of the classic Helmholtz equation also respects the condition above.

The objective of this paragraph is to give some ideas on how to construct a such radiation condition for the hyperbolic equation (3.29). More precisely, we want to design a condition with the following pattern:

$$R(\boldsymbol{x})\left|\mathbb{t}(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) - i\omega g(\boldsymbol{x})u(\boldsymbol{x})\right| \xrightarrow[r\to+\infty]{} 0$$

where $\mathbb{t}(\boldsymbol{x})$ is a 3×3 matrix, $\boldsymbol{n}(\boldsymbol{x})$ the normal to some surface, and $R(\boldsymbol{x})$, $g(\boldsymbol{x})$ are two functions. For example, in the case of the classic Helmholtz equation, we would have $\mathbb{t} = \mathbb{I}_3$ and $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{x}/|\boldsymbol{x}|$, and $g(\boldsymbol{x}) = 1$. This condition is expected to be compatible with the fundamental solution (3.38). Indeed, we can rewrite its derivative as

$$\nabla\mathscr{G}^\beta_{\omega,\pm}(\boldsymbol{x}) = -\mathscr{G}^\beta_{\omega,\pm}\frac{\beta\mathbb{e}\boldsymbol{x}}{d_\beta(\boldsymbol{x})^2} \pm i\omega\mathscr{G}^\beta_{\omega,\pm}(\boldsymbol{x})\times\begin{cases}\dfrac{\beta(\omega)\mathbb{e}^{-1}\boldsymbol{x}}{d_\beta(\boldsymbol{x})}, & \text{if } \boldsymbol{x}\in C^\beta_p,\\[2ex] \mp i\dfrac{\beta(\omega)\mathbb{e}^{-1}\boldsymbol{x}}{d_\beta(\boldsymbol{x})}, & \text{if } \boldsymbol{x}\in C^\beta_e.\end{cases}$$

Therefore, in the view of this expression, if we want to select the outgoing solution, it seems natural to set

$$\mathbb{t}(\boldsymbol{x}) = \beta^{-1}\mathbb{e} = \begin{pmatrix}\beta^{-1} & 0 & 0\\ 0 & \beta^{-1} & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad \text{and} \qquad g(\boldsymbol{x}) = \begin{cases}\dfrac{\boldsymbol{x}\cdot\boldsymbol{n}(\boldsymbol{x})}{d_\beta(\boldsymbol{x})}, & \text{if } \boldsymbol{x}\in C^\beta_p,\\[2ex] -i\dfrac{\boldsymbol{x}\cdot\boldsymbol{n}(\boldsymbol{x})}{d_\beta(\boldsymbol{x})}, & \text{if } \boldsymbol{x}\in C^\beta_e,\end{cases}$$

[2]Actually $\partial_{|z|}\mathscr{F}_{x,y}\left[\mathscr{G}^\beta_{\omega,+}\right] = i\sqrt{\omega^2-\beta(\omega)^{-1}|\mathbf{k}_\parallel|^2}\,\mathscr{F}_{x,y}\left[\mathscr{G}^\beta_{\omega,+}\right]$.

with $d_\beta(\boldsymbol{x}) = \left|z^2 - |\beta|\rho^2\right|^{1/2}$, $\rho = \sqrt{x^2+y^2}$, so that a simple computation gives

$$\beta^{-1}\mathbb{e}\nabla\mathscr{G}^{\beta}_{\omega,+}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) - i\omega g(\boldsymbol{x})\mathscr{G}^{\beta}_{\omega,+}(\boldsymbol{x}) = -\mathscr{G}^{\beta}_{\omega,+}(\boldsymbol{x})\frac{\boldsymbol{x}\cdot\boldsymbol{n}(\boldsymbol{x})}{d_\beta(\boldsymbol{x})^2}.$$

Then, given a direction $\boldsymbol{e}\in\mathbb{R}^3$, $|\boldsymbol{e}| = 1$, not parallel to the boundary of the cones C_p^β, C_e^β, we obviously have

$$|\boldsymbol{x}|\left|\beta^{-1}\mathbb{e}\nabla\mathscr{G}^{\beta}_{\omega,+}(|\boldsymbol{x}|\boldsymbol{e})\cdot\boldsymbol{n}(|\boldsymbol{x}|\boldsymbol{e}) - i\omega g(|\boldsymbol{x}|\boldsymbol{e})\mathscr{G}^{\beta}_{\omega,+}(|\boldsymbol{x}|\boldsymbol{e})\right| \leq \frac{1}{4\pi|\boldsymbol{x}|d_\beta(\boldsymbol{e})^3} \xrightarrow[|\boldsymbol{x}|\to+\infty]{} 0. \tag{3.51}$$

On the other hand, this conditions is not satisfied by $\mathscr{G}^{\beta}_{\omega,-}$.

The Rellich's lemma ensures the uniqueness of the Helmholtz via the control of the limit of $\|u\|_{L^2(S_R)}$ as $R\to+\infty$ where S_R is the sphere of radius R. Unfortunately, the quantity $\left\|\mathscr{G}^{\beta}_{\omega,+}\right\|_{L^2(S_R)}$ is not defined since $\mathscr{G}^{\beta}_{\omega,+}$ is not L^2-integrable on the sphere, for the exact same reason as $\nabla\mathscr{G}^{\beta}_{\omega,\pm}\notin L^1_{loc}\left(\mathbb{R}^3\right)^3$, see Lemma 3.3.8. Therefore, instead of a simple L^2-norm on the sphere S_R, we will consider a weighted L^2-norm on the boundary $\partial\Omega$ of some bounded domain $\Omega\subset\mathbb{R}^3$.

Let $u\in\mathscr{C}^2\left(\mathbb{R}^3\right)$ be a solution of the equation (3.29) with $f=0$ and $W(\boldsymbol{x})$ a measurable real weight function. Then, using the divergence form (3.32) of the equation, we have

$$\begin{aligned}
0 &= -\operatorname{Im}\int_\Omega \operatorname{div}\left(\beta^{-1}\mathbb{e}\nabla u\right)\bar{u}\,W(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
&= \operatorname{Im}\int_\Omega \bar{u}\beta^{-1}\mathbb{e}\nabla u\cdot\nabla W(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \operatorname{Im}\int_{\partial\Omega}\bar{u}\beta^{-1}\mathbb{e}\nabla u\cdot\boldsymbol{n}(\boldsymbol{x})\,W(\boldsymbol{x})\mathrm{d}s(\boldsymbol{x}).
\end{aligned}$$

Notice the appearance of $\operatorname{Im}\left(\bar{u}\beta^{-1}\mathbb{e}\nabla u\right)$ which must be compared with the vector $\operatorname{Im}\left(\bar{u}\nabla u\right)$. Therefore, following the idea of dominating the weighted L^2-norm on $\partial\Omega$, the Cauchy-Schwarz inequality leads to

$$\begin{aligned}
\omega&\int_{\partial\Omega}|u|^2|g(\boldsymbol{x})|W(\boldsymbol{x})\mathrm{d}s(\boldsymbol{x})\\
&\leq\left|\operatorname{Im}\int_{\partial\Omega}\bar{u}\left(\beta^{-1}\mathbb{e}\nabla u\cdot\boldsymbol{n}(\boldsymbol{x}) - i\omega g(\boldsymbol{x})u\right)\,W(\boldsymbol{x})\mathrm{d}s(\boldsymbol{x}) - \operatorname{Im}\int_\Omega\bar{u}\beta^{-1}\mathbb{e}\nabla u\cdot\nabla W(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\\
&\leq\left(\int_{\partial\Omega}|u|^2|g(\boldsymbol{x})|W(\boldsymbol{x})\mathrm{d}s(\boldsymbol{x})\right)^{1/2}\left(\int_{\partial\Omega}\left|\beta^{-1}\mathbb{e}\nabla u\cdot\boldsymbol{n}(\boldsymbol{x}) - i\omega g(\boldsymbol{x})u\right|^2\frac{W(\boldsymbol{x})}{|g(\boldsymbol{x})|}\mathrm{d}s(\boldsymbol{x})\right)^{1/2}\\
&\qquad\qquad+\left|\int_\Omega\bar{u}\beta^{-1}\mathbb{e}\nabla u\cdot\nabla W(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|.
\end{aligned}$$

In order to have an idea of the weight, let us set $W(\boldsymbol{x}) = \left(\frac{d_\beta(\boldsymbol{x})}{|\boldsymbol{x}|}\right)^\mu$ with $\mu\in\mathbb{R}$ some exponent to determine, and $\Omega = \overline{K_{p,R}\cup K_{e,R}}$. Notice that W is defined such that it only depends on $\boldsymbol{x}/|\boldsymbol{x}|$. Then, following computations like in the second step of the proof of Proposition 3.3.7, and focusing only on the problematic terms, we have

$$\begin{aligned}
\int_{\partial\Omega}\left|\mathscr{G}^{\beta}_{\omega,+}\right|^2|g(\boldsymbol{x})|W(\boldsymbol{x})\mathrm{d}s(\boldsymbol{x}) &\approx \int_{\partial\Omega}\left(d_\beta(\boldsymbol{x})\right)^{\mu-3}\mathrm{d}s(\boldsymbol{x})<\infty,\\
\int_{\partial\Omega}\left|\beta^{-1}\mathbb{e}\nabla\mathscr{G}^{\beta}_{\omega,+}\cdot\boldsymbol{n}(\boldsymbol{x}) - i\omega g(\boldsymbol{x})\mathscr{G}^{\beta}_{\omega,+}\right|^2\frac{W(\boldsymbol{x})}{|g(\boldsymbol{x})|}\mathrm{d}s(\boldsymbol{x}) &\approx \int_{\partial\Omega}\left(d_\beta(\boldsymbol{x})\right)^{\mu-5}\mathrm{d}s(\boldsymbol{x})<\infty
\end{aligned}$$

if $\mu>1$ for the first integral and $\mu>3$ for the second.

To end this paragraph, the ideas above are premises to write a condition radiation without the use of the partial Fourier transform. However, it was not possible to develop further these ideas because two quantities must be estimated for a regular non-vanishing solution u of the equation with a vanishing source term:

- the volume integral $\int_\Omega \bar{u}\beta^{-1}\mathbb{c}\nabla u \cdot \nabla W(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$;
- the growth of the surface integral $\int_{\partial\Omega} |u|^2 |g(\boldsymbol{x})| W(\boldsymbol{x})\mathrm{ds}(\boldsymbol{x})$.

In principle, once the behavior of both these integrals is known, a proof of the uniqueness of the solution following the same steps as the classic Helmholtz equation could be written.

The two next sections are devoted to the derivation of the well-posedness result for the reduced TM problem (3.22). The strategy is the same as the TE problem : we first establish the existence of classic solutions, then a radiation condition is discussed.

3.3.2 Existence of solutions

The fundamental solution of the hyperbolic problem (3.12) cannot be computed like int eh case of the Maxwell system in vacuum because of the presence of the tensor $\mathbb{c}$. On the other hand, in spite of its complexity due to the high number of differential operator, the reduced formalism introduced in section 3.1.4 allows us to compute the solution. We look for a pair of distributions $(\mathbb{E}_\parallel, \tilde{\mathbb{B}}_\perp) \in \boldsymbol{\mathcal{S}}' \times \tilde{\boldsymbol{\mathcal{S}}}'$ with $\boldsymbol{\mathcal{S}}' = \mathcal{S}'\left(\mathbb{R}^3\right)^{3\times 3} \times \mathcal{S}'\left(\mathbb{R}^3\right)^{3\times 2}$ and $\tilde{\boldsymbol{\mathcal{S}}}' = \mathcal{S}'\left(\mathbb{R}^3\right)^{2\times 3} \times \mathcal{S}'\left(\mathbb{R}^3\right)^{2\times 2}$ such that

$$\left|\begin{aligned} & i\omega\mathbb{c}\mathbb{E}_\parallel + \boldsymbol{curl}_\perp \tilde{\mathbb{B}}_\perp = \mathbb{j}_\parallel, \\ & -i\omega\tilde{\mathbb{B}}_\perp + \widetilde{\mathbf{curl}}\,\mathbb{E}_\parallel = \tilde{\mathbb{m}}_\perp, \end{aligned}\right. \tag{3.52}$$

where $\mathbb{j}_\parallel = (\delta_0\mathbb{I}_3, 0_{3\times 2})$ and $\tilde{\mathbb{m}}_\perp = (0_{2\times 3}, \delta_0\mathbb{I}_2)$. In the view of the one-unknown equation (3.23) and identity (3.15), the operator $\Delta_\beta = \beta^{-1}(\partial_x^2 + \partial_y^2) + \partial_z^2$ with $\beta(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$ plays a capital role in the resolution of this problem. The following proposition illustrates this fact.

Proposition 3.3.13. *For $\omega \in \mathbb{C}\setminus\mathbb{R}$, the fundamental solution of the TM problem (3.52) is decomposed in two parts*

$$\mathbb{E}_\parallel = \mathbb{E}_\parallel^{reg} + \mathbb{E}_\parallel^{sing}, \qquad \tilde{\mathbb{B}}_\perp = \tilde{\mathbb{B}}_\perp^{reg} + \tilde{\mathbb{B}}_\perp^{sing},$$

with

$$\mathbb{E}_\parallel^{reg} = \left(i\omega\mathscr{G}_\omega^\beta\mathbb{c}^{-1} - \frac{1}{i\omega\beta}\mathrm{Hess}\,\mathscr{G}_\omega^\beta,\ \mathbb{c}^{-1}\,\boldsymbol{curl}_\perp\left(\mathscr{G}_\omega^\beta\mathbb{I}_2\right)\right), \qquad \tilde{\mathbb{B}}_\perp^{reg} = \left(\widetilde{\mathbf{curl}}\left(\mathscr{G}_\omega^\beta\mathbb{c}^{-1}\right),\ -i\omega\mathscr{G}_\omega^\beta\mathbb{I}_2\right),$$

and

$$\begin{aligned} \mathbb{E}_\parallel^{sing} &= \left(\frac{i\omega}{\beta}\boldsymbol{curl}_\perp\,\mathrm{curl}_\perp\left(\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}} + \omega^2\mathscr{G}_\omega^\beta\right)\mathbb{I}_3\right),\ \frac{1}{\beta}\boldsymbol{curl}_\perp\,\mathrm{div}_\perp\,\partial_z\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right)\right), \\ \tilde{\mathbb{B}}_\perp^{sing} &= \left(\frac{1}{\beta}\tilde{\nabla}\,\mathrm{curl}_\perp\,\partial_z\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}}\mathbb{c}^{-1}\right),\ \frac{i\omega}{\beta}\tilde{\nabla}\,\mathrm{div}_\perp\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right)\right), \end{aligned}$$

where $\underline{\mathscr{G}_\omega^{1D}} = \delta_0(x,y) \otimes \mathscr{G}_\omega^{1D}(z)$ and $\mathscr{G}_\omega^{1D}(z) = -\gamma\frac{e^{\gamma i\omega|x|}}{2i\omega}$, $\gamma = \mathrm{sign}\,\mathrm{Im}\,\omega$.

Proof. Using identity (3.15) in the one-unknown equation (3.23) yields

$$-\Delta_\beta \tilde{\mathbb{B}}_\perp - \omega^2 \tilde{\mathbb{B}}_\perp = \widetilde{\mathbf{curl}}(\varepsilon^{-1}\mathbb{j}_\parallel) - i\omega\tilde{\mathbb{m}}_\perp - \beta^{-1}\tilde{\nabla}\operatorname{div}_\perp \tilde{\mathbb{B}}_\perp.$$

Similarly to the proof of proposition 3.2.1, $\operatorname{div}_\perp \tilde{\mathbb{B}}_\perp$ solves this Helmholtz equation:

$$-\partial_z^2 \operatorname{div}_\perp \tilde{\mathbb{B}}_\perp - \omega^2 \operatorname{div}_\perp \tilde{\mathbb{B}}_\perp = -\partial_z \operatorname{curl}_\perp(\varepsilon^{-1}\mathbb{j}_\parallel) - i\omega \operatorname{div}_\perp \tilde{\mathbb{m}}_\perp.$$

Therefore, for $\omega \in \mathbb{C}\setminus\mathbb{R}$, the unique solution to this equation is

$$\begin{aligned}\operatorname{div}_\perp \tilde{\mathbb{B}}_\perp &= \underline{\mathscr{G}_\omega^{1D}} * \left(-\partial_z \operatorname{curl}_\perp(\varepsilon^{-1}\mathbb{j}_\parallel) - i\omega \operatorname{div}_\perp \tilde{\mathbb{m}}_\perp\right)\\ &= \left(-\partial_z \operatorname{curl}_\perp\left(\underline{\mathscr{G}_\omega^{1D}}\varepsilon^{-1}\right),\ \ -i\omega \operatorname{div}_\perp\left(\underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right)\right),\end{aligned}$$

where $\underline{\mathscr{G}_\omega^{1D}} = \delta_0(x,y)\otimes\mathscr{G}_\omega^{1D}(z)$ and $\mathscr{G}_\omega^{1D}$ is the fundamental solution of the 1D Helmholtz equation, see Lemma A.3.1. Thanks to the Proposition 3.3.2, we obtain

$$\begin{aligned}\tilde{\mathbb{B}}_\perp &= \mathscr{G}_\omega^\beta * \left(\widetilde{\mathbf{curl}}(\varepsilon^{-1}\mathbb{j}_\parallel) - i\omega\tilde{\mathbb{m}}_\perp - \beta^{-1}\tilde{\nabla}\operatorname{div}_\perp \tilde{\mathbb{B}}_\perp\right)\\ &= \left(\widetilde{\mathbf{curl}}\left(\mathscr{G}_\omega^\beta\varepsilon^{-1}\right) + \frac{1}{\beta}\tilde{\nabla}\operatorname{curl}_\perp\partial_z\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}}\varepsilon^{-1}\right),\ \ -i\omega\mathscr{G}_\omega^\beta\mathbb{I}_2 + \frac{i\omega}{\beta}\tilde{\nabla}\operatorname{div}_\perp\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right)\right)\end{aligned}$$

Due to the structure of ε, we have $\varepsilon^{-1}\,\boldsymbol{curl}_\perp f = \boldsymbol{curl}_\perp f$ and $\varepsilon^{-1}\,\boldsymbol{curl}_\perp\tilde{\nabla} f = \boldsymbol{curl}_\perp\tilde{\nabla} f$. Finally, using identity (A.6), Proposition 3.3.2, the identity $\boldsymbol{curl}_\perp\tilde{\nabla} f = -\boldsymbol{curl}_\perp\partial_z f$, it yields

$$\begin{aligned}\mathbb{E}_\parallel &= \frac{\varepsilon^{-1}}{i\omega}\left(\mathbb{j}_\parallel - \boldsymbol{curl}_\perp\tilde{\mathbb{B}}_\perp\right)\\ &= \left(i\omega\mathscr{G}_\omega^\beta\varepsilon^{-1} - \frac{1}{i\omega\beta}\operatorname{Hess}\mathscr{G}_\omega^\beta + \frac{i\omega}{\beta}\boldsymbol{curl}_\perp\operatorname{curl}_\perp\left(\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}} + \omega^2\mathscr{G}_\omega^\beta\right)\mathbb{I}_3\right),\right.\\ &\qquad\left.\varepsilon^{-1}\boldsymbol{curl}_\perp\left(\mathscr{G}_\omega^\beta\mathbb{I}_2\right) + \frac{1}{\beta}\boldsymbol{curl}_\perp\operatorname{div}_\perp\partial_z\left(\mathscr{G}_\omega^\beta * \underline{\mathscr{G}_\omega^{1D}}\mathbb{I}_2\right)\right).\end{aligned}$$

□

Finally, the following proposition gives an integral representation of the solution to the hyperbolic problem. We do not give the proof, since it is the same as the elliptic case, see Proposition 3.2.3

Proposition 3.3.14. *For $\omega \in \mathbb{C}\setminus\mathbb{R}$, given $\boldsymbol{j}_\parallel \in \boldsymbol{H}(\operatorname{curl}_\perp 0;\mathbb{R}^3) \cap \boldsymbol{H}(\operatorname{div};\mathbb{R}^3)$, $\tilde{\mathbf{m}}_\perp \in \tilde{\mathrm{H}}(\operatorname{div}_\perp 0;\mathbb{R}^3) \cap \tilde{\mathrm{H}}(\boldsymbol{curl}_\perp;\mathbb{R}^3)$, the unique solution of the TM problem (3.22) is*

$$\mathbf{E}_\parallel = \mathbb{E}_\parallel^{reg} * \begin{pmatrix}\boldsymbol{j}_\parallel\\ \tilde{\mathbf{m}}_\perp\end{pmatrix} = \mathscr{G}_\omega^\beta * \left(i\omega\varepsilon^{-1}\boldsymbol{j}_\parallel + \varepsilon^{-1}\boldsymbol{curl}_\perp\tilde{\mathbf{m}}_\perp\right) - \frac{1}{i\omega\beta}\nabla\mathscr{G}_\omega^\beta * \operatorname{div}\boldsymbol{j}_\parallel,$$

$$\tilde{\boldsymbol{B}}_\perp = \tilde{\mathbb{B}}_\perp^{reg} * \begin{pmatrix}\boldsymbol{j}_\parallel\\ \tilde{\mathbf{m}}_\perp\end{pmatrix} = \mathscr{G}_\omega^\beta * \left(\widetilde{\mathbf{curl}}\left(\varepsilon^{-1}\boldsymbol{j}_\parallel\right) - i\omega\tilde{\mathbf{m}}_\perp\right).$$

In order to prove the existence of classic solutions for $\omega \in (0,\omega_p)$, we would pass to the limit the expression of $(\mathbf{E}_\parallel, \tilde{\boldsymbol{B}}_\perp)$ given in the previous proposition. However, since $\nabla\mathscr{G}_{\omega,+}^\beta \notin \left(L^1\left(\mathbb{R}^3\right)\right)^3$, as it has been stated in Lemma 3.3.8, we need to impose more regularity on $\boldsymbol{j}_\parallel$ specifically.

Proposition 3.3.15. *Let $\omega \in (0,\omega_p)$, $\boldsymbol{j}_\parallel \in \boldsymbol{H}(\operatorname{curl}_\perp 0;\mathbb{R}^3) \cap \left(\mathscr{C}^3\left(\mathbb{R}^3\right)\right)^3$, $\tilde{\mathbf{m}}_\perp \in \tilde{\mathrm{H}}(\operatorname{div}_\perp 0;\mathbb{R}^3) \cap \left(\mathscr{C}^2\left(\mathbb{R}^3\right)\right)^2$. Then,*

$$\boldsymbol{E}_\parallel = \mathscr{G}_{\omega,+}^\beta * \left(i\omega\varepsilon^{-1}\boldsymbol{j}_\parallel + \varepsilon^{-1}\boldsymbol{curl}_\perp\tilde{\mathbf{m}}_\perp - \frac{1}{i\omega\beta}\nabla\operatorname{div}\boldsymbol{j}_\parallel\right), \qquad \tilde{\mathbf{B}}_\perp = \mathscr{G}_{\omega,+}^\beta * \left(\widetilde{\mathbf{curl}}\left(\varepsilon^{-1}\boldsymbol{j}_\parallel\right) - i\omega\tilde{\mathbf{m}}_\perp\right)$$

are such that $\boldsymbol{E}_\parallel \in \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^3$, $\tilde{\mathbf{B}}_\perp \in \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^2$ and solve the TM problem (3.22) in a strong sense.

3.3.3 Fourier Silver-Müller radiation condition

Like the TE problem, the uniqueness of the reduced TM is ensured by a radiation condition. The radiation condition (3.3.11) of the scalar problem associated to the TM problem is written with the partial Fourier in the x, y-direction. Therefore, it is natural to extend this condition to the TM problem within this framework.

Firstly, like radiation condition (3.3.11), we need the existence of the partial Fourier transform of $\boldsymbol{E}_\parallel$ and $\tilde{\mathbf{B}}_\perp$:

$$\left| \begin{array}{ll} \left[(x,y) \in \mathbb{R}^2 \mapsto \boldsymbol{E}_\parallel(x,y,z)\right] \in \left(L^2(\mathbb{R}^2)\right)^3, & \text{a.e. } z \in \mathbb{R}, \\ \left[(x,y) \in \mathbb{R}^2 \mapsto \tilde{\mathbf{B}}_\perp(x,y,z)\right] \in \left(L^2(\mathbb{R}^2)\right)^2, & \text{a.e. } z \in \mathbb{R}. \end{array} \right. \tag{3.53}$$

Next, given $\left(k_x, k_y\right) \in \mathbb{R}^2$, we impose the decrease of some combination of $\hat{\boldsymbol{E}}_\parallel = \mathscr{F}_{x,y}\left[\boldsymbol{E}_\parallel\right]$ and $\widehat{\tilde{\mathbf{B}}}_\perp = \mathscr{F}_{x,y}\left[\tilde{\mathbf{B}}_\perp\right]$ when $z \to \pm\infty$:

$$\left| \begin{array}{ll} \left|\hat{E}_{\parallel,y}\operatorname{sign}(z) + a_\omega(k_x,k_y)\hat{B}_{\perp,x}\right| \xrightarrow[|z|\to\infty]{} 0, & \text{a.e. } (k_x,k_y) \in \mathbb{R}^2, \\ \left|\hat{E}_{\parallel,x}\operatorname{sign}(z) - a_\omega(k_x,k_y)\hat{B}_{\perp,y}\right| \xrightarrow[|z|\to\infty]{} 0, & \text{a.e. } (k_x,k_y) \in \mathbb{R}^2, \end{array} \right. \tag{3.54}$$

with $a_\omega(k_x,k_y) = \sqrt{1 - \frac{|k_x|^2+|k_y|^2}{\omega^2\beta(\omega)}}$.

Remark 3.3.16. The last condition can be written within the vector formalism, with $\widehat{\boldsymbol{B}}_\perp = \left(\widehat{\tilde{\mathbf{B}}}_\perp, 0\right)$:

$$\left|\hat{\boldsymbol{E}}_\parallel \times \frac{z}{|z|}\boldsymbol{e}_z + a_\omega(k_x,k_y)\widehat{\boldsymbol{B}}_\perp\right| \xrightarrow[|z|\to\infty]{} 0, \quad \text{a.e. } (k_x,k_y) \in \mathbb{R}^2.$$

Definition 3.3.17 (outgoing Fourier Silver-Müller condition)**.** A pair of vector fields $\boldsymbol{E}_\parallel \in \boldsymbol{H}_{loc}\left(\operatorname{curl}_\perp 0; \mathbb{R}^3\right) \cap \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^3$ and $\tilde{\mathbf{B}}_\perp \in \mathbf{H}_{loc}\left(\operatorname{div}_\perp 0; \mathbb{R}^3\right) \cap \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^2$ satisfies the outgoing Fourier Silver-Müller condition if it satisfies (3.53) and (3.54).

These considerations lead to the following theorem.

Theorem 3.3.18. *Let $\omega \in (0, \omega_p)$. If $\boldsymbol{E}_\parallel \in \boldsymbol{H}_{loc}\left(\operatorname{curl}_\perp 0; \mathbb{R}^3\right) \cap \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^3$ and $\tilde{\mathbf{B}}_\perp \in \tilde{\mathbf{H}}_{loc}\left(\operatorname{div}_\perp 0; \mathbb{R}^3\right) \cap \left(\mathscr{C}^1\left(\mathbb{R}^3\right)\right)^2$ solve the homogeneous TM problem (3.22) and satisfies the condition (3.3.17), then $\boldsymbol{E}_\parallel = 0$ and $\tilde{\mathbf{B}}_\perp = 0$.*

Proof. Using (3.15) and (3.23), $\tilde{\mathbf{B}}_\perp$ solves

$$-\Delta_\beta \tilde{\mathbf{B}}_\perp - \omega^2 \tilde{\mathbf{B}}_\perp = 0.$$

Then, the application of the partial Fourier transform in the x, y-direction leads to the following 1D Helmholtz equation along the z-direction:

$$-\partial_z^2 \widehat{\tilde{\mathbf{B}}}_\perp - \left(\omega a_\omega(k_x,k_y)\right)^2 \widehat{\tilde{\mathbf{B}}}_\perp = 0, \quad \text{with} \quad a_\omega(k_x,k_y) = \sqrt{1 - \frac{|k_x|^2+|k_y|^2}{\omega^2\beta(\omega)}}, \ (k_x,k_y) \in \mathbb{R}^2.$$

Then, there are two vectors $\tilde{\mathbf{B}}^+, \tilde{\mathbf{B}}^- \in \mathbb{R}^2$ such that $\widehat{\tilde{\mathbf{B}}}_\perp = \tilde{\mathbf{B}}^+ e^{i\omega a_\omega(k_x,k_y)z} + \tilde{\mathbf{B}}^- e^{-i\omega a_\omega(k_x,k_y)z}$. On the other hand, applying the partial Fourier transform to $\boldsymbol{curl}_\perp \tilde{\mathbf{B}}_\perp + i\omega\varepsilon \boldsymbol{E}_\parallel = 0$ gives the following two identities:

$$\partial_z \hat{B}_{\perp,x} = -i\omega \hat{E}_{\parallel,y}, \qquad \partial_z \hat{B}_{\perp,y} = i\omega \hat{E}_{\parallel,x}.$$

Then, using the first identity with the first line of (3.54) imposes $B_y^+ = B_y^- = 0$, and the second identity with the second line of (3.54) gives $B_x^+ = B_x^- = 0$. □

Remark 3.3.19. Notice that there is a priori no condition on $\widehat{E_z}$, the third component of $\hat{\boldsymbol{E}}_\parallel$. Indeed, the above Fourier Silver-Müller condition controls the energy of $\mathbf{B}_\perp$ and $\boldsymbol{E}_\parallel$, solutions of the TM problem (3.22), via the flux through the planes $\{z = \pm R\}$ of the Poynting vector $\hat{\boldsymbol{P}} = \hat{\boldsymbol{E}}_\parallel \times \overline{\widehat{\boldsymbol{B}}_\perp}$. In particular, with $\Omega = \{(x, y, z) \in \mathbb{R}^3,\ z \in (-R, R)\}$, one can show that

$$\operatorname{Re} \int_{\partial\Omega} \hat{\boldsymbol{P}} \cdot \boldsymbol{n} \mathrm{d}\boldsymbol{x} = 0.$$

Since $\hat{E}_z$ is not involved in the quantity $\hat{\boldsymbol{P}} \cdot \boldsymbol{e}_z$, it is naturally not involved in (3.54).

3.4 Conclusions

This chapter provides some results about the existence and the uniqueness of the solution to the hyperbolic 3D Maxwell problem in free space (3.2). This was achieved in three main steps:

- the splitting of the original problem into the reduced TE problem (3.20) and the reduced TM problem (3.22), where we justified the equivalence between these problems ;
- the existence of smooth solutions of the TE problem (3.20) and their uniqueness via a classic Silver-Müller radiation condition ;
- the existence of smooth solutions of the TM problem (3.22) and their uniqueness via the establishment a Silver-Müller radiation condition expressed with partial Fourier transforms.

Moreover, some results of [22], originally stated for the 2D hyperbolic problems, have been extended to 3D. On the other hand, we pointed out some difficulties in establishing a radiation condition without the use of partial Fourier transforms.

Although the uniqueness or the existence are well established, we still lack the results on the control of the solutions in well-fitted norms (in particular those which would account for the propagation of singularities along characteristics). Our next step would be establishing the boundary integral equation framework for this problem, which would allow us to consider more complicated cases of half-bounded, and, eventually, bounded domains.

APPENDIX A

Appendix

A.1 Proof of Lemma 3.1.5

Lemma A.1.1 ([6]). *The dispersion function is written:*

$$\begin{aligned} F_\omega(\boldsymbol{k}) &= \left(\omega^2 - \omega_\perp(\boldsymbol{k})^2\right)\left(\omega^2 - \omega_\parallel^+(\boldsymbol{k})^2\right)\left(\omega^2 - \omega_\parallel^-(\boldsymbol{k})^2\right) \\ &= \left(\omega^2 - \omega_p^2\right)\left(\omega^2 - |\boldsymbol{k}|^2\right)\left(\omega^2 - \beta(\omega)^{-1}|\boldsymbol{k}_\parallel|^2 - k_z^2\right), \end{aligned} \tag{A.1}$$

where $\omega_\perp(\boldsymbol{k})^2 = |\boldsymbol{k}|^2$, *and* $\omega_\parallel^\pm(\boldsymbol{k})^2 = \frac{1}{2}\left(\omega_p^2 + |\boldsymbol{k}|^2 \pm \sqrt{\Delta(\boldsymbol{k})}\right)$ *with* $\Delta(\boldsymbol{k}) = \left(\omega_p^2 + |\boldsymbol{k}|^2\right)^2 - 4k_z^2\omega_p^2 \leq 0$. *Then* F_ω *vanishes if:*

1. $\omega_\perp(\boldsymbol{k})^2 = \omega^2$ *and the associated eigenspace is* $\operatorname{span}\left(\boldsymbol{k}_\parallel, \boldsymbol{e}_z\right)^\perp$.
2. $\omega_\parallel^\pm(\boldsymbol{k})^2 = \omega^2$ *and the associated eigenspaces are subset of* $\operatorname{span}\left(\boldsymbol{k}_\parallel, \boldsymbol{e}_z\right)$.

Proof. We reproduce here the proof given in [6]. The matrix

$$\mathbb{A}(\boldsymbol{k}) = |\boldsymbol{k}|^2\mathbb{I}_3 - \boldsymbol{k}\boldsymbol{k}^\top + \omega_p^2\boldsymbol{e}_z\boldsymbol{e}_z^\top$$

is symmetric, so its eigenvectors are orthogonal.

We first assume that $|\boldsymbol{k}_\parallel|^2 = k_x^2 + k_y^2 \neq 0$. Thus, a first eigenvector is $\boldsymbol{e}_z \times \boldsymbol{k} \neq 0$:

$$\mathbb{A}(\boldsymbol{k})(\boldsymbol{e}_z \times \boldsymbol{k}) = \left[|\boldsymbol{k}|^2\mathbb{I}_3 - \boldsymbol{k}\boldsymbol{k}^\top + \omega_p^2\boldsymbol{e}_z\boldsymbol{e}_z^\top\right](\boldsymbol{e}_z \times \boldsymbol{k}) = |\boldsymbol{k}|^2(\boldsymbol{e}_z \times \boldsymbol{k}).$$

Thus, the first eigenvalue is $\omega_\perp(\boldsymbol{k})^2 = |\boldsymbol{k}|^2$, and the associated space is $\operatorname{span}(\boldsymbol{k}_\perp)$.

The other two eigenvectors of $\mathbb{A}(\boldsymbol{k})$ are therefore in $\operatorname{span}(\boldsymbol{k}, \boldsymbol{e}_z) = \operatorname{span}(\boldsymbol{k}_\parallel, \boldsymbol{e}_z)$. Let $\boldsymbol{v} \in \operatorname{span}(\boldsymbol{k}_\parallel, \boldsymbol{e}_z) \setminus \{0\}$ be an eigenvector of the system. We have:

$$\begin{aligned} (\mathbb{A}(\boldsymbol{k})\boldsymbol{v}) \cdot \boldsymbol{k} &= \omega_p^2 k_z v_z = \omega^2 \boldsymbol{v} \cdot \boldsymbol{k}, \\ (\mathbb{A}(\boldsymbol{k})\boldsymbol{v}) \cdot \boldsymbol{e}_z &= \left(|\boldsymbol{k}|^2 + \omega_p^2\right) v_z - k_z(\boldsymbol{k} \cdot \boldsymbol{v}) = \omega^2 v_z. \end{aligned}$$

Notice that $v_z \neq 0$: by contradiction, if $v_z = 0$ then $\boldsymbol{v} \cdot \boldsymbol{k} = 0$ which is not possible. By multiplying the second equation by ω^2 and replacing $\omega^2\boldsymbol{v} \cdot \boldsymbol{k}$ by $\omega_p^2 k_z v_z$, we obtain the equation:

$$\omega^4 - \omega^2\left(|\boldsymbol{k}|^2 + \omega_p^2\right) + \omega_p^2 k_z^2 = 0. \tag{A.2}$$

The solutions of this equation are the two other eigenvalues:

$$\omega_{\parallel}^{\pm}(\boldsymbol{k})^2 = \frac{\omega_p^2 + |\boldsymbol{k}|^2 \pm \sqrt{\Delta(\boldsymbol{k})}}{2}, \quad \text{with} \quad \Delta(\boldsymbol{k}) = \left(\omega_p^2 + |\boldsymbol{k}|^2\right)^2 - 4k_z^2\omega_p^2.$$

By factoring the equation (A.2) by $(\omega^2 - \omega_p^2)$, this equation can be rewritten in the form:

$$\left(\omega^2 - \omega_p^2\right)\left(\omega^2 - \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1} |\boldsymbol{k}_\parallel|^2 - k_z^2\right) = 0.$$

Finally, if $\boldsymbol{k}_\parallel = 0$, i.e., $\boldsymbol{k} = k_z\boldsymbol{e}_z$, then $\mathbb{A}(\boldsymbol{k}) = k_z^2\mathbb{I}_3 + \left(\omega_p^2 - k_z^2\right)\boldsymbol{e}_z\boldsymbol{e}_z^\top$. Thus, its eigenvalue-vector pairs are $(\omega_p^2, \boldsymbol{e}_z)$, $(k_z^2, \boldsymbol{e}_x)$ and $(k_z^2, \boldsymbol{e}_y)$. Notice that $\omega_\perp(k_z\boldsymbol{e}_z)^2 = k_z^2$, $\omega_\parallel^+(k_z\boldsymbol{e}_z)^2 = \max\left(\omega_p^2, k_z^2\right)$ and $\omega_\parallel^-(k_z\boldsymbol{e}_z)^2 = \min\left(\omega_p^2, k_z^2\right)$. Moreover, the associated eigenspace is $\operatorname{span}(\boldsymbol{e}_z)^\perp$ when $k_z^2 = \omega^2$. □

A.2 Reduced differential operators

The following operators are introduced in section 3.1.4:

$$\widetilde{\operatorname{curl}}\,\boldsymbol{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \end{pmatrix}, \qquad \operatorname{curl}_\perp \boldsymbol{F} = \partial_x F_y - \partial_y F_x, \qquad \operatorname{div}_\perp \boldsymbol{F} = \partial_x F_x + \partial_y F_y,$$

$$\boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \begin{pmatrix} -\partial_z F_y \\ \partial_z F_x \\ \partial_x F_y - \partial_y F_x \end{pmatrix}, \qquad \operatorname{curl}_\perp \tilde{\mathrm{F}} = \partial_x F_y - \partial_y F_x, \qquad \operatorname{div}_\perp \tilde{\mathrm{F}} = \partial_x F_x + \partial_y F_y,$$

$$\boldsymbol{curl}_\perp f = \boldsymbol{curl}(f\mathbf{e}_z) = \begin{pmatrix} \partial_y f \\ -\partial_x f \\ 0 \end{pmatrix}, \qquad \tilde{\nabla} f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} \qquad \Delta_\perp f = \operatorname{div}_\perp \tilde{\nabla} f = \operatorname{div}_\perp \nabla f = \partial_x^2 f + \partial_y^2 f,$$

where $\boldsymbol{F} = (F_x, F_y, F_z)^\top$ and $\tilde{\mathrm{F}} = (F_x, F_y)^\top$. Notice that the operators are consistent with the classical ***curl***-operator since

$$\boldsymbol{curl}\,\boldsymbol{curl}\,\boldsymbol{F} = \boldsymbol{curl}_\perp \widetilde{\operatorname{curl}}\,\boldsymbol{F} + \boldsymbol{curl}_\perp \operatorname{curl}_\perp \boldsymbol{F}.$$

Then, we have the following identities:

$$\widetilde{\operatorname{curl}}\,\boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \tilde{\nabla}\operatorname{div}_\perp \tilde{\mathrm{F}} - \Delta\tilde{\mathrm{F}}, \tag{A.3}$$

$$\boldsymbol{curl}_\perp \widetilde{\operatorname{curl}}\,\boldsymbol{F} = \nabla \operatorname{div}\boldsymbol{F} - \boldsymbol{\Delta F} - \boldsymbol{curl}_\perp \operatorname{curl}_\perp \boldsymbol{F}, \tag{A.4}$$

$$\widetilde{\operatorname{curl}}\,\mathbb{e}^{-1}\,\boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \beta^{-1}\,\tilde{\nabla}\operatorname{div}_\perp \tilde{\mathrm{F}} - \Delta_\beta\tilde{\mathrm{F}}, \tag{A.5}$$

$$\boldsymbol{curl}_\perp \widetilde{\operatorname{curl}}\,\mathbb{e}^{-1}\boldsymbol{F} = \beta^{-1}\mathbb{e}\,\nabla \operatorname{div}\boldsymbol{F} - \Delta_\beta\boldsymbol{F} - \beta^{-1}\,\boldsymbol{curl}_\perp \operatorname{curl}_\perp \boldsymbol{F}, \tag{A.6}$$

$$\operatorname{curl}_\perp \widetilde{\operatorname{curl}}\,\boldsymbol{F} = \partial_z \operatorname{div}_\perp \boldsymbol{F} - \Delta_\perp F_z, \tag{A.7}$$

$$\operatorname{curl}_\perp \boldsymbol{curl}_\perp \tilde{\mathrm{F}} = \partial_z \operatorname{div}_\perp \tilde{\mathrm{F}}, \tag{A.8}$$

Additionally, we have:

$$\operatorname{div}_\perp \widetilde{\operatorname{curl}}\,\boldsymbol{F} = -\partial_z \operatorname{curl}_\perp \boldsymbol{F}, \tag{A.9}$$

$$\operatorname{div}_\perp \boldsymbol{curl}_\perp \tilde{\mathrm{F}} = -\partial_z \operatorname{curl}_\perp \tilde{\mathrm{F}}. \tag{A.10}$$

A.3 Reminder on Helmholtz Fundamental Solutions

Let us recall some technical lemmas related to the classical Helmholtz equation in $\mathbb{R}$ and $\mathbb{R}^3$.

Lemma A.3.1. *For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the unique fundamental solution in $\mathcal{S}'(\mathbb{R})$ to the 1D Helmholtz equation $-\mathcal{G}_\omega'' - \omega^2 \mathcal{G}_\omega = \delta_0$ is*

$$\mathcal{G}_\omega(x) = -\gamma \frac{e^{\gamma i \omega |x|}}{2i\omega}, \qquad \text{with } \gamma = \operatorname{sign} \operatorname{Im} \omega. \tag{A.11}$$

Proof. The application of the Fourier transform to the equation gives $\left(k^2 - \omega^2\right) \mathcal{F}\mathcal{G}_\omega = 1$ in $\mathcal{S}'(\mathbb{R})$. Since $\omega \in \mathbb{C} \setminus \mathbb{R}$, the function $k \mapsto \left(k^2 - \omega^2\right)^{-1}$ belongs to $\mathcal{C}^\infty(\mathbb{R})$, so that its singular support is empty. Therefore, the multiplication of $\left(k^2 - \omega^2\right)^{-1}$ and $\left(k^2 - \omega^2\right) \mathcal{F}\mathcal{G}_\omega$ is valid[1] in $\mathcal{S}'(\mathbb{R})$. Then, we have $\mathcal{F}\mathcal{G}_\omega(k) = \left(k^2 - \omega^2\right)^{-1} \in L^1(\mathbb{R})$ and

$$\mathcal{G}_\omega(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ikx}}{k^2 - \omega^2} \mathrm{d}k.$$

For $\operatorname{Im} \omega > 0$, the application of the residue theorem on $[-R, R] \cup \{Re^{i\theta}, \theta \in [0, \pi]\}$ for $x > 0$ or $[-R, R] \cup \{Re^{i\theta}, \theta \in [-\pi, 0]\}$ for $x < 0$ yields $\mathcal{G}_\omega(x) = -\frac{\exp(i\omega x)}{2i\omega}$. A similar argument for $\operatorname{Im} \omega < 0$ results in $\mathcal{G}_\omega(x) = \frac{\exp(-i\omega x)}{2i\omega}$. □

Lemma A.3.2. *For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the unique fundamental solution in $\mathcal{S}'(\mathbb{R}^3)$ to the 3D Helmholtz equation $-\Delta \mathcal{G}_\omega - \omega^2 \mathcal{G}_\omega = \delta_0$ is*

$$\mathcal{G}_\omega(\boldsymbol{x}) = \frac{e^{\gamma i \omega |\boldsymbol{x}|}}{4\pi |\boldsymbol{x}|}, \qquad \text{with } \gamma = \operatorname{sign} \operatorname{Im} \omega. \tag{A.12}$$

Proof. Applying the Fourier transform to the equation, one obtains that $\left(|\boldsymbol{k}|^2 - \omega^2\right) \mathcal{F}\mathcal{G}_\omega = 1$. Since $\omega \in \mathbb{C} \setminus \mathbb{R}$, the function $\boldsymbol{k} \mapsto \left(|\boldsymbol{k}|^2 - \omega^2\right)^{-1}$ belongs to $\mathcal{C}^\infty(\mathbb{R}^3)$, and its singular support is empty. Therefore, we have $\mathcal{F}\mathcal{G}_\omega(\boldsymbol{k}) = \left(|\boldsymbol{k}|^2 - \omega^2\right)^{-1} \in L^2(\mathbb{R}^3)$ and

$$\mathcal{G}_\omega(\boldsymbol{x}) = \lim_{R \to +\infty} \frac{1}{(2\pi)^3} \int_{B_R} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2 - \omega^2} \mathrm{d}\boldsymbol{k}.$$

Therefore, a simple computation gives

$$\begin{aligned}
\frac{1}{(2\pi)^3} \int_{B_R} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2 - \omega^2} \mathrm{d}\boldsymbol{k} &= \frac{1}{(2\pi)^3} \int_{\rho=0}^{R} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{\rho^2 \sin\theta e^{i\rho|\boldsymbol{x}|\cos\theta}}{\rho^2 - \omega^2} \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\varphi \\
&= \frac{2}{(2\pi)^2 |\boldsymbol{x}|} \int_0^R \frac{\rho \sin(\rho|\boldsymbol{x}|)}{\rho^2 - \omega^2} \mathrm{d}\rho \\
&= \frac{1}{(2i\pi)(4\pi|\boldsymbol{x}|)} \int_{-R}^{R} \frac{\rho e^{i\rho|\boldsymbol{x}|}}{\rho^2 - \omega^2} \mathrm{d}\rho.
\end{aligned}$$

Finally, by applying the residue theorem on $[-R, R] \cup \{Re^{i\theta}, \theta \in [0, \pi]\}$ and taking the limit as $R \to +\infty$, we obtain the expected result. □

[1]This step guarantees the uniqueness of the fundamental solution. The multiplication is obviously not valid if $\omega \in \mathbb{R}$.

Another useful lemma concerns the scaled Helmholtz equation.

Corollary A.3.3. *The fundamental solution of the scaled Helmholtz equation*

$$-\tilde{\beta}^{-1}\left(\partial_x^2+\partial_y^2\right)u-\partial_z^2 u-\omega^2 u=\delta_0,$$

with $\tilde{\beta}>0$ *and* $\omega\in\mathbb{C}\setminus\mathbb{R}$ *is*

$$\mathscr{G}_\omega^{\tilde{\beta}}(x,y,z)=\tilde{\beta}\frac{\exp\left(\gamma i\omega\sqrt{\tilde{\beta}(x^2+y^2)+z^2}\right)}{4\pi\sqrt{\tilde{\beta}(x^2+y^2)+z^2}},\qquad \gamma=\operatorname{sign}(\operatorname{Im}\omega). \tag{A.13}$$

Proof. The application of change of variables $(x,y,z)=\left(\tilde{\beta}^{-1/2}x_1,\tilde{\beta}^{-1/2}x_2,x_3\right)$ to the scaled Helmholtz equation leads back to the classical 3D Helmholtz equation

$$-\Delta\mathscr{G}_\omega^{\tilde{\beta}}-\omega^2\mathscr{G}_\omega^{\tilde{\beta}}=\tilde{\beta}\delta_0.$$

One can verify [34, example 6.1.3] with $f(x_1,x_2,x_3)=\left(\tilde{\beta}^{-1/2}x_1,\tilde{\beta}^{-1/2}x_2,x_3\right)$ for the scaling in front of the Dirac measure. Finally, applying Lemma A.3.2 is sufficient to conclude. □

A.4 Hyperbolic coordinates

The 2D Helmholtz equation can be naturally written with the polar coordinates:

$$-\frac{1}{r}\partial_r(r\partial_r u)-\frac{1}{r^2}\partial_\theta^2 u-\omega^2 u=f.$$

However, the polar coordinates are clearly not adapted to the 2D hyperbolic Helmholtz equation studied in [22]:

$$-\beta^{-1}\partial_x^2 u-\partial_z^2 u-\omega^2 u=f.$$

Let us assume $\beta=-1$ for this paragraph. Then, making the change of variable $\xi=z-x$, $\eta=z+x$ leads to the equation

$$4\partial_{\xi\eta}^2 u+\omega^2 u=f,$$

with some abuse of notation. But this change of variable is not possible in the 3D coordinates since it corresponds to a rotation of the xz-plane of a $\pi/4$-angle. Another system of coordinates in which the equation could be studied is the following:

$$x=\rho\sinh\theta,\qquad y=\rho\cosh\theta,$$

with $\rho>0$ and $\theta\in\mathbb{R}$. This change of coordinates maps the upper cone $\left\{(x,y)\in\mathbb{R}^2\,:\,y>|x|\right\}$ onto $(0,+\infty)\times\mathbb{R}$. In this system of coordinate, the equation becomes

$$-\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho u\right)+\frac{1}{\rho^2}\partial_\theta^2 u-\omega^2 u=f.$$

Then, this equation is very similar with the classical Helmholtz equation in polar coordinates except the presence of a minus sign in front of the second term $\rho^{-2}\partial_\theta^2 u$. Then, the equivalent change in 3D is

$$x=\rho\sinh\theta\cos\varphi,\qquad y=\rho\sinh\theta\sin\varphi,\qquad z=\rho\cosh\theta,$$

with $\rho>0$, $\theta>0$ and $\varphi\in(-\pi,\pi)$.

PART II

Degenerate piecewise elliptic equation

CHAPTER 4

State of the art

Contents

The propagation of electromagnetic waves inside the plasma has already been investigated in [49]. In particular, a numerical method based on a mixed variational formulation was proposed in [49]. This method is summarized in §4.2. From the physics viewpoint, There is a logarithmic singularity that is responsible for plasma heating phenomenon [25, 26]. In what follows, we study a mathematical model that allows to recover this singular behavior.

We consider a bounded Lipschitz domain D in $\mathbb{R}^2$. Let $\lambda > 0$ and $f \in L^2(\partial D)$. We study the following boundary value problem:

$$\left| \begin{array}{ll} -\operatorname{div}_\perp(\mathbb{\alpha}_0 \nabla_\perp B_3) - \omega^2 B_3 = 0 & \text{in } D, \\ (\mathbb{\alpha}_0 \nabla_\perp B_3)\cdot \mathbf{n}_\perp + i\lambda B_3 = f & \text{on } \partial D, \end{array} \right. \tag{4.1}$$

where $\mathbf{n}_\perp$ denotes the outward unit vector field to ∂D. According to our model, it holds that $\mathbb{\alpha}_0(\mathbf{x}_\perp) = \alpha_0(\mathbf{x}_\perp)\mathbb{H}(\mathbf{x}_\perp)$, where the scalar field α_0 and the hermitian matrix field $\mathbb{H}$ are $\mathscr{C}^2(\overline{D})$-regular (cf. (2.7)). We set $D_p = \{\mathbf{x}_\perp \in D : \alpha_0(\mathbf{x}_\perp) > 0\}$, $D_n = \{\mathbf{x}_\perp \in D : \alpha_0(\mathbf{x}_\perp) < 0\}$, and recall that the interface $I = \{\mathbf{x}_\perp \in D : \alpha_0(\mathbf{x}_\perp) = 0\}$ is a C^1-loop (without self-intersections). We assume here that $\operatorname{meas}(D_{p,n}) > 0$, and that I does not intersect ∂D. Observe that outside every neighborhood of I we are solving a classic second-order elliptic PDE with smooth coefficients. Hence, following the classical theory, we shall look for a solution that belongs to H^1 outside this neighborhood. To fix ideas, we consider the case where that D is a *tubular neighborhood of I*. Finally, we recall that $|\alpha|$ behaves like $\operatorname{dist}(\cdot, I)$ in a neighborhood of the interface.

Remark 4.0.1. The limit conditions have been chosen so that it mimics the classical absorbing condition used to solve Maxwell's system in bounded domain, see [1] for example. Instinctively, the problem is a priori solvable for any frequency ω, and this choice allows us to focus on the appearance of the singular behavior of the solutions at the interface.

4.1 Mathematical setting

Like in [49], we focus on the problem (4.1) posed in the neighborhood of the interface. Let $\Omega = (-a,a)\times(0,L)$ be a subset of $\mathbb{R}^2$, with the normalized orthogonal coordinates (x,y). Introduce the bijective transform $\psi\ :\ \mathbf{x}=(x,y)\rightarrow \mathbf{x}_\perp$ which maps $\overline{\Omega}$ to $\overline{D}$ with the following properties, see Figure 4.1:

- the preimage of the interface I is the straight line $\Sigma = \{0\}\times[0,L)$;
- the preimage of the subregion D_n is the rectangle $\Omega_n = (-a,0)\times(0,L)$;
- the preimage of the subregion D_p is the rectangle $\Omega_p = (0,a)\times(0,L)$;
- the preimage of $\partial D_n \setminus I$ is the straight line $\{-a\}\times[0,L)$;
- the preimage of $\partial D_p \setminus I$ is the straight line $\{a\}\times[0,L)$;
- the image of $(-a,a)\times\{0\}$ is equal to the image of $(-a,a)\times\{L\}$.

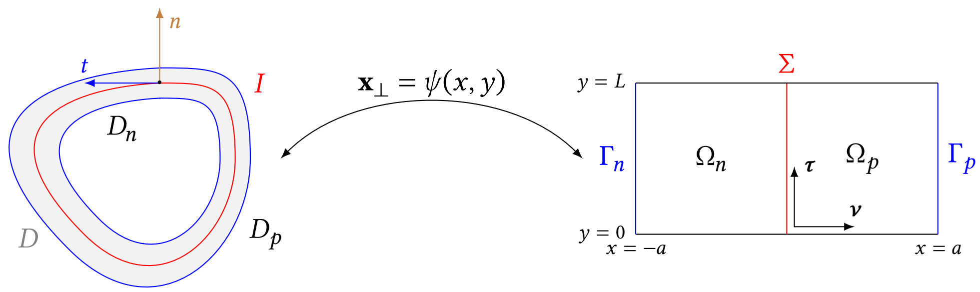


Figure 4.1: [Left] The tubular neighborhood D of I. [Right] The domain $\Omega = (-a,a)\times(0,L)$. [Center] The transform $\psi\ :\ \overline{\Omega}\rightarrow\overline{D}$ with $\psi(\Sigma) = I$, $\psi(\Omega_{p,n}) = D_{p,n}$ and $\psi(\Gamma_{p,n}) = \partial D_{p,n}\setminus I$.

We split the boundary of Ω into 4 components:

$$\Gamma_p = \{a\}\times(0,L),\quad \Gamma_n = \{-a\}\times(0,L),\quad \Gamma_1 = (-a,a)\times\{0\},\quad \Gamma_2 = (-a,a)\times\{L\}.$$

Let $u := B_3\circ\psi$. Then, we have $\nabla B_3(\mathbf{x}_\perp) = [D\psi(\mathbf{x})]^{-t}\,\nabla u(\mathbf{x})$ where $\mathbf{x}_\perp = \psi(\mathbf{x})$. Then, expressing (4.1) variationally and following [9, §2.1.3], we have for all $v\in\mathscr{C}^1(\overline{\Omega})$

$$\int_D \left\{\mathbb{a}_0(\mathbf{x}_\perp)\nabla B_3(\mathbf{x}_\perp)\cdot\nabla v(\mathbf{x}_\perp) - \omega^2 B_3(\mathbf{x}_\perp)v(\mathbf{x}_\perp)\right\}\mathrm{d}\mathbf{x}_\perp = \int_\Omega \left\{\underline{\mathbb{a}}(\mathbf{x})\nabla u(\mathbf{x})\cdot\nabla w(\mathbf{x}) - \omega^2 J(\mathbf{x})u(\mathbf{x})w(\mathbf{x})\right\}\mathrm{d}\mathbf{x}$$

where $w = v\circ\psi$, $J(\mathbf{x}) = |\det D\psi(\mathbf{x})|$ and $\underline{\mathbb{a}}(\mathbf{x}) = J(\mathbf{x})[D\psi(\mathbf{x})]^{-1}\mathbb{a}_0(\mathbf{x}_\perp)[D\psi(\mathbf{x})]^{-t}$ with the correspondence $\mathbf{x}_\perp = \psi(x,y)$. Next, on the interface ∂D, we have for all $v\in\mathscr{C}^1(\overline{\Omega})$

$$\int_{\partial D} \{\mathbb{a}_0(\mathbf{x}_\perp)\nabla B_3(\mathbf{x}_\perp)\cdot\mathbf{n}_\perp(\mathbf{x}_\perp) + i\lambda B_3(\mathbf{x}_\perp)\}\,v(\mathbf{x}_\perp)\mathrm{d}s(\mathbf{x}_\perp) = \int_{\Gamma_n\cup\Gamma_p} \{\mathbb{a}(\mathbf{x})\nabla u(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) + i\lambda J_{\mathbf{n}}(\mathbf{x})u(\mathbf{x})\}\,w(\mathbf{x})\mathrm{d}s(\mathbf{x})$$

where $J_{\mathbf{n}}(\mathbf{x}) = J(\mathbf{x}) \left\| [D\psi(\mathbf{x})]^{-t} \mathbf{n} \right\|_{\mathbb{R}^2}$ and $\mathbf{n}$ denotes the outward unit vector field to $\partial\Omega$. Thus, the problem on u is

$$\left| \begin{array}{ll} -\operatorname{div}(\underline{\alpha}\nabla u) - \omega^2 J(x)u = 0 & \text{in } \Omega, \\ \underline{\alpha}\nabla u \cdot \mathbf{n} + i\lambda J_{\mathbf{n}} u = \tilde{f} & \text{on } \Gamma_n \cup \Gamma_p, \\ u(x,0) = u(x,L), \quad (\underline{\alpha}\nabla u \cdot \mathbf{e}_y)(x,0) = (\underline{\alpha}\nabla u \cdot \mathbf{e}_y)(x,L), & x \in (-a,\, a), \end{array} \right.$$

with $\tilde{f}(\mathbf{x}) = J_{\mathbf{n}}(\mathbf{x}) f \circ \psi(\mathbf{x})$. Above, the divergence and gradient operators are the classical 2D operators. The last condition accounts for periodicity.

We can assume that the transform is volume preserving so that $J(\mathbf{x}) = 1$ and $J_{\mathbf{n}}(\mathbf{x}) = 1$. This requirement does not reduce the scope of the study since there exist two constants $C_{\min}, C_{\max} > 0$ such that $C_{\min} \leq J(\mathbf{x}) \leq C_{\max}$ for all $\mathbf{x} \in \overline{\Omega}$, and $C_{\min} \leq J_{\mathbf{n}}(\mathbf{x}) \leq C_{\max}$ for all $\mathbf{x} \in \Gamma_n \cup \Gamma_p$. Therefore, the definition of $\underline{\alpha}(\mathbf{x})$ in (4.1) changes to $\underline{\alpha}(\mathbf{x}) = [D\psi(\mathbf{x})]^{-1} \alpha_0(\mathbf{x}_\perp)[D\psi(\mathbf{x})]^{-t}$.

In what follows, we make two simplifying technical assumptions. First, that $\underline{\alpha}$ is pointwise proportional to the identity matrix, that is $\underline{\alpha}(\mathbf{x}) = \alpha(\mathbf{x}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ everywhere in $\overline{\Omega}$. Here we use, with an abuse of notation, the same letter for the new coefficient $\alpha(\mathbf{x})$ and the coefficient $\alpha(\mathbf{x}_\perp)$ in (2.6); those however are not to be confused. In this situation, the model can be recast as

$$\left| \begin{array}{ll} -\operatorname{div}(\alpha\nabla u) - \omega^2 u = 0 & \text{in } \Omega, \\ \alpha\partial_n u + i\lambda u = f & \text{on } \Gamma_n \cup \Gamma_p, \\ u(x,0) = u(x,L), \quad (\alpha\partial_y)u(x,0) = (\alpha\partial_y)u(x,L), & x \in (-a,\, a). \end{array} \right. \tag{4.2}$$

Let $\mathtt{r}(y) := \partial_y \alpha(0,y)$. Notice that $\mathtt{r} \in \mathscr{C}^1_{per}(0,L)$. We assume that the sign change of α does not degenerate, i.e., $\mathtt{r}(y) > 0$ for all $y \in (0,L)$. Due to these assumptions, the behavior of α near the interface $\Sigma = \{(x,y) \,:\, x = 0\}$ is simple:

$$\alpha(x,y) = \mathtt{r}(y)x + \mathcal{O}(x^2).$$

Because the coefficient α is a scalar, we observe that the interface Σ is now described by $\{(x,y) \,:\, \alpha(x,y) = 0\}$, while the two subdomains are respectively described by $\Omega_p = \{(x,y) \,:\, \alpha(x,y) > 0\}$ and $\Omega_n = \{(x,y) \,:\, \alpha(x,y) < 0\}$.

There remains to specify the requested regularity of u, so as to allow for the modelling of plasma heating. In this manuscript, we look for limiting absorption solutions of the above problem, namely, we look for u being an L^2-weak limit of u^ν, as $\nu \to 0^+$, where $u^\nu \in H^1(\Omega)$ is solution of the following limiting absorption problem:

$$\left| \begin{array}{ll} \text{find } u^\nu \in H^1(\Omega) \text{ s.t.} & \\ -\operatorname{div}((\alpha + i\nu)\nabla u^\nu) - \omega^2 u^\nu = 0 & \text{in } \Omega, \\ (\alpha + i\nu)\,\partial_n u^\nu + i\lambda u^\nu = f & \text{on } \Gamma_n \cup \Gamma_p, \\ u^\nu(x,0) = u^\nu(x,L), \quad \left((\alpha + i\nu)\,\partial_y\right) u^\nu(x,0) = \left((\alpha + i\nu)\,\partial_y\right) u^\nu(x,L), & x \in (-a,\, a). \end{array} \right. \tag{4.3}$$

Proposition 4.1.1. *The problem (4.3) is well-posed for all $\omega \in \mathbb{R}$ and for all ν positive. Moreover, for small enough $\nu > 0$, there is a constant independent of ν such that its unique solution verifies*

$$\|u^\nu\|_{H^1(\Omega)} \leq \frac{C}{\nu} \|f\|_{L^2(\Gamma_p \cup \Gamma_n)} \cdot$$

Proof. The variational form of the problem (4.3) is

$$\text{find } u^\nu \in H^1_{per,y}(\Omega) \text{ s.t.}$$
$$\underbrace{\int_\Omega \left[(\alpha + i\nu)\,\nabla u^\nu \cdot \overline{\nabla v} - \omega^2 u^\nu \overline{v}\right] \mathrm{dx} + i\lambda \int_{\Gamma_p \cup \Gamma_n} u^\nu \overline{v} \mathrm{ds}}_{b^\nu(u^\nu, v)} = \underbrace{\int_{\Gamma_p \cup \Gamma_n} f \overline{v} \mathrm{ds}}_{\ell(v)}, \text{ for all } v \in H^1_{per,y}(\Omega).$$

Since the $L^2(\Gamma_p \cup \Gamma_n)$-norm is controlled by the $H^1(\Omega)$-norm, b^ν and ℓ are continuous in $H^1(\Omega)$. Given that for all $u \in H^1_{per,y}(\Omega)$

$$\operatorname{Re} b^\nu(u,u) \leq \|\alpha\|_\infty \|\nabla u\|^2_{L^2(\Omega)} - \omega^2 \|u\|^2_{L^2(\Omega)}, \qquad \operatorname{Im} b^\nu(u,u) = \nu \|\nabla u\|^2_{L^2(\Omega)} + \lambda \|u\|^2_{L^2(\Gamma_p \cup \Gamma_n)},$$

we easily obtain for $\omega \neq 0$ that

$$\|u\|^2_{H^1(\Omega)} \leq \operatorname{Re}\left(-\left(1 + \frac{i\left(\|\alpha\|_\infty + \omega^2\right)}{\nu\omega^2}\right) b^\nu(u,u)\right)$$

On the other hand, if $\omega = 0$, then we use Poincaré-Friedrichs inequality (see e.g., [54, example 2.7])

$$\|u\|^2_{L^2(\Omega)} \leq C\left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma_p \cup \Gamma_n)}\right)$$

for some $C > 0$, so that

$$\|u\|^2_{H^1(\Omega)} \leq \operatorname{Im}\left(\frac{\lambda(1+C) + \nu C}{\nu C} b^\nu(u,u)\right).$$

The Lax-Milgram theorem allows us to conclude in both cases. □

Up to our knowledge, the limiting absorption principle can be justified in 1D, as well as for particular values of $\alpha(x,y)$ in slab geometries, cf. [25, 26]. Let us provide an illuminating example whose goal is two-fold. On one hand, we will show how the limiting absorption principle leads to the occurrence of a logarithmic singularity in the solution. On the other hand, we will highlight the difficulty in the choice of the functional framework that would accommodate such singular solutions. Consider the 1D boundary-value problem: given c_1, $c_2 \in \mathbb{R}$, find u solving

$$-(xu')' = 0 \text{ on } \mathcal{I} := (-a, a), \quad u(-a) = c_1,\ u(a) = c_2.$$

In this 1D setting, $\Sigma = \{0\}$. We could have looked for the solution to the above problem in the space $\mathcal{H}^1_{1/2}(\mathcal{I}) = \overline{C^\infty(\overline{\mathcal{I}})}^{\|.\|_{|\alpha|^{1/2}}}$, where

$$\|v\|^2_{|\alpha|^{1/2}} = \int_{\mathcal{I}} |v|^2 + \int_{\mathcal{I}} |x||v'|^2.$$

In the definition of the norm above, the choice of the weight in front of $|v'|^2$ is motivated by the ODE itself, since after multiplying it by any admissible function v supported away from 0 in either $\mathcal{I}_p = (0,a)$ or $\mathcal{I}_n = (-a, 0)$ and integrating by parts, one gets a volume term like $\pm \int_{\mathcal{I}_{p,n}} |x| u' v'$. In this space, the bilinear form associated to the above equation is obviously continuous. A straightforward computation shows that in this case u is a piecewise constant function

$$u = c_1 \text{ on } (-a, 0), \quad u = c_2 \text{ on } (0, a).$$

We see that the above solution does not contain any singularity other than the jump at the origin.

On the other hand, we can have a look at the limiting absorption solution to the above equation, where the absorption solution solves

$$-((x+i\nu)(u^\nu)')' = 0 \text{ on } \mathcal{I}, \quad u^\nu(-a) = c_1,\, u^\nu(a) = c_2.$$

In particular, for each $\nu > 0$ the $H^1(\mathcal{I})$-solution to this problem is unique. With $z \mapsto \log z$ defined by its principal value (i.e., $\log(z) = \log|z| + i\,\mathrm{Arg}(z)$, $\mathrm{Arg}\, z \in (-\pi,\pi)$), we then compute the solution to the above equation

$$u^\nu = a_\nu \log(x+i\nu) + b_\nu, \quad \text{with } a_\nu = \frac{c_2 - c_1}{\log(a+i\nu) - \log(-a+i\nu)}, \quad \text{and } b_\nu = c_2 - a_\nu \log(a+i\nu).$$

Since $\log(x+i\nu) \xrightarrow[\nu\to 0^+]{} \log|x| + i\pi\mathbb{1}_{x<0}$ for $x \in \mathbb{R}^*$, the limiting absorption solution $u^+(x) = \lim\limits_{\nu\to 0+} u^\nu$ is given by the pointwise limit

$$u^+(x) = a_+\left(\log|x| + i\pi\mathbb{1}_{x<0}\right) + b_+, \quad a_+ = \frac{c_1 - c_2}{i\pi}, \quad b_+ = c_2 - a_+ \log a.$$

Note that $\int_{\mathcal{I}} |x||(u^+)'|^2 = +\infty$ as soon as $a_+ \neq 0$. We thus see the difference between two solutions $u \in \mathcal{H}^1_{1/2}(\mathcal{I})$ and u^+: the first one has a jump singularity only, while the second one has both a logarithmic and a jump singularities. Therefore, we focus on approximating the latter solution that includes the jump and logarithmic singularities for the 2D model (4.2). From now on, we use the following notation to describe the singularity:

$$\mathrm{S}(x) = \log|x| + i\pi\mathbb{1}_{x<0}.$$

Let $C^\infty_{per,y}(\overline{\Omega}_j) = \{v \in C^\infty(\overline{\Omega}_j) \,:\, \partial_y^m v(x,0) = \partial_y^m v(x,L), \forall m\}$, $j \in \{p,n\}$. Introduce the two spaces $H^1_{1/2}(\Omega_j) = \overline{C^\infty_{per,y}(\overline{\Omega}_j)}^{\|\cdot\|_{|\alpha|^{1/2}}}$, $j \in \{p,n\}$, with associated norm

$$\|v\|^2_{|\alpha|^{1/2}} = \int_{\Omega_j} |v|^2 + \int_{\Omega_j} |\alpha||\nabla v|^2, \quad j \in \{p,n\}.$$

Defining the above spaces is motivated by the same observation as above: multiplying the second-order PDE by any admissible function v supported either in Ω_p or Ω_n and integrating by parts, one gets a volume term like $\pm \int_{\Omega_{p,n}} |\alpha|\nabla u \cdot \nabla v$. It has been proved in [49, Proposition 4] that problem (4.2) admits a unique solution in $H^1_{1/2}(\Omega_p) \times H^1_{1/2}(\Omega_n)$. On the other hand, $\mathrm{S} \notin H^1_{1/2}(\Omega_j)$, $j \in \{p,n\}$ because of the logarithmic singularity.

In light of the 1D example, we consider from now on that one can recast the 2D model with solution u as follows.

Assumption 4.1.2. *The family of solutions $(u^\nu)_{\nu>0}$ of (4.3) converges in $L^2(\Omega)$ to the limiting absorption solution $u^+ \in L^2(\Omega)$*

$$u^\nu \xrightarrow[\nu\to 0+]{L^2(\Omega)} u^+. \tag{4.4}$$

Moreover, u^+ can be represented as

$$u^+ = u^+_{reg} + u^+_{sing},$$

where the pair (u^+_{reg}, u^+_{sing}) is such that $u^+_{reg|\Omega_{p,n}} \in H^1_{1/2}(\Omega_{p,n})$ and $u^+_{sing}(x,y) = g^+(y)\mathrm{S}(x)$ with $g^+ \in H^1_{per}(\Sigma)$.

Regarding the convergence, the validity of this assumption will be studied in chapter 5. On the other hand, the mathematical relevance of assumption 4.1.2 is discussed in the next proposition.

Proposition 4.1.3. *Let u^+ be governed by (4.2).*

In the framework of Assumption 4.1.2, there holds that $u^+ \in H^1(\Omega \backslash \mathscr{V}(\Sigma))$, for each neighborhood of the interface $\mathscr{V}(\Sigma)$.

Conversely, if u can be decomposed as $u^+ = u^+_{reg} + u^+_{sing}$ where $u^+_{reg}\big|_{\Omega_{p,n}} \in H^1_{1/2}(\Omega_{p,n})$ is periodic in y-direction and where $u^+_{sing}(x,y) = g^+(y)\mathrm{S}(x)$, and if there holds that $u^+ \in H^1(\Omega \setminus \mathscr{V}(\Sigma))$ for each neighborhood of the interface $\mathscr{V}(\Sigma)$, then $g^+ \in H^1_{per}(\Sigma)$.

Proof. Away from the interface Σ, the norms of $H^1_{1/2}(\Omega_{p,n} \backslash \mathscr{V}(\Sigma))$ and $H^1(\Omega_{p,n} \backslash \mathscr{V}(\Sigma))$ are equivalent. Therefore, given u^+ within the framework of Assumption 4.1.2, we clearly have $u^+_{reg}, u^+_{sing} \in H^1(\Omega_{p,n} \backslash \mathscr{V}(\Sigma))$.

On the other hand, if $u^+ \in H^1(\Omega \setminus \mathscr{V}(\Sigma))$ and $u^+_{reg}\big|_{\Omega_{p,n}} \in H^1_{1/2}(\Omega_{p,n})$ for all neighborhood of the interface $\mathscr{V}(\Sigma)$, then $u^+_{sing} \in H^1(\Omega \setminus \mathscr{V}(\Sigma))$. In particular for $\Omega^\varepsilon_p = \left\{(x,y) \in \Omega_p \,:\, x > \varepsilon\right\}$, we have that

$$\left\|u^+_{sing}\right\|^2_{H^1(\Omega^\varepsilon_p)} = \left\|g^+\right\|^2_{L^2(\Sigma)} \left\|\mathrm{S}\right\|^2_{H^1(\varepsilon,a)} + \left\|g^{+\prime}\right\|^2_{L^2(\Sigma)} \left\|\mathrm{S}\right\|^2_{L^2(\varepsilon,a)},$$

which shows that $g^+ \in H^1(\Sigma)$. The periodic conditions come from the periodicity of u^+ and u^+_{reg}. □

Starting from (4.4), it is easy to verify that the limiting absorption solution u^+ is governed by the 2D model

$$\begin{cases} -\operatorname{div}\left(\alpha \nabla u^+\right) - \omega^2 u^+ = 0 & \text{in } \Omega_{p,n}, \\ \alpha \partial_n u^+ + i\lambda u^+ = f & \text{on } \Gamma_{p,n}, \\ u^+(x,0) = u^+(x,L), \quad (\alpha\partial_y)u^+(x,0) = (\alpha\partial_y)u^+(x,L), & x \in (-a,\, a) \text{ a.e.} \end{cases} \tag{4.5}$$

We identify the function u^+_{reg} with a pair

$$\mathbf{u}^+ = (u^+_{reg|\Omega_p}, u^+_{reg|\Omega_n}) \in Q := H^1_{1/2}(\Omega_p) \times H^1_{1/2}(\Omega_n). \tag{4.6}$$

For generic $g(y)$, we use the notation

$$s_g(x,y) := g(y)\mathrm{S}(x). \tag{4.7}$$

As noticed in the 1D example, when the *singular coefficient* g^+ does not vanish, s_{g^+} does not belong to the space $H^1_{1/2}(\Omega_p) \times H^1_{1/2}(\Omega_n)$, hence the notation s_{g^+}, with s for "singular".

Given $\mathbf{u}^+ = (u^+_p, u^+_n) \in H^1_{1/2}(\Omega_p) \times H^1_{1/2}(\Omega_n)$ and $g^+ \in H^1_{per}(\Sigma)$ a solution of the system (4.5), no transmission condition through Σ is imposed a priori between u^+_p and u^+_n. On the other hand, the convergence assumption (4.4) contains a hidden transmission condition through Σ, as we will see in chapter 6.

Remark 4.1.4. One could also examine a problem with a non-smooth sign-changing coefficient α. For example, $\alpha(x) = \operatorname{sign}(x)|x|^\beta$ with $\beta > 0$. Then, one could add some absorption and solve $-\left((\alpha(x) + i\nu)u^{\nu\prime}\right)' = 0$. Then, using [51, (15.6.1)], and the help of some formal computation software, the solution writes $u^\nu(x) = \frac{x}{i\nu} F\left(1, \frac{1}{\beta}; 1 + \frac{1}{\beta}; -\frac{\operatorname{sign}(x)|x|^\beta}{i\nu}\right)$ where F is the hypergeometric function, see [51, (15.2.1)].

Remark 4.1.5. It is unclear if we can use $H^1_{1/2}(\Omega) := \overline{C^\infty_{per,y}(\overline{\Omega})}^{\|\cdot\|_{|\alpha|^{1/2}}}$. Indeed, it might contain functions which are not distributions. Therefore, we use separate spaces on both side of the interface, as suggested in [41].

4.2 The method from Nicolopoulos et al.

We now recall the main ingredients that were used by Nicolopoulos, Campos Pinto, et al. in [49, 50] to build a numerical approximation to problem (4.2), with its solution split into a regular and a singular part. We emphasize that the derivation of the mathematical model is formal, cf. [49, theorem 2], whereas we propose a mathematical derivation based on assumption 4.1.2. Also, a stronger assumption on the singular part was used in [49], namely that $g \in H^2_{per}(\Sigma)$, which has strong consequences numerically, see §4.2.2.

4.2.1 Main ideas and formulation of the method

In order to explain the method of [49], let us introduce the following functions, which we describe as "singularities with absorption"

$$s^\nu_g(x, y) := g(y) \log\left(x + \frac{i\nu}{\mathtt{r}(y)}\right) \quad \text{with} \quad \nu > 0,\ g \in H^1_{per}(\Sigma). \tag{4.8}$$

The absorption parameter scaled by $1/\mathtt{r}(y)$ will ensure some nice convergence properties on second order derivatives. We also introduce the weighted $L^2(\Sigma)$-norm

$$\|g\|_{\mathtt{r}} := \left(\int_\Sigma |g(y)|^2 \mathtt{r}(y) dy\right)^{1/2} \tag{4.9}$$

and its associated inner product is denoted by $(\cdot,\cdot)_{\mathtt{r}}$. Note that for all $g \in L^2(\Sigma)$, it is easily checked that $s^\nu_g \to s_g$ in $L^2(\Omega)$ as $\nu \to 0+$. One needs the following three technical results, whose proofs are given in section 6.2.1.

Lemma 4.2.1. *Given $g \in H^1_{per}(\Sigma)$, the following limits hold in $L^2(\Omega)$ as $\nu \to 0+$:*

$$\begin{aligned} &s^\nu_g \to s_g, && \partial_y s^\nu_g \to \partial_y s_g,\\ &(\alpha + i\nu)\partial_x s^\nu_g \to \alpha\partial_x s_g, && \partial_x((\alpha + i\nu)\partial_x s^\nu_g) \to \partial_x(\alpha\partial_x s_g). \end{aligned}$$

Let φ be a truncation function satisfying

Definition 4.2.2. Given $\varphi_1 \in C^1_0((-a, a); \mathbb{R})$ and $\varphi_1 = 1$ in the vicinity of $x = 0$, let $\varphi(x, y) = \varphi_1(x)$.

This function is used to localize the contribution near the interface. The method of [49] relies on the observation that, for $g \neq 0$, the singular ansatz s_g does not belong to Q.

Lemma 4.2.3. *Let $g \in H^1_{per}(\Sigma)$ and a truncation function φ as in definition 4.2.2. Then the following limit holds:*

$$\lim_{\nu\to 0+} \int_\Omega \nu|\nabla s^\nu_g|^2 \varphi\, d\mathbf{x} = \pi\|g\|^2_{\mathtt{r}} > 0.$$

Physically, the above identity is related to the plasma heating [26, 50]. Let $u^\nu \in H^1(\Omega)$ be the unique solution of (4.3). By assumption 4.1.2, $u^\nu \to u^+_{reg} + s^+_g$ in $L^2(\Omega)$, as $\nu \to 0+$. We then split $u^\nu = u^\nu_{reg} + s^\nu_{g^+}$; evidently, $s^\nu_{g^+} \to s^+_g$ in $L^2(\Omega)$, and $u^\nu_{reg} \to u^+_{reg}$ in $L^2(\Omega)$. Recall that u^+_{reg} is identified with a pair of functions $\mathbf{u}^+ = (u^+_p, u^+_n)$.

Lemma 4.2.4. *Let $(u^\nu)_{\nu>0}$ be a family governed by (4.3) fulfilling assumption 4.1.2. Then,*

$$\lim_{\nu\to 0+} \int_\Omega \nu \left|\nabla u^\nu_{reg}\right|^2 \varphi \, \mathrm{d}\mathbf{x} = 0. \tag{4.10}$$

The above observations serve as a basis to construct a functional to minimize; this minimization procedure will yield a desired variational formulation. As a matter of fact, if we consider the function s^ν_h, with $h \in H^2_{per}(\Sigma)$ being an artificial variable: we observe that

$$\int_\Omega \nu \left|\nabla\left(u^\nu_{reg} + s^\nu_{g^+-h}\right)\right|^2 \varphi \, \mathrm{d}\mathbf{x}$$

goes to 0 as ν goes to 0 if $h = g^+$. The difficulty is now to link this result with the problem at hand. Interestingly, one may rewrite the above integral as

$$\int_\Omega \nu \left|\nabla\left(u^\nu_{reg} + s^\nu_{g^+-h}\right)\right|^2 \varphi \, \mathrm{d}\mathbf{x} = \mathrm{Im}\left(\int_\Omega (\alpha(x,y) + i\nu)\left|\nabla\left(u^\nu_{reg} + s^\nu_{g^+} - s^\nu_h\right)\right|^2 \varphi \, \mathrm{d}\mathbf{x}\right).$$

Hence, if we define the following energy functional:

$$\mathcal{J}^\nu(u^\nu_{reg}, g^+, h) := \mathrm{Im}\left(\int_\Omega (\alpha(x,y) + i\nu)\left|\nabla\left(u^\nu_{reg} + s^\nu_{g^+} - s^\nu_h\right)\right|^2 \varphi \, \mathrm{d}\mathbf{x}\right), \tag{4.11}$$

we have that $\mathcal{J}^\nu(u^\nu_{reg}, g^+, h)$ converges to the limit $\pi\|g^+ - h\|^2_\Gamma$ when $\nu \to 0+$. We obviously observe that $\lim_{\nu\to 0^+} \mathcal{J}^\nu(u^\nu_{reg}, g^+, g^+) = 0$. Hence, the limit of (u^ν_{reg}, g^+, g^+) should be a solution to a minimization problem expressed variationally. Introducing $V^{(2)} := Q \times H^2_{per}(\Sigma) \times H^2_{per}(\Sigma)$, we find that the limit is governed by the following mixed variational formulation (see chapter 6 for a complete derivation):

$$\text{Find } (\mathbf{u}, g, h) \in V^{(2)} \text{ and } \boldsymbol{\lambda} \in Q \text{ such that}$$
$$\begin{cases} a^{(2)}((\mathbf{u},g,h),(\mathbf{v},k,l)) - \overline{b^{(2)}((\mathbf{v},k,l),\boldsymbol{\lambda})} = 0, & \forall (\mathbf{v},k,l) \in V^{(2)}, \\ b^{(2)}((\mathbf{u},g,h),\boldsymbol{\mu}) = \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q. \end{cases} \tag{4.12}$$

First, the form $a^{(2)} : V^{(2)} \times V^{(2)} \to \mathbb{C}$ can be recast as

$$\begin{aligned} a^{(2)}((\mathbf{u},g,h),(\mathbf{v},k,l)) := \sum_{j\in\{p,n\}} &\int_{\Omega_j} \alpha(u_j + s_{g-h})\partial_x\overline{(v_j + s_{k-l})}\partial_x\varphi \mathrm{d}\mathbf{x} \\ &- \int_{\Omega_j} \alpha\overline{(v_j + s_{k-l})}\partial_x(u_j + s_{g-h})\partial_x\varphi \mathrm{d}\mathbf{x} \\ &- \int_{\Omega_j} \left(-\operatorname{div}\alpha\nabla s_h - \omega^2 s_h\right)\overline{(v_j + s_{k-l})}\varphi \mathrm{d}\mathbf{x} \\ &+ \int_{\Omega_j} \overline{\left(-\operatorname{div}\alpha\nabla s_l - \omega^2 s_l\right)}(u_j + s_{g-h})\varphi \mathrm{d}\mathbf{x}. \end{aligned} \tag{4.13}$$

The sesquilinear form $b^{(2)} : V^{(2)} \times Q \to \mathbb{C}$ is, in its turn, given by

$$b^{(2)}((\mathbf{u},g,h),\mathbf{v}) = b^{(2)}_{reg}(\mathbf{u},\mathbf{v}) + b^{(2)}_{sing}(g,\mathbf{v}), \tag{4.14}$$

where, for all $\mathbf{u}, \mathbf{v} \in Q$, $g \in H^2_{per}(\Sigma)$

$$b^{(2)}_{reg}(\mathbf{u}, \mathbf{v}) := \sum_{j\in\{p,n\}} \int_{\Omega_j} \left(\alpha\nabla u_j \cdot \overline{\nabla v_j} - \omega^2 u_j \overline{v_j}\right) \mathrm{d}\mathbf{x} + \int_{\Gamma_j} i\lambda u_j \overline{v_j} \mathrm{d}s, \tag{4.15}$$

$$b^{(2)}_{sing}(g, \mathbf{v}) := \sum_{j\in\{p,n\}} \int_{\Omega_j} \left(-\operatorname{div}(\alpha\nabla s_g) - \omega^2 s_g\right) \overline{v_j} \mathrm{d}\mathbf{x} + \int_{\Gamma_j} (\alpha\partial_n s_g + i\lambda s_g)\overline{v_j} \mathrm{d}s. \tag{4.16}$$

Finally, the antilinear form $\ell^{(2)}(\boldsymbol{\mu})$ is defined as

$$\ell^{(2)}(\boldsymbol{\mu}) = \sum_{j\in\{p,n\}} \int_{\Gamma_j} f\overline{\boldsymbol{\mu}} \, \mathrm{d}s.$$

In order to guarantee the well-posedness, two stabilization terms are added to (4.12). More precisely, one considers

$$\text{Find } (\mathbf{u}, g, h) \in V^{(2)} \text{ and } \boldsymbol{\lambda} \in Q \text{ such that}$$
$$\begin{cases} a^{(2)}_{\rho,\mu}((\mathbf{u}, g, h), (\mathbf{v}, k, l)) - \overline{b^{(2)}((\mathbf{v}, k, l), \boldsymbol{\lambda})} = 0, & \forall (\mathbf{v}, k, l) \in V^{(2)}, \\ b^{(2)}((\mathbf{u}, g, h), \boldsymbol{\mu}) = \ell^{(2)}(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q, \end{cases} \tag{4.17}$$

where

$$a^{(2)}_{\rho,\mu}((\mathbf{u}, g, h), (\mathbf{v}, k, l)) = a^{(2)}((\mathbf{u}, g, h), (\mathbf{v}, k, l)) + i\left(-\rho(g, k)_{H^2(\Sigma)} + \mu(\partial_{yy}h, \partial_{yy}l)_{L^2(\Sigma)}\right),$$

with $\rho, \mu > 0$. The form of the stabilization terms follows from the T-coercivity requirements on the first sesquilinear form $a^{(2)}$, see [18] for the definition of the T-coercivity. It is shown in [49, Theorem 16] that for $\rho, \mu > 0$, and $f \in L^2(\Gamma_p \cup \Gamma_n)$, the problem (4.17) is well-posed. However, up to our knowledge, there exists no proof that the solution to (4.17) is a limiting absorption solution of the original problem.

4.2.2 Numerical experiments and comments

In [49], a conforming discretization of (4.17) was proposed, with $V^{(2)}_{h_1,h_2} = Q_{h_1} \times H^2_{h_2} \times H^2_{h_2}$,

$$Q_{h_1} = \{v_{h_1} \in Q \ : \ v_{h_1}\big|_K \in P_1(K), \text{ for all } K \in \mathcal{T}^{\Omega}_{h_1}\},$$
$$H^2_{h_2} = \{p_{h_2} \in H^2_{per}(\Sigma) \ : \ p_{h_2}\big|_K \in H_m(K), \text{ for all } K \in \mathcal{T}^{\Sigma}_{h_2}\},$$

where $H_m(K)$ is Hermite finite element of order m, $\mathcal{T}^{\Omega}_{h_1}$ is a triangulation of Ω with meshsize h_1 that is conforming with respect to the interface Σ (for all $K \in \mathcal{T}_{h_1}$, $\operatorname{int}(K) \cap \Sigma = \emptyset$), and $\mathcal{T}^{\Sigma}_{h_2}$ is a triangulation of Σ with meshsize h_2. Notice that the restriction to Ω_p (respectively Ω_n) of elements of Q_{h_1} belongs to $H^1(\Omega_p)$ (resp. $H^1(\Omega_n)$). On the other hand, there is no matching condition at the interface for elements of Q_{h_1}.

The discretization of (4.17) leads to the linear system $\boldsymbol{A}U_{h_1,h_2} = L$ where:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}^{(2)} & -\boldsymbol{B}^{(2)^\dagger} \\ \boldsymbol{B}^{(2)} & 0 \end{pmatrix},$$

$$\boldsymbol{A}^{(2)} = \begin{pmatrix} \boldsymbol{A}_p & 0 & -\boldsymbol{A}_{p,\Sigma_g}{}^\dagger & -\boldsymbol{A}_{p,\Sigma_h}{}^\dagger \\ 0 & \boldsymbol{A}_n & -\boldsymbol{A}_{n,\Sigma_g}{}^\dagger & -\boldsymbol{A}_{n,\Sigma_h}{}^\dagger \\ \boldsymbol{A}_{p,\Sigma_g} & \boldsymbol{A}_{n,\Sigma_g} & \boldsymbol{A}_{\Sigma_g} & -\boldsymbol{A}_{\Sigma_g,\Sigma_h}{}^\dagger \\ \boldsymbol{A}_{p,\Sigma_h} & \boldsymbol{A}_{n,\Sigma_h} & \boldsymbol{A}_{\Sigma_g,\Sigma_h} & \boldsymbol{A}_{\Sigma_h} \end{pmatrix}, \qquad \boldsymbol{B}^{(2)} = \begin{pmatrix} \boldsymbol{B}_p & 0 & \boldsymbol{B}_{\Sigma_g,p} & 0 \\ 0 & \boldsymbol{B}_n & \boldsymbol{B}_{\Sigma_g,n} & 0 \end{pmatrix},$$

$$U_{h_1,h_2} = \begin{pmatrix} U_{p,h_1} & U_{n,h_1} & G_{h_2} & H_{h_2} & \Lambda_{p,h_1} & \Lambda_{n,h_2} \end{pmatrix}^\top, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 & L_p & L_n \end{pmatrix}^\top,$$

where the stars stand for minus transpose conjugate, so that $\boldsymbol{A} = -\boldsymbol{A}^\dagger$, $\boldsymbol{A}^{(2)} = -\boldsymbol{A}^{(2)^\dagger}$. This holds because the sesquilinear form $a^{(2)}_{\rho,\mu}$ is anti-hermitian[1].

In the original paper [49], the numerical experiments were done for a single discretization. Structured meshes were used for both regular and singular parts, with $h_2 = 4h_1$. In particular, the question of the convergence of the discrete solution to the continuous one was not addressed. The goal of this section is to provide insight into this question, by letting h_1 vary and keeping $h_2 = 4h_1$.

We consider the case $\alpha(x, y) = x$, $\omega = 0$ and perform two experiments with $L = 2$ on the domain $\Omega = (-1, 1) \times (-1, 1)$ with known exact solutions. We choose the boundary data so that, in the first case, the exact solution is purely regular and equal to $u(x, y) = 1$, and in the second case it is given by $u(x, y) = \log|x| + i\pi\mathbb{1}_{x<0}$. In the first case $g(y) = 0$, while in the second case, there is a non-zero singular part $s_g(x, y)$ with $g(y) = 1$. Notice that the stabilization parameters ρ, μ are taken equal $\rho = \mu = 10^{-5}$.

We use the code provided by A. Nicolopoulos written in `FreeFem++` [32]. The amplitude of the singular part g was discretized with the 2D HCT finite elements penalized along the normal direction, i.e., we add to $\boldsymbol{A}_{\Sigma_g}$ and $\boldsymbol{A}_{\Sigma_h}$ the corresponding term to the discretization of the following sesquilinear form, with a coefficient $Z > 0$ large enough:

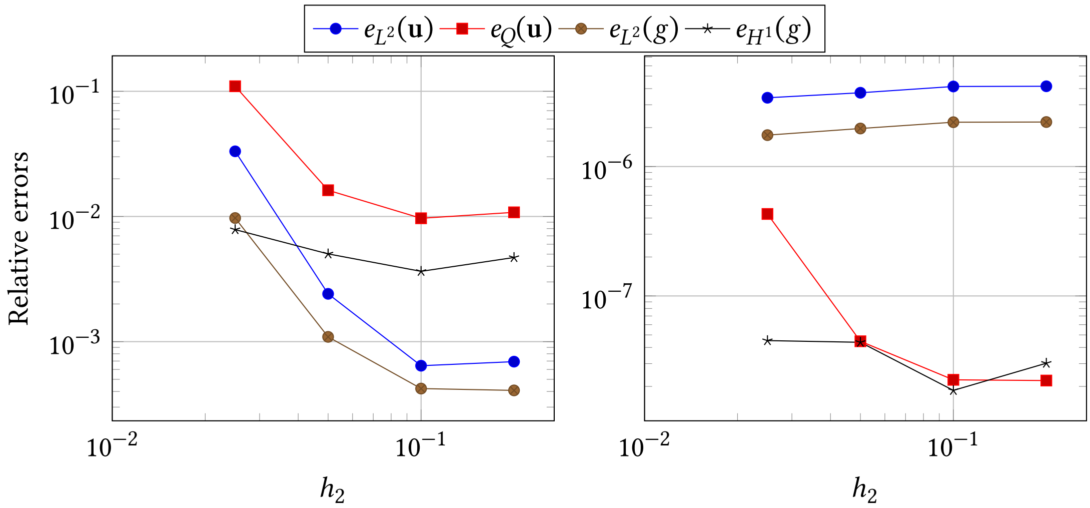
$$Z\, i \int_\Omega \partial_x g\, \overline{\partial_x k}\, \mathrm{d}\mathbf{x}.$$

Notice that results given below may vary depending on the mesh used to discretize the singular part g.

We denote by $e_{L^2}(\mathbf{u})$, resp. $e_Q(\mathbf{u})$, the relative error of the regular part in $L^2(\Omega)$-norm, resp. in $\|\cdot\|_Q$ norm. And we denote by $e_{L^2}(g)$ the relative error of the singular coefficient in $L^2(\Sigma)$-norm. Note that, when measuring volume errors, we do not take into account the cells that touch the interface because we observed that the errors were strongly localized in these cells. This phenomenon is clearly linked to the singular behavior of α near the interface.

Although $u = 1$ belongs to the discrete space, the computed solution does not seem to converge, see figure 4.2a on a log-log scale. This phenomenon happens regardless of the value of the stabilization parameters ρ, μ from 10^{-2} to 10^{-7}. On the other hand, the results with $u = \log|x| + i\pi\mathbb{1}_{x<0}$ are promising in the sense that the relative error is small, although it does not converge; see Figure 4.2b.

[1] A sesquilinear form $b : V \times V$ is anti-hermitian if $b(u, v) = -\overline{b(v, u)}$.



(a) $u = 1$.

(b) $u = \log|x| + i\pi\mathbb{1}_{x<0}$.

Figure 4.2: Relative errors with structured meshes for regular and singular parts.

These experiments seem to indicate that the numerical method of [49] does not converge numerically. We do not know whether the source of the instability is intrinsic to the numerical variational formulation itself, or is due to the penalization of the HCT elements in the normal direction, used in the implementation. In this manuscript, we will not dwell on the precise reason for this instability. Instead, we propose in chapter 6 an alternative method.

CHAPTER 5

Fourier analysis

Contents

5.1 Problem setting and definitions

Let $\Omega = (-1, 1) \times (0, L)$, with the notations of Chapter 4. The aim of this chapter is the study of the problem

$$\left| \begin{array}{ll} \text{find } u \text{ such that} & \\ -\operatorname{div}(\alpha \nabla u) - \omega^2 u = f_\Omega & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_n \cup \Gamma_p, \\ u(x,0) = u(x,L), \quad \left(\alpha \partial_y\right) u(x,0) = \left(\alpha \partial_y\right) u(x,L), & x \in (-1,1), \end{array} \right. \tag{5.1}$$

where $f_\Omega \in L^2(\Omega)$. Finding desirable properties of f_Ω from which the regularity of solutions can be inferred a priori is actually one of the issues of this chapter.

Therefore, the method used below consists in decomposing the solution u into an appropriate function basis. In order to find the appropriate function basis, we also formulate the following assumption on the pattern of α.

Assumption 5.1.1. *Given* $\mathtt{r}(y) \in \mathscr{C}^1_{per}([0, L]; (0, +\infty))$, *the function* α *can be written as* $\alpha(x, y) = \mathtt{r}(y)x$.

The interface in the sense of Chapter 4 is thus $\Sigma = \{0\} \times [0, L)$. Moreover, we divide the domain into two parts $\Omega_p = (0, 1) \times (0, L)$ and $\Omega_n = (-1, 0) \times (0, L)$. Aside from this assumption, we have switched from inhomogeneous Robin's boundary conditions to homogeneous Dirichlet's boundary conditions. This change is not restrictive. Indeed, given u the solution to the problem with Robin's boundary conditions, it is assumed H^1 away of the interface Σ. Therefore, it is also a solution to the problem with Dirichlet's boundary conditions formulated on a subset of the original domain and the subset would still contain the interface. This modification simplifies the computations and enable us to concentrate on the behavior of the problem near the interface. Besides, homogeneous Dirichlet's boundary conditions are not restrictive since it is always possible to lift these boundary conditions. Then, such a lifting is taken into account into the volume source term and the boundary conditions become homogeneous. It is always possible to construct lifting whose support does not intersect the interface.

Indeed, notice the appearance of the volume source term f_Ω, in comparison with Chapter 4. Since this source term may induce a behavior of the solution u "more" singular than the expected logarithmic plus jump singularity, the influence of the source term will be carefully examined throughout this chapter. Up to our knowledge, only vanishing source term with non-homogeneous boundary conditions where considered in the study of lower hybrid resonance. As noticed in the previous paragraph, such boundary conditions can be modeled by a source term with a disjoint support from the interface.

This chapter begins with a precise explicit description of the solutions of the problem. Then, since the computations are explicit, basic regularity properties are studied. Finally, the limiting absorption principle is proved. The argument is developed only for $\omega = 0$, even though the same method also applies for $\omega \neq 0$. In the view of the method used, the results obtained are obviously valid in any dimension $\mathbb{R}^{d+1}$, assuming the interface is a d-dimensional manifold.

Let us outline the method used. If $\mathtt{r}(y) = 1$, then we solve

$$-\operatorname{div}(x\nabla u) = f \quad \text{in} \quad \Omega.$$

We can consider the Fourier's expansion of u with respect to the variable y, so that we can decompose $u(x, y) = \sum_{k\in\mathbb{Z}} u_k(x) \exp\left(\frac{2ik\pi y}{L}\right)$. Then one can derive an equation for each $u_k(x)$, solve it, etc...This process leads to an explicit formula ready to be studied.

In a general case, $\mathtt{r}(y)$ is not a constant. Therefore, one has to decompose $u(x, y)$ on a basis that is adapted to $\mathtt{r}(y) \in \mathscr{C}^1_{per}\left([0, L]; \mathbb{R}^+_*\right)$. To that aim, we define the following complex Hilbert space $L^2_{\mathtt{r}}(0, L) \equiv L^2(0, L)$ with its inner product

$$(u, v)_{\mathtt{r}} = \int_0^L u(y)\overline{v(y)}\mathtt{r}(y)\mathrm{dy},$$

and the following functions, given by the spectral theorem.

Definition 5.1.2. Let $(\psi_k)_{k\in\mathbb{N}}$ be the eigenvectors of the operator $-\mathtt{r}^{-1}\partial_y\left(\mathtt{r}\partial_y\cdot\right)$ on $(0, L)$ with periodic boundary conditions, such that $(\psi_k)_{k\in\mathbb{N}}$ constitute a normalized orthogonal Hilbert basis of $L^2_{\mathtt{r}}(0, L)$. Each eigenvector ψ_k is associated to a real positive eigenvalue λ_k^2 such that the sequence $(\lambda_k)_{k\geq 0}$ increases and $\lim_{k\to+\infty} \lambda_k = +\infty$. In particular $\lambda_0 = 0$ and ψ_0 is constant.

If $u \in L^2_{\mathtt{r}}(0,L)$, then we can decompose it as $u(y) = \sum_{k\in\mathbb{N}} u_k\psi_k(y)$. Parseval's identity ensures that

$$\|u\|^2_{L^2_{\mathtt{r}}(0,L)} = \sum_{k\in\mathbb{N}} |u_k|^2.$$

Lemma 5.1.3. *Let $u \in L^2_{\mathtt{r}}(0,L)$ be such that $\partial_y u \in L^2_{\mathtt{r}}(0,L)$. Then*

$$\left\|\partial_y u\right\|^2_{L^2_{\mathtt{r}}(0,L)} = \sum_{k\in\mathbb{N}} |\lambda_k u_k|^2.$$

Proof. By definition, for all $k \in \mathbb{N}$, the eigenvector ψ_k verifies

$$\left(\partial_y\psi_k, \partial_y v\right)_{L^2_{\mathtt{r}}(0,L)} = \lambda_k^2 \left(\psi_k, v\right)_{L^2_{\mathtt{r}}(0,L)}, \qquad \forall v \in H^1_{per}(0,1).$$

On one hand, this yields $\left\|\partial_y\psi_k\right\|^2_{L^2_{\mathtt{r}}(0,L)} = \lambda_k^2 \left\|\psi_k\right\|^2_{L^2_{\mathtt{r}}(0,L)} = \lambda_k^2$ because $(\psi_k)_k$ are normalized, and on the other hand, $\left(\partial_y\psi_k, \partial_y\psi_l\right)_{L^2_{\mathtt{r}}(0,L)} = 0$, for $k \neq l$, because $(\psi_k)_k$ are orthogonal. Thus, we have that

$$\left\|\partial_y u\right\|^2_{L^2_{\mathtt{r}}(0,L)} = \sum_{k\in\mathbb{N}} |u_k|^2 \left\|\partial_y\psi_k\right\|^2_{L^2_{\mathtt{r}}(0,L)} = \sum_{k\in\mathbb{N}} |\lambda_k u_k|^2.$$

□

From this point, we define the following fractional Sobolev spaces for $s \geq 0$ with their associated norms:

$$H^s_{\mathtt{r}}(0,L) := \left\{u \in L^2_{\mathtt{r}}(0,L) \; : \; \sum_{k\in\mathbb{N}} \left|\lambda_k^s u_k\right|^2 < \infty\right\}, \qquad \|u\|^2_{H^s_{\mathtt{r}}} = \sum_{k\in\mathbb{N}} \left(1+\lambda_k^2\right)^s |u_k|^2.$$

Finally, we define the following classical Sobolev space on Ω:

$$H^1_{\mathtt{r}}(\Omega) = \left\{u \in L^2_{\mathtt{r}}(\Omega) \; : \; \nabla u \in L^2_{\mathtt{r}}(\Omega)\right\}, \qquad \|u\|^2_{H^1_{\mathtt{r}}(\Omega)} = \int_\Omega \left[|u|^2 + |\nabla u|^2\right] \mathtt{r}(y)\mathrm{d}\mathbf{x}.$$

Then, it is naturally connected with $H^1_{\mathtt{r}}(0,L)$, so that we can decompose $u \in H^1(\Omega, \mathtt{r})$ as $u(x,y) = \sum_{k\in\mathbb{N}} u_k(x)\psi_k(y)$. Moreover, this provides the following equivalent norm:

$$\|u\|^2_{H^1(\Omega,\mathtt{r})} = \sum_{k\in\mathbb{N}} \left[\left(1+\lambda_k^2\right)\|u_k\|^2_{L^2(-1,1)} + \|\partial_x u_k\|^2_{L^2(-1,1)}\right].$$

5.2 Limiting absorption solution

Recall that we focus on the case $\omega = 0$. It has been observed in [50] that the equation of problem (5.1) with $\mathtt{r}(y) = 1$ and $L = 2\pi$ can be solved using the classical Fourier's expansion. This leads to solve the very well-known *modified Bessel equation*:

$$-x\partial_x\left(x\partial_x u_k\right) + (kx)^2 u_k = x f_k. \tag{5.2}$$

A feature of this ODE is that it is singular at the point $x = 0$. A general solution of the homogeneous equation can be written as

$$u_k(x) = \begin{cases} a_{k,+}I_0(kx) + b_{k,+}K_0(kx) & \text{if } x > 0, \\ a_{k,-}I_0(kx) + b_{k,-}K_0(kx) & \text{if } x < 0. \end{cases}$$

However, the two boundary conditions do not provide enough equations to close the problem, i.e., to completely determine the coefficient $a_{k,\pm}$ and $b_{k,\pm}$. Thus, the aim of this part is to derive a new equation in order to "glue" together the solutions on either side of the point $x = 0$. Like in chapter 4, we proceed by introducing small absorption and then letting this absorption tend to 0. This approach is common in literature, for example Campos Pinto and Després used this method in [14] to extract variational relations.

Therefore, we look for a solution u of (5.1) such that it is also the limiting absorption solution, i.e., u the limit in some sense of $(u^\nu)_{\nu>0}$ as $\nu \to 0+$ and u^ν solves the following problem.

$$\left|\begin{array}{ll} \text{Find } u^\nu \text{ such that} & \\ -\operatorname{div}\left(\mathtt{r}(y)\,(x+i\nu)\,\nabla u^\nu\right) = \mathtt{r}(y) f & \text{in } \Omega, \\ u^\nu = 0 & \text{on } \Gamma_n \cup \Gamma_p, \\ u^\nu(x,0) = u^\nu(x,L), \quad \left(\mathtt{r}(y)\,(x+i\nu)\,\partial_y\right) u^\nu(x,0) = \left(\mathtt{r}(y)\,(x+i\nu)\,\partial_y\right) u^\nu(x,L), & x \in (-1,1). \end{array}\right. \tag{5.3}$$

Notice the change of source term $f = f_\Omega/\mathtt{r} \in L^2(\Omega)$, in order to simplify the computations below. One can prove via the Lax-Milgram theorem that this problem is well-posed in $H^1(\Omega)$.

5.2.1 Solution with absorption

The first result of this part consists in a precise description of the solution with a small absorption $\nu > 0$.

Proposition 5.2.1. *Let $u^\nu \in H^1(\Omega)$ be the unique solution of the problem with absorption (5.3). Then, there exist $\left(u_k^\nu\right)_k \subset H^1(-1,1)$ be such that $u^\nu(x,y) = \sum_{k\in\mathbb{N}} u_k^\nu(x)\psi_k(y)$ with*

$$u_0^\nu(x) = a_0^\nu + b_0^\nu \log(x+i\nu) + \int_1^x \log(t+i\nu)\, f_0(t)\mathrm{d}t + \log(x+i\nu)\int_0^x f_0(t)\mathrm{d}t,$$

and, for $k \geq 1$,

$$\begin{aligned} u_k^\nu(x) &= a_k^\nu I_0\left(\lambda_k(x+i\nu)\right) - b_k^\nu K_0\left(\lambda_k(x+i\nu)\right) \\ &\quad - I_0\left(\lambda_k(x+i\nu)\right)\int_1^x K_0\left(\lambda_k(t+i\nu)\right) f_k(t)\mathrm{d}t + K_0\left(\lambda_k(x+i\nu)\right)\int_0^x I_0\left(\lambda_k(t+i\nu)\right) f_k(t)\mathrm{d}t, \end{aligned}$$

where a_k^ν and b_k^ν are given below, see (5.8), (5.9), (5.10) and (5.11).

Proof. Easy manipulations on the problem (5.3) allow us to rewrite it under the form:

$$-\partial_x\left((x+i\nu)\,\partial_x u^\nu\right) - \frac{x+i\nu}{\mathtt{r}}\partial_y(\mathtt{r}\partial_y u^\nu) = f. \tag{5.4}$$

Since the eigenvectors $(\psi_k)_k$ of the operator $-\mathtt{r}^{-1}\partial_y(\mathtt{r}\partial_y\cdot)$ constitute a Hilbert basis of $L^2_{\mathtt{r}}(0,L)$, we are allowed to decompose $u^\nu \in H^1_{\mathtt{r}}(\Omega)$ and $f \in L^2_{\mathtt{r}}(\Omega)$ as

$$u^\nu(x,y) = \sum_{k\in\mathbb{N}} u_k^\nu(x)\psi_k(y), \qquad f(x,y) = \sum_{k\in\mathbb{N}} f_k(x)\psi_k(y).$$

Then, the equation (5.4) can be written as a system of ODE parametrized by $k \in \mathbb{N}$:

$$\left|\begin{array}{l} -\partial_x\left((x+i\nu)\,\partial_x u_k^\nu\right) + \lambda_k^2\,(x+i\nu)\,u_k^\nu = f_k, \text{ in } (-1,1), \\ u_k^\nu(1) = u_k^\nu(-1) = 0. \end{array}\right. \tag{5.5}$$

The general solution of these equations is written as

$$u_k^\nu(x) = u_k^{\nu,0}(x) + \hat{u}_k^\nu(x), \qquad \text{with} \quad u_k^{\nu,0}(x) = a_k^\nu u_k^{\nu,1}(x) + b_k^\nu u_k^{\nu,2}(x), \tag{5.6}$$

where $u_k^{\nu,0}$ is a solution of the homogeneous equation and $\hat{u}_k^\nu$ a particular solution. The solution of the homogeneous equation is a linear combination of a fundamental pair of solutions $u_k^{\nu,1}$ and $u_k^{\nu,2}$, and the coefficients a_k^ν and b_k^ν are computed using the boundary conditions. As a memento, the Wronskian $\mathscr{W}\left\{u_k^{\nu,1}, u_k^{\nu,2}\right\}$ of this pair of solutions can easily be computed (see [51, (1.13.5)]), and the fundamental pair will be normalized so that

$$\mathscr{W}\left\{u_k^{\nu,1}, u_k^{\nu,2}\right\}(x) = u_k^{\nu,1}(x)\partial_x u_k^{\nu,2}(x) - \partial_x u_k^{\nu,1}(x) u_k^{\nu,2}(x) = \frac{1}{x+i\nu}. \tag{5.7}$$

With the change of variable $z = (x + i\nu)$, (5.5) becomes a modified Bessel equation [51, §10.25]:

$$-z\partial_z\left(z\partial_z u_k\right) + \left(\lambda_k z\right)^2 u_k = z f_k.$$

Therefore, for $k \geq 0$, the basis $\left(u_k^{\nu,1}, u_k^{\nu,2}\right)$ of solutions of the homogeneous equation associated to (5.5) are

$$\begin{aligned}
&u_0^{\nu,1}(x) = 1, && u_0^{\nu,2}(x) = \log(x+i\nu), && k = 0,\\
&u_k^{\nu,1}(x) = I_0\left(\lambda_k(x+i\nu)\right), && u_k^{\nu,2}(x) = -K_0\left(\lambda_k(x+i\nu)\right), && k > 0.
\end{aligned}$$

One can consult [51, §10.25] for the definitions of the modified Bessel functions I_0 and K_0. We consider here the principal value of $z \mapsto \log z$ and $z \mapsto K_0(z)$, with a branch cut at $(-\infty, 0]$. The Wronskian of the pairs is (5.7), see [51, (10.28.2)]. The computation of a particular solution uses the variation of parameters method (see e.g., [51, (1.13.10)]). Then, particular solutions can simply be written as

$$\begin{aligned}
\hat{u}_k^\nu(x) &= u_k^{\nu,1}(x)\int_1^x \frac{u_k^{\nu,2}(t)}{\mathscr{W}\left\{u_k^{\nu,1}, u_k^{\nu,2}\right\}(t)}\left(\frac{1}{t+i\nu} f_k(t)\right)\mathrm{d}t - u_k^{\nu,2}(x)\int_0^x \frac{u_k^{\nu,1}(t)}{\mathscr{W}\left\{u_k^{\nu,1}, u_k^{\nu,2}\right\}(t)}\left(\frac{1}{t+i\nu} f_k(t)\right)\mathrm{d}t\\
&= u_k^{\nu,1}(x)\int_1^x u_k^{\nu,2}(t) f_k(t)\mathrm{d}t - u_k^{\nu,2}(x)\int_0^x u_k^{\nu,1}(t) f_k(t)\mathrm{d}t,
\end{aligned}$$

where the value of the Wronskian (5.7) has been used. Next, using the homogeneous boundary condition allows to compute the expected values of a_k^ν and b_k^ν. In the case of $k = 0$, we obtain that

$$u_0^\nu(x) = a_0^\nu + b_0^\nu \log\left(x+i\nu\right) + \int_1^x \log\left(t+i\nu\right) f_0(t)\mathrm{d}t - \log\left(x+i\nu\right)\int_0^x f_0(t)\mathrm{d}t,$$

$$a_0^\nu = \frac{\log(1+i\nu)}{\Delta_0^\nu}\int_{-1}^1 \left[\log(-1+i\nu) - \log(t+i\nu)\right] f_0(t)\mathrm{d}t, \tag{5.8}$$

$$b_0^\nu = \frac{1}{\Delta_0^\nu}\int_{-1}^1 \log(t+i\nu) f_0(t)\mathrm{d}t - \int_{-1}^0 f_0(t)\mathrm{d}t - \frac{\log(1+i\nu)}{\Delta_0^\nu}\int_{-1}^1 f_0(t)\mathrm{d}t, \tag{5.9}$$

$$\Delta_0^\nu = \log(-1+i\nu) - \log(1+i\nu) = i\left(\pi - 2\arctan(\nu)\right).$$

Recall that in the case of $k \geq 1$, we have

$$\begin{aligned}
u_k^\nu(x) &= a_k^\nu I_0\left(\lambda_k(x+i\nu)\right) - b_k^\nu K_0\left(\lambda_k(x+i\nu)\right)\\
&\quad - I_0\left(\lambda_k(x+i\nu)\right)\int_1^x K_0\left(\lambda_k(t+i\nu)\right) f_k(t)\mathrm{d}t + K_0\left(\lambda_k(x+i\nu)\right)\int_0^x I_0\left(\lambda_k(t+i\nu)\right) f_k(t)\mathrm{d}t
\end{aligned}$$

Like in the case $k = 0$, the coefficients a_k^ν and b_k^ν are then computed using the Dirichlet conditions at $x = \pm 1$:

$$\begin{aligned} a_k^\nu = & \frac{K_0\left(\lambda_k(1+i\nu)\right) K_0\left(\lambda_k(-1+i\nu)\right)}{\Delta_k^\nu} \int_{-1}^{1} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & - \frac{K_0\left(\lambda_k(1+i\nu)\right) I_0\left(\lambda_k(-1+i\nu)\right)}{\Delta_k^\nu} \int_{-1}^{1} K_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \end{aligned} \tag{5.10}$$

$$\begin{aligned} b_k^\nu = & \frac{K_0\left(\lambda_k(1+i\nu)\right) I_0\left(\lambda_k(-1+i\nu)\right)}{\Delta_k^\nu} \int_{0}^{1} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & + \frac{I_0\left(\lambda_k(1+i\nu)\right) K_0\left(\lambda_k(-1+i\nu)\right)}{\Delta_k^\nu} \int_{-1}^{0} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & - \frac{I_0\left(\lambda_k(1+i\nu)\right) I_0\left(\lambda_k(-1+i\nu)\right)}{\Delta_k^\nu} \int_{-1}^{1} K_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \end{aligned} \tag{5.11}$$

$$\Delta_k^\nu = K_0\left(\lambda_k(1+i\nu)\right) I_0\left(\lambda_k(-1+i\nu)\right) - I_0\left(\lambda_k(1+i\nu)\right) K_0\left(\lambda_k(-1+i\nu)\right) \tag{5.12}$$

Finally, it is easy to check that $u_k^\nu \in H^1(-1,1)$ for all $k \in \mathbb{N}$ and $\nu > 0$. □

Remark 5.2.2. Since the functions $u_k^{\nu,1}$ are smooth for any values of $\nu \geq 0$, the integration bounds of $\int^x u_k^{\nu,2} f_k \mathrm{d}t$ do not really matter, and have been chosen in order to facilitate the computations. On the other hand, $\lim_{\nu\to 0+} u_k^{\nu,2}$ is not smooth at the point $x = 0$, which is the reason why the lower bound of $\int^x u_k^{\nu,1} f_k \mathrm{d}t$ has been chosen in order to compensate the singular behavior of $u_k^{\nu,2}$.

Remark 5.2.3. With the identities from [51, §10.34], we have for any $a, b > 0$,

$$I_0(-a+ib) = \overline{I_0(a+ib)}, \tag{5.13}$$

$$K_0(-a+ib) = \overline{K_0(a+ib)} - i\pi\overline{I_0(a+ib)}. \tag{5.14}$$

Then these two identities can be used to reexpress a_k^ν and b_k^ν as:

$$\begin{aligned} a_k^\nu = & \frac{\left|K_0\left(\lambda_k(1+i\nu)\right)\right|^2}{\Delta_k^\nu} \int_{-1}^{1} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & - \frac{i\pi K_0\left(\lambda_k(1+i\nu)\right) \overline{I_0\left(\lambda_k(1+i\nu)\right)}}{\Delta_k^\nu} \int_{0}^{1} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & - \frac{K_0\left(\lambda_k(1+i\nu)\right) \overline{I_0\left(\lambda_k(1+i\nu)\right)}}{\Delta_k^\nu} \left(\int_{-1}^{0} \overline{K_0\left(\lambda_k(|t|+i\nu)\right)} f_k(t) \mathrm{d}t + \int_{0}^{1} K_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \right), \end{aligned} \tag{5.15}$$

$$\begin{aligned} b_k^\nu = & \frac{K_0\left(\lambda_k(1+i\nu)\right) \overline{I_0\left(\lambda_k(1+i\nu)\right)}}{\Delta_k^\nu} \int_{0}^{1} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & + \frac{\overline{K_0\left(\lambda_k(1+i\nu)\right)} I_0\left(\lambda_k(1+i\nu)\right)}{\Delta_k^\nu} \int_{-1}^{0} I_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \\ & - \frac{\left|I_0\left(\lambda_k(1+i\nu)\right)\right|^2}{\Delta_k^\nu} \left(\int_{-1}^{0} \overline{K_0\left(\lambda_k(|t|+i\nu)\right)} f_k(t) \mathrm{d}t + \int_{0}^{1} K_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \right) \end{aligned} \tag{5.16}$$

$$\Delta_k^\nu = i\pi \left|I_0\left(\lambda_k(1+i\nu)\right)\right|^2 + 2i \operatorname{Im}\left(\overline{I_0\left(\lambda_k(1+i\nu)\right)} K_0\left(\lambda_k(1+i\nu)\right) \right). \tag{5.17}$$

In spite of their apparent sophistication, these formulas are easier to manipulate and estimate.

5.2.2 Solution without absorption

Next, it is natural to define the limiting absorption solution at a point $(x, y) \in \Omega \setminus \{0\} \times (0, L)$ as

$$u^+(x, y) = \sum_{k\in\mathbb{N}} \lim_{\nu\to 0+} u_k^\nu(x)\psi_k(y).$$

Then, as noticed earlier in this paragraph, $u_k^{\nu,2}$ does not converge toward a smooth solution. This is due to the fact that $z \mapsto \log z$ and $z \mapsto K_0(z)$ are not entire functions ; the branch cut of their principal values is $(-\infty, 0]$. Therefore, as $\nu \to 0+$, the identity (5.14) is important to consider. In particular, for $x \neq 0$, we have

$$\lim_{\nu\to 0+} K_0(x + i\nu) = K_0(|x|) - i\pi I_0(x)\mathbb{1}_{x<0}.$$

Then, it is natural to extend the definition of K_0 to the negative real numbers as

$$K_0(x) := \begin{cases} K_0(x) & \text{if } x > 0, \\ K_0(-x) - i\pi I_0(x) & \text{if } x < 0. \end{cases} \tag{5.18}$$

It is worth noting that this is the standard convention adopted by mathematical software. Another useful property is the parity of I_0: $I_0(x) = I_0(-x)$ for all $x \in \mathbb{R}$. Finally, $I_0(x), K_0(x) > 0$ for all $x > 0$.

Lemma 5.2.4. *The formal limit of* $(u^\nu)_{\nu>0}$ *is* $u^+ = \sum_{k\in\mathbb{N}} u_k^+\psi_k$ *with*

$$u_0^+(x) = b_0^+ \left(\log|x| + i\pi\mathbb{1}_{x<0}\right) + \int_1^x \left(\log|t| + i\pi\mathbb{1}_{t<0}\right) f_0(t)\mathrm{d}t + \left(\log|x| + i\pi\mathbb{1}_{x<0}\right)\int_0^x f_0(t)\mathrm{d}t,$$
$$u_k^+(x) = a_k^+ I_0(\lambda_k x) - b_k^+ K_0(\lambda_k x) - I_0(\lambda_k x)\int_1^x K_0(\lambda_k t) f_k(t)\mathrm{d}t + K_0(\lambda_k x)\int_0^x I_0(\lambda_k t) f_k(t)\mathrm{d}t,$$

and where

$$b_0^+ = \int_{-1}^1 \log|t| f_0(t)\mathrm{d}t,$$
$$a_k^+ = \frac{K_0(\lambda_k)^2}{i\pi I_0(\lambda_k)^2}\int_{-1}^1 I_0(\lambda_k t) f_k(t)\mathrm{d}t - \frac{K_0(\lambda_k)}{i\pi I_0(\lambda_k)}\int_{-1}^1 K_0(|\lambda_k t|) f_k(t)\mathrm{d}t - \frac{K_0(\lambda_k)}{I_0(\lambda_k)}\int_0^1 I_0(\lambda_k t) f_k(t)\mathrm{d}t,$$
$$b_k^+ = \frac{K_0(\lambda_k)}{i\pi I_0(\lambda_k)}\int_{-1}^1 I_0(\lambda_k t) f_k(t)\mathrm{d}t - \frac{1}{i\pi}\int_{-1}^1 K_0(|\lambda_k t|) f_k(t)\mathrm{d}t.$$

Proof. The formal limits are easily computed with the help of

$$\lim_{\nu\to 0+} \log(x + i\nu) = \log|x| + i\pi\mathbb{1}_{x<0}, \qquad \lim_{\nu\to 0+} I_0(\lambda_k(x + i\nu)) = I_0(\lambda_k x),$$
$$\lim_{\nu\to 0+} K_0(\lambda_k(x + i\nu)) = K_0(\lambda_k x) = K_0(\lambda_k|x|) - i\pi I_0(\lambda_k x)\mathbb{1}_{x<0},$$

The limits of a_0^ν and b_0^ν are straightforward, and the limits of a_k^ν and b_k^ν are computed using (5.15) and (5.16) respectively. □

In the view of the structure of u_k^+, we define

$$u_k^{+,0} = \begin{cases} b_0^+ \left(\log|x| + i\pi\mathbb{1}_{x<0}\right) & \text{if } k = 0, \\ a_k^+ I_0(\lambda_k x) - b_k^+ K_0(\lambda_k x) & \text{if } k \geq 1, \end{cases}$$

and

$$\hat{u}_k^+ = \begin{cases} \displaystyle\int_1^x (\log|t| + i\pi \mathbb{1}_{t<0}) f_0(t)\mathrm{d}t + (\log|x| + i\pi\mathbb{1}_{x<0}) \int_0^x f_0(t)\mathrm{d}t & \text{if } k = 0, \\ \displaystyle -I_0(\lambda_k x) \int_1^x K_0(\lambda_k t) f_k(t)\mathrm{d}t + K_0(\lambda_k x) \int_0^x I_0(\lambda_k t) f_k(t)\mathrm{d}t & \text{if } k \geq 1, \end{cases}$$

so that $u_k^+ = u_k^{+,0} + \hat{u}_k^+$. It is easy to check that $u_k^{+,0}$ is a solution of the homogeneous modified Bessel equation (5.2) modulo the change of variables, and $\hat{u}_k^+$ is a particular solution of the same equation. This particular solution satisfies the following proposition.

Proposition 5.2.5. *$u^+ \in L^2_\mathfrak{r}(\Omega)$ and $\partial_y u^+ \in L^2_\mathfrak{r}(\Omega)$.*

This proposition relies on the following lemmas based on the asymptotics of modified Bessel functions.

Lemma 5.2.6 ([51, (10.25.3), §10.30])**.** *The modified Bessel functions satisfies*

$$I_0(z) \underset{z\to 0}{\sim} 1 \qquad K_0(z) \underset{z\to 0}{\sim} -\log(z)$$
$$I_0(z) \underset{|z|\to+\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}} \qquad K_0(z) \underset{|z|\to+\infty}{\sim} \sqrt{\frac{\pi}{2z}} e^{-z},$$

for $|\arg z| < \pi/2 - \delta$, for any $\delta > 0$ arbitrary small.

Then, the following lemma allows us to estimate the integrals and coefficients a_k^+, b_k^+ that are involved in u_k^+.

Lemma 5.2.7. *Let $f \in L^2(0,1)$ and $k \geq 1$. For all $x \in (0,1)$ such that $0 \leq \lambda_k x \leq 1$, we have*

$$\left|\int_0^x I_0(\lambda_k t) f(t)\mathrm{d}t\right| = \mathcal{O}\left(\sqrt{x}\|f\|_{L^2(0,1)}\right), \qquad \left|\int_x^1 K_0(\lambda_k t) f(t)\mathrm{d}t\right| = \mathcal{O}\left(\frac{\|f\|_{L^2(0,1)}}{\sqrt{\lambda_k}}\right).$$

For all $x \in (0,1)$ such that $\lambda_k x \geq 1$, we have

$$\left|\int_0^x I_0(\lambda_k t) f(t)\mathrm{d}t\right| = \mathcal{O}\left(\frac{e^{\lambda_k x}}{\lambda_k\sqrt{x}}\|f\|_{L^2(0,1)}\right), \qquad \left|\int_x^1 K_0(\lambda_k t) f(t)\mathrm{d}t\right| = \mathcal{O}\left(\frac{e^{-\lambda_k x}}{\lambda_k\sqrt{x}}\|f\|_{L^2(0,1)}\right).$$

Proof. In what follows, $a \lesssim b$ means that there exist a constant $C > 0$ independent of x and k such that $a \leq Cb$. Firstly, since $K_0 \in L^2(0,\infty)$ and $I_0 \in L^2(0,1)$, then for $x \in (0,1)$ such that $0 \leq \lambda_k x \leq 1$, we have with the help of the Cauchy-Schwarz inequality and change of variables

$$\left|\int_x^1 K_0(\lambda_k t) f(t)\mathrm{d}t\right| \leq \left(\frac{1}{\lambda_k}\int_{\lambda_k x}^{\lambda_k} K_0(t)^2\mathrm{d}t\right)^{1/2} \|f\|_{L^2(x,1)} \leq \frac{\|K_0\|_{L^2(0,\infty)}\|f\|_{L^2(0,1)}}{\sqrt{\lambda_k}},$$

and

$$\left|\int_0^x I_0(\lambda_k t) f(t)\mathrm{d}t\right| \leq \left(\frac{1}{\lambda_k}\int_0^{\lambda_k x} I_0(t)^2\mathrm{d}t\right)^{1/2} \|f\|_{L^2(0,x)} \leq \|I_0\|_{L^\infty(0,1)}\sqrt{x}\|f\|_{L^2(0,1)}.$$

On the other hand, for $x \in (0,1)$ such that $\lambda_k x \geq 1$, we have

$$\begin{aligned}\left|\int_x^1 K_0(\lambda_k t) f(t)\mathrm{d}t\right| &\leq \frac{1}{\sqrt{\lambda_k}}\left(\int_{\lambda_k x}^{+\infty} K_0(t)^2\mathrm{d}t\right)^{1/2} \|f\|_{L^2(0,1)} \\ &\lesssim \frac{1}{\sqrt{\lambda_k}}\left(\int_{\lambda_k x}^{+\infty} \frac{e^{-2t}}{t}\mathrm{d}t\right)^{1/2} \|f\|_{L^2(0,1)} \lesssim \frac{e^{-\lambda_k x}}{\lambda_k\sqrt{x}}\|f\|_{L^2(0,1)},\end{aligned}$$

and, using the fact[1] that $\int_1^{x'} \frac{e^{2t}}{t}\mathrm{d}t \lesssim \frac{e^{2x'}}{x'}$,

$$\left|\int_0^x I_0(\lambda_k t) f(t)\mathrm{d}t\right| \leq \frac{1}{\sqrt{\lambda_k}} \left(\int_0^{\lambda_k x} I_0(t)^2 \mathrm{d}t\right)^{1/2} \|f\|_{L^2(0,1)}$$
$$\lesssim \frac{1}{\sqrt{\lambda_k}} \left(\int_1^{\lambda_k x} \frac{e^{2t}}{t} \mathrm{d}t\right)^{1/2} \|f\|_{L^2(0,1)} \lesssim \frac{e^{\lambda_k x}}{\lambda_k \sqrt{x}} \|f\|_{L^2(0,1)}\,.$$

□

Remark 5.2.8. Notice that the behavior of $\int_x^1 K_0(\lambda_k t) f(t)\mathrm{d}t$ is optimal in the sense that

$$\sup_{k>0} \sup_{f \in L^2(0,1)} \frac{\sqrt{\lambda_k} \int_0^1 K_0(\lambda_k t) f(t)\mathrm{d}t}{\|f\|_{L^2(0,1)}} = \|K_0\|_{L^2(0,\infty)}\,.$$

This can be easily check by computing the limit of the above quantity with $f(t) = K_0(\lambda_k t)$.

A direct corollary of the asymptotics of the modified Bessel functions is the following.

Lemma 5.2.9. *We have, for $k > 0$,*

$$a_k^+ = \mathcal{O}\left(\frac{e^{-\lambda_k}}{\lambda_k} \|f_k\|_{L^2(-1,1)}\right), \qquad b_k^+ = \mathcal{O}\left(\frac{1}{\sqrt{\lambda_k}} \|f_k\|_{L^2(-1,1)}\right). \tag{5.19}$$

Proof. Using the expressions of a_k^+ and b_k^+ in Lemma 5.2.4 and the two Lemmas 5.2.6 and 5.2.7, we directly have

$$|a_k^+| \lesssim e^{-4\lambda_k} \frac{e^{\lambda_k}}{\lambda_k} \|f_k\|_{L^2(-1,1)} + e^{-2\lambda_k} \frac{\|f_k\|_{L^2(-1,1)}}{\sqrt{\lambda_k}} + e^{-2\lambda_k} \frac{e^{\lambda_k}}{\lambda_k} \|f_k\|_{L^2(0,1)} \lesssim \frac{e^{-\lambda_k}}{\lambda_k} \|f_k\|_{L^2(-1,1)}\,,$$
$$|b_k^+| \lesssim e^{-2\lambda_k} \frac{e^{\lambda_k}}{\lambda_k} \|f_k\|_{L^2(-1,1)} + \frac{\|f_k\|_{L^2(-1,1)}}{\sqrt{\lambda_k}} \lesssim \frac{\|f_k\|_{L^2(-1,1)}}{\sqrt{\lambda_k}}.$$

□

Proof of Proposition 5.2.5. According to paragraph 5.1, showing that $u^+ \in L^2_{\mathrm{r}}(\Omega)$ consists in bounding from above

$$\sum_{k \in N} (1 + \lambda_k^2) \|u_k^+\|^2_{L^2(-1,0)}\,, \quad \text{and} \quad \sum_{k \in N} (1 + \lambda_k^2) \|u_k^+\|^2_{L^2(0,1)}\,.$$

Estimation on $(0,1)$: We first estimate $\hat{u}_k^+$. Using Lemmas 5.2.6 and 5.2.7, the first part of $\hat{u}_k^+$ is estimated as

$$\int_0^1 \left| I_0(\lambda_k x) \int_x^1 K_0(\lambda_k t) f_k(t)\mathrm{d}t\right|^2 \mathrm{d}x$$
$$= \int_0^{1/\lambda_k} \left| I_0(\lambda_k x) \int_x^1 K_0(\lambda_k t) f_k(t)\mathrm{d}t\right|^2 \mathrm{d}x + \int_{1/\lambda_k}^1 \left| I_0(\lambda_k x) \int_x^1 K_0(\lambda_k t) f_k(t)\mathrm{d}t\right|^2 \mathrm{d}x$$
$$\lesssim \int_0^{1/\lambda_k} \left|\frac{\|f\|_{L^2(0,1)}}{\sqrt{\lambda_k}}\right|^2 \mathrm{d}x + \int_{1/\lambda_k}^1 \left|\frac{e^{\lambda_k x}}{\sqrt{x}} \frac{e^{-\lambda_k x}}{\lambda_k \sqrt{x}} \|f\|_{L^2(0,1)}\right|^2 \mathrm{d}x \lesssim \frac{\|f_k\|^2_{L^2(0,1)}}{\lambda_k^2},$$

[1] Notice that $\int_1^{x'} \frac{e^{2t}}{t}\mathrm{d}t = \left[\frac{e^{2t}}{2t}\right]_1^{x'} + \int_1^{x'} \frac{e^{2t}}{2t^2}\mathrm{d}t$ so that $\int_1^{x'} \frac{e^{2t}}{t}\mathrm{d}t \lesssim \frac{e^{2x'}}{x'}$.

Repeating the same argument for the second part of $\hat{u}_k^+$ gives the same estimation, which results in

$$\|\hat{u}_k^+\|_{L^2(0,1)}^2 \lesssim \frac{\|f_k\|_{L^2(-1,1)}^2}{\lambda_k^2}.$$

Next, in order to estimate $u_k^{+,0}$, we use Lemma 5.2.9, so that

$$\left\|u_k^{+,0}\right\|_{L^2(0,1)}^2 \lesssim \frac{e^{-2\lambda_k}}{\lambda_k^2}\|f_k\|_{L^2(-1,1)}^2 \int_0^1 |I_0(\lambda_k x)|^2 \mathrm{d}x + \frac{\|f_k\|_{L^2(-1,1)}^2}{\lambda_k}\int_0^1 |K_0(\lambda_k x)|^2 \mathrm{d}x \lesssim \frac{\|f_k\|_{L^2(-1,1)}^2}{\lambda_k^2}.$$

Then, summing up the last two equations gives that $(1+\lambda_k^2)\left\|u_k^+\right\|_{L^2(0,1)}^2 \lesssim \|f_k\|_{L^2(-1,1)}^2$, and the result holds on $(0,1)$ since $f \in L_\Gamma^2(\Omega)$.

Estimation on $(-1,0)$**:** We can no longer estimate $\hat{u}_k^+$ and $u_k^{+,0}$ separately on $(-1,0)$ because of the apparition of terms like

$$b_k^+ I_0(\lambda_k x), \qquad I_0(\lambda_k x)\int_x^1 K_0(\lambda_k|t|) f_k(t)\mathrm{d}t, \qquad \text{and} \qquad I_0(\lambda_k x)\int_0^x I_0(\lambda_k t) f_k(t)\mathrm{d}t,$$

which do not behave nicely as $k \to +\infty$. Therefore, using the expression (5.18) of K_0, we have for $x \in (-1,0)$:

$$\begin{aligned} u_k^+(x) = &\left(a_k^+ + \int_{-1}^1 K_0(|\lambda_k t|) f_k(t)\mathrm{d}t + i\pi b_k^+\right) I_0(\lambda_k x) - b_k^+ K_0(|\lambda_k x|) \\ &- I_0(\lambda_k x)\int_{-1}^x K_0(|\lambda_k t|) f_k(t)\mathrm{d}t + K_0(|\lambda_k x|)\int_0^x I_0(\lambda_k t)\, f_k(t)\mathrm{d}t. \end{aligned} \tag{5.20}$$

Then, it is easy to check as in Corollary 5.2.9 that

$$\int_{-1}^1 K_0(|\lambda_k t|) f_k(t)\mathrm{d}t + i\pi b_k^+ = \frac{K_0(\lambda_k)}{I_0(\lambda_k)}\int_{-1}^1 I_0(\lambda_k t) f_k(t)\mathrm{d}t = \mathcal{O}\left(\frac{e^{-\lambda_k}}{\lambda_k}\|f\|_{L^2(-1,1)}\right).$$

Therefore, estimating u_k^+ as before leads to the expected result. □

Although $u^+ \in L^2(\Omega)$ is expected, $\partial_y u^+ \in L^2(\Omega)$ is not. Indeed, as noticed in Chapter 4, the natural norm of the problem is

$$\left(\int_\Omega |u|^2 \mathbf{dx} + \int_\Omega |x|\, |\nabla u|^2 \mathbf{dx}\right)^{1/2}, \tag{5.21}$$

which a priori does not control the L^2-norm of $\partial_y u^+$. On the other hand, the solution is not regular along the x-direction. To see this, it suffices to consider $f \in L^2(\Omega)$ such that $f(x,y) \equiv f(x)$. With a such f, $f_0 = f$ and $f_k = 0$ for $k \geq 0$, which leads to a tensor-like example like the one exposed in section 4.1. Moreover, in the view of Lemma 5.2.4,

$$\int_\Omega |x|\, |\partial_x u|^2 \mathbf{dx} = \infty,$$

due to the presence of the logarithm. Notice that this behavior is independent of whether the support of the source term touches the interface or not.

Remark 5.2.10. On the other hand, the behavior of the source term has an influence on the regularity of the solution. For example, assume that the support of f is a subset of $\Omega \setminus (-l,l)\times(0,L)$. Then, the estimation b_k can be significantly improved, see next section.

5.3 Regularity matters

5.3.1 Regular and singular parts of the solution

As noticed in the previous paragraph, the derivative of the solution $\partial_x u^+$ is not regular. This is due to the logarithmic behavior of its component u_k^+. Indeed, in view of Remark 5.2.2 and the behavior of K_0 near 0, we can split u_k^+ in to parts $u_k^+ = u_{k,sing}^+ + u_{k,reg}^+$ with

$$u_{k,sing}^+(x) = b_k^+ \left(\log|x| + i\pi \mathbb{1}_{x<0}\right),$$
$$u_{k,reg}^+(x) = u_k^+(x) - b_k^+ \left(\log|x| + i\pi \mathbb{1}_{x<0}\right).$$

Therefore, it is natural to define the singular part of the solution as:

$$u_{sing}^+(x,y) = g^+(y)\left(\log|x| + i\pi \mathbb{1}_{x<0}\right), \qquad \text{with} \qquad g^+(y) := \sum_{k\in\mathbb{N}} b_k^+ \psi_k(y), \tag{5.22}$$

and the regular part as

$$u_{reg}^+(x,y) = \sum_{k\in\mathbb{N}} u_{k,reg}^+(x)\psi_k(y). \tag{5.23}$$

Let us give the following two regularity results.

Proposition 5.3.1. *Given $f \in L_{\mathrm{r}}^2(\Omega)$, it holds that $g^+ \in H_{\mathrm{r}}^{1/2}(\Sigma)$. In addition, $\|g^+\|_{H_{\mathrm{r}}^{1/2}(\Sigma)} \lesssim \|f_\Omega\|_{L_{\mathrm{r}}^2(\Omega)}$.*

Proof. This is a direct consequence of Lemma 5.2.7, and Remark 5.2.10. □

Proposition 5.3.2. *Given $f \in L_{\mathrm{r}}^2(\Omega)$, $\partial_x u_{reg}^+ \in L_{\mathrm{r}}^2(\Omega_j)$ for $j \in \{p,n\}$. In addition, $\left\|\partial_x u_{reg}^+\right\|_{L_{\mathrm{r}}^2(\Omega_j)} \lesssim \|f_\Omega\|_{L_{\mathrm{r}}^2(\Omega)}$.*

Like for Proposition 5.2.5, we need some results about the asymptotics of the modified Bessel functions. Recall that $I_0'(x) = I_1(x)$ and $K_0'(x) = -K_1(x)$.

Lemma 5.3.3 ([51, §10.30, (10.25.3), (10.31.1)])**.** *The modified Bessel functions satisfies*

$$I_1(z) \underset{z\to 0}{\sim} \frac{z}{2}, \qquad K_1(z) - \frac{1}{z} \underset{z\to 0}{\sim} \frac{z}{2}\log(z),$$
$$I_1(z) \underset{|z|\to+\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}}, \qquad K_1(z) \underset{|z|\to+\infty}{\sim} \sqrt{\frac{\pi}{2z}} e^{-z},$$

for $|\arg z| < \pi/2 - \delta$, for any $\delta > 0$ arbitrary small.

Proof. The derivative $\partial_x u_{reg}^+$ is easily computed inside Ω_j, $j \in \{p,n\}$:

$$\begin{aligned}\partial_x u_{k,reg}^+(x) =& a_k^+ \lambda_k I_1(\lambda_k x) + b_k^+ \lambda_k \left(K_1(\lambda_k x) - \frac{1}{\lambda_k x}\right) \\ &- \lambda_k I_1(\lambda_k x) \int_1^x K_0(\lambda_k t) f_k(t) \mathrm{d}t - \lambda_k K_1(\lambda_k x) \int_0^x I_0(\lambda_k t) f_k(t) \mathrm{d}t.\end{aligned}$$

Estimation on $(0,1)$**.** Most of the integrals can easily be bounded like in the proof of Proposition 5.2.5, except

$$\int_0^{1/\lambda_k} \left|\lambda_k K_1(\lambda_k x) \int_0^x I_0(\lambda_k t)\, f_k(t) \mathrm{d}t\right|^2 \mathrm{d}x \lesssim \int_0^{1/\lambda_k} \left|\frac{\|f_k\|_{L^2(0,1)}}{\sqrt{x}}\right|^2 \mathrm{d}x = +\infty$$

if one uses naively the Lemmas 5.2.7 and 5.3.3. Fortunately, the Hardy's inequality [52] allows us to conclude. Firstly, there is a constant $C_{K_1} > 0$ such that $K_1(x) \le \frac{C_{K_1}}{x}$ for all $x \in (0,1)$. Then,

$$\int_0^{1/\lambda_k} \left|\lambda_k K_1(\lambda_k x) \int_0^x I_0(\lambda_k t)\, f_k(t) \mathrm{d}t\right|^2 \mathrm{d}x \le C_{K_1} \int_0^{1/\lambda_k} \left|\frac{1}{x}\int_0^x I_0(\lambda_k t)\, f_k(t) \mathrm{d}t\right|^2 \mathrm{d}x$$

Next, using the Hardy's inequality gives

$$\begin{aligned}\int_0^{1/\lambda_k} \left|\lambda_k K_1(\lambda_k x) \int_0^x I_0(\lambda_k t)\, f_k(t) \mathrm{d}t\right|^2 \mathrm{d}x &\le 4C_{K_1} \int_0^{1/\lambda_k} \left|I_0(\lambda_k x)\, f_k(x)\right|^2 \mathrm{d}x \\ &\le 4C_{K_1} \|I_0\|^2_{L^\infty(0,1)} \|f_k\|^2_{L^2\left(0,\frac{1}{\lambda_k}\right)}.\end{aligned}$$

These estimations lead to $\left\|\partial_x u^+_{k,reg}\right\|_{L^2(0,1)} \lesssim \|f_k\|_{L^2(0,1)}$. Since the result holds for all k, we have $\left\|\partial_x u^+_{reg}\right\|_{L^2_\Gamma(\Omega_p)} \lesssim \|f_\Omega\|_{L^2_\Gamma(\Omega)}$, and $\partial_x u^+_{reg} \in L^2_\Gamma(\Omega_p)$.

Estimation on $(-1,0)$**.** In the view of the expression (5.20) of u^+_k on $(-1,0)$, we have

$$\begin{aligned}\partial_x u^+_{k,reg}(x) =& \lambda_k \left(a^+_k + \int_{-1}^1 K_0(|\lambda_k t|) f_k(t)\mathrm{d}t + i\pi b^+_k\right) I_1(\lambda_k x) - b^+_k \lambda_k \left(K_1(|\lambda_k x|) - \frac{1}{|\lambda_k x|}\right) \\ &- \lambda_k I_1(\lambda_k x) \int_{-1}^x K_0(|\lambda_k t|) f_k(t)\mathrm{d}t + \lambda_k K_1 |\lambda_k x| \int_0^x I_0(\lambda_k t)\, f_k(t)\mathrm{d}t.\end{aligned}$$

Therefore, the estimations follows exactly like the previous case and $\partial_x u^+_{reg} \in L^2_\Gamma(\Omega_n)$ with the bound $\partial_x u^+_{reg} \in L^2_\Gamma(\Omega_n)$. □

We proved that $g^+ \in H^{1/2}_\Gamma(\Sigma)$, which does not imply that $\partial_y u^+_{sing} \in L^2(\Omega)$. Similarly, we did not prove that $\partial_y u^+_{reg} \in L^2(\Omega)$. Therefore, u^+_{reg} may not have classical traces on the interface. Indeed, this can easily be seen on the trace: using that $K_0(x) + \log(x) \to \log(2) - \gamma$ as $x \to 0+$, where γ is the Euler-Mascheroni constant, we find that

$$\lim_{x\to 0+} u^+_{k,reg} = \lim_{x\to 0-} u^+_{k,reg} = a^+_k - b^+_k (\log \lambda_k + \log(2) - \gamma) + \int_0^1 K_0(\lambda_k t) f_k(t)\mathrm{d}t.$$

Then, using the estimation of a^+_k and b^+_k for Lemma 5.2.9, which is optimal, see Remark 5.2.8, the dominating term is $b^+_k \log \lambda_k$, and we can only conclude that

$$u^+_{reg}\big|_\Sigma \in H^{1/2-\varepsilon}_\Gamma(\Sigma), \quad \forall \varepsilon > 0.$$

5.3.2 Going back to the jump of solutions

On the other hand, given that the limits above coincide on each side of the interface, one could argue that the jump between $u^+_{reg}\big|_{\Omega_p}$ and $u^+_{reg}\big|_{\Omega_n}$ vanishes. Notice this holds in particular if the source term goes up to the interface, i.e., $\operatorname{supp} f_\Omega \cap \Sigma \neq \emptyset$.

Going back to Assumption 4.1.2, the Sobolev space $Q = H^1_{1/2}(\Omega_p) \times H^1_{1/2}(\Omega_n)$, see (4.6), dissociate Ω_p and Ω_n in order to ensure completeness results. It is in this Sobolev space in which the sesquilinear form of the problem is continuous and in which the regular part is measured. Then, there is a priori no relation between the two parts of the solution at the interface. Consequently, a weak notion of jump for functions in Q is addressed in Chapter 6.

5.3.3 Refined regularity estimation

From the above considerations, we see that the key quantity of the regularity analysis is the behavior of the integral

$$\int_{-1}^{1} K_0(|\lambda_k t|) f_k(t)\mathrm{d}t. \tag{5.24}$$

Thus, it appears that imposing conditions on the source term may improve the regularity of both the regular and the singular parts. A very simple way to eliminate the last integral (5.24) is to consider $f_\Omega \in L^2_{\mathrm{r}}(\Omega)$ such that every f_k are odd for all $k \in \mathbb{N}$ sufficiently large. However, this takes advantages of the symmetry of the domain, which is not guaranteed. Another simple and more realistic condition is that the support of the source term is far from the interface. Actually, this is the behavior of the source term in the original problem introduced in Chapter 4.

Therefore, the results of this section are supported by the following lemma, which is an improvement of Lemma 5.2.7.

Lemma 5.3.4. *Let $f \in L^2(-1,1)$ be such that there is $\underline{x} > 0$ such that $\operatorname{supp} f \cap (-\underline{x}, \underline{x}) = \emptyset$. Then,*

$$\left|\int_{-1}^{1} K_0(|\lambda_k t|) f(t)\mathrm{d}t\right| = \mathcal{O}\left(\frac{e^{-\lambda_k l}}{\lambda_k \sqrt{l}} \|f\|_{L^2(0,1)}\right).$$

Proof. Since $\lambda_k \to +\infty$ as $k \to \infty$, it suffices to apply the second part of Lemma 5.2.7 as soon as $\lambda_k l > 1$. $\square$

Then, implementing the previously outlined approach significantly enhances the behavior of b^+_k since

$$b^+_k = \mathcal{O}\left(\frac{e^{-\lambda_k l}}{\lambda_k \sqrt{l}} \|f\|_{L^2(0,1)}\right),$$

whose consequences are summarized in the following proposition.

Proposition 5.3.5. *Given $f \in L^2_{\mathrm{r}}(\Omega)$ be such that $\operatorname{supp} f \cap \Sigma = \emptyset$, we have $g^+ \in H^1_{\mathrm{r}}(\Sigma)$ and $u^+_{reg} \in H^1_{\mathrm{r}}(\Omega)$. In addition, $\|g^+\|_{H^1_{\mathrm{r}}(\Sigma)} + \left\|u^+_{reg}\right\|_{H^1_{\mathrm{r}}(\Omega)} \lesssim \|f_\Omega\|_{L^2_{\mathrm{r}}(\Omega)}$.*

In that case, the regular part has more than a vanishing jump at the interface, since it also has a classical trace.

It is legitimate to look at the case $\omega \neq 0$. However, the result above cannot be straightforwardly extended to the case of $\omega \neq 0$ via a bootstrap argument since even if the support of source term is disjoint with the interface, this is obviously not the case for a solution of the problem. Nevertheless, it is possible to replicate all the approach now with $\omega \neq 0$. In that case, a Whittaker equation appears instead of a modified Bessel equation. Its resolution provides solutions with very similar behavior to the modified Bessel functions, in particular the logarithmic behavior of the singular part.

Remark 5.3.6. In the case of $\operatorname{supp} f \cap \Sigma = \emptyset$, according to Lemma 5.2.7, the exponential behavior of b_k^+ actually leads to $g^+ \in \mathscr{C}^\infty_{per}(\Sigma)$.

5.4 Limiting absorption principle

Since both the formal limiting absorption solution u^+ and the solution with absorption u^ν belong to $L^2(\Omega)$, one can verify the following theorem.

Theorem 5.4.1 (Limiting absorption principle)**.** *The following convergence holds:*

$$u^\nu \xrightarrow[\nu \to 0+]{L^2(\Omega)} u^+.$$

The proof of this theorem relies on the following two lemmas, in which actually all the mathematical difficulties lie.

Lemma 5.4.2. *The series* $\nu \mapsto \|u^\nu\|^2_{L^2(\Omega)} = \sum_{k\in\mathbb{N}} \|u_k^\nu\|^2_{L^2(-1,1)}$ *is normally convergent for* $\nu \in (0,1)$*.*

Lemma 5.4.3. *For all* $k \in \mathbb{N}$, $u_k^\nu \to u_k^+$ *as* $\nu \to 0+$ *in* $L^2(-1,1)$*.*

Proof of Theorem 5.4.1. It suffices to prove that the series $\sum_{k\in\mathbb{N}} \|u_k^\nu - u_k^+\|^2_{L^2(-1,1)}$ goes to 0 as $\nu \to 0+$. Obviously, by Lemma 5.4.2, this series is normally convergent for $\nu \in (0,1)$. Therefore, we can swap the sum and the limit symbols. Then, we conclude with Lemma 5.4.3:

$$\lim_{\nu\to 0+} \sum_{k\in\mathbb{N}} \|u_k^\nu - u_k^+\|^2_{L^2(-1,1)} = \sum_{k\in\mathbb{N}} \lim_{\nu\to 0+} \|u_k^\nu - u_k^+\|^2_{L^2(-1,1)} = 0.$$

□

Both previous lemmas need the following estimations of the modified Bessel functions with complex argument.

Lemma 5.4.4. *We have for all* $x > 0$ *and* $y \in \mathbb{R}$

$$|I_0(x+iy)| \le I_0(x), \qquad \text{and} \qquad |K_0(x+iy)| \le K_0(x).$$

Proof. Both these inequalities use the integral representation of modified Bessel functions, refer to [51, §10.32]. Therefore, we have

$$|I_0(x+iy)| \le \frac{1}{\pi}\int_0^\pi \left|e^{(x+iy)\cos\theta}\right| \mathrm{d}\theta = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\mathrm{d}\theta = I_0(x).$$

The inequality on K_0 follows in the same way. □

Proof of Lemma 5.4.2. The proof consists in estimating $\|u_k^\nu\|^2_{L^2(-1,1)}$. Like in the proof of Proposition 5.2.5, we separate the study over the intervals $(-1,0)$ and $(0,1)$. Since this study on both intervals follows the same process, we will only describe it on $(0,1)$. Therefore, we have

$$\|u_k^\nu\|_{L^2(0,1)} \leq |a_k^\nu| \, \|I_0(\lambda_k(x+i\nu))\|_{L^2(0,1)} + |b_k^\nu| \, \|K_0(\lambda_k(x+i\nu))\|_{L^2(0,1)} + \|\hat{u}_k^\nu\|_{L^2(0,1)} \, .$$

Firstly, Lemma 5.4.4 yields

$$\|I_0(\lambda_k(x+i\nu))\|_{L^2(0,1)} \leq \|I_0(\lambda_k x)\|_{L^2(0,1)} \, , \qquad \text{and} \qquad \|K_0(\lambda_k(x+i\nu))\|_{L^2(0,1)} \leq \|K_0(\lambda_k x)\|_{L^2(0,1)} \, .$$

With this observation, it suffices to show that a_k^ν and b_k^ν follow behavior similar to the ones described in Lemma 5.2.9, and similarly for $\|\hat{u}_k^\nu\|_{L^2(0,1)}$ with $\|\hat{u}_k^+\|_{L^2(0,1)}$. The latter is simple since it follows the proof of Proposition 5.2.5, for example

$$\int_0^1 \left| I_0\left(\lambda_k(x+i\nu)\right) \int_x^1 K_0\left(\lambda_k(t+i\nu)\right) f_k(t) \mathrm{d}t \right|^2 \mathrm{d}x \leq \int_0^1 \left| I_0\left(\lambda_k x\right) \int_x^1 \left| K_0\left(\lambda_k x\right) f_k(t) \right| \mathrm{d}t \right|^2 \mathrm{d}x \lesssim \frac{\|f_k\|^2_{L^2(0,1)}}{\lambda_k^2},$$

as for Proposition 5.2.5. Therefore, we have $\|\hat{u}_k^\nu\|_{L^2(0,1)} \lesssim \lambda_k^{-1} \, \|f_k\|_{L^2(0,1)}$. In the view of the expressions (5.15) and (5.16) of a_k^ν and b_k^ν, we first estimate $\left(\Delta_k^\nu\right)^{-1}$, more precisely, we compare it to $\Delta_k^0 = i\pi I_0(\lambda_k)^2$. Let us prove that

$$(\lambda,\nu) \in [\lambda_1,+\infty) \times [0,1] \mapsto \left| \frac{i\pi I_0(\lambda)^2}{\Delta_k^\nu} \right|$$

is bounded. Using the asymptotics of modified Bessel function, summarized in Lemma 5.2.6, there is $C > 0$ large enough such that for all $\lambda > C$ and $\nu \in (0,1)$, the function above is bounded on $[C,+\infty] \times [0,1]$. On the other hand, this function is continuous on the compact set $[\lambda_1, C] \times [0,1]$, which implies that it is bounded on this set, and so on $[\lambda_1,+\infty] \times [0,1]$. Afterwards, with the expressions of a_k^ν and b_k^ν, Lemma 5.4.4 and the last boundedness result, we retrieve the behavior of Lemma 5.2.9, uniformly with respect to $\nu \in (0,1)$:

$$|a_k^\nu| \lesssim \frac{e^{-\lambda_k}}{\lambda_k} \|f_k\|_{L^2(-1,1)} \, , \qquad \text{and} \qquad |b_k^\nu| \lesssim \frac{1}{\sqrt{\lambda_k}} \|f_k\|_{L^2(-1,1)} \, . \tag{5.25}$$

Finally, like for Proposition 5.2.5, it follows that $\|u_k^\nu\|_{L^2(0,1)} \leq C\lambda_k^{-1} \, \|f_k\|_{L^2(0,1)}$ where C is independent of $\nu \in (0,1)$ and λ_k. □

The estimations above allows us to show the Lemma 5.4.3.

Proof of Lemma 5.4.3. Using Lemma 5.4.4, we easily find $g_k \in L^2(-1,1)$ be such that $|u_k^\nu(x)| < g_k(x)$ for $\nu \in (0,1)$. Finally, since $u_k^\nu(x) \to u_k^+(x)$ as $\nu \to 0+$ for almost every $x \in (0,1)$, the Lebesgue's dominated convergence theorem concludes the proof. □

5.5 Conclusions

The observation of the present chapter justify the use in Chapter 6 of Assumption 4.1.2 requiring a source term with support disjoint from the interface. This also shows that assuming $g \in H^2_{per}(\Sigma)$ is too restrictive.

On the other hand, in the case of a general source term, the fact that the amplitude g^+ of the singular part (5.22) only belongs to $H^{1/2}_{\mathfrak{r}}(\Sigma)$ suggests that it does not trap all the singularities of the problem. An idea would be to consider an amplitude of the singular part with more degrees of freedom, for example, a lifting $\tilde{g}^+ \in H^1_{\mathfrak{r}}(\Omega)$ of g^+, with a singular part now equal to $\tilde{u}^+_{sing}(x,y) = \tilde{g}^+(x,y)\,(\log|x| + i\pi \mathbb{1}_{x<0})$. The counterpart of this approach is that the regular part would be less regular since $\nabla\tilde{u}_{reg} \notin L^2_{\mathfrak{r}}(\Omega)$ but rather $x^{\varepsilon}\nabla\tilde{u}_{reg} \in L^2_{\mathfrak{r}}(\Omega)$ for all $\varepsilon > 0$. Then, the following assumption would be more pertinent.

Assumption 5.5.1. *The family of solutions $(u^\nu)_{\nu>0}$ of the problem with absorption converges in $L^2(\Omega)$ to the limiting absorption solution $u^+ \in L^2(\Omega)$*

$$u^\nu \xrightarrow[\nu\to 0+]{L^2(\Omega)} u^+. \tag{5.26}$$

Moreover, u^+ can be represented as

$$u^+ = \tilde{u}^+_{reg} + \tilde{u}^+_{sing},$$

where the pair $(\tilde{u}^+_{reg}, \tilde{u}^+_{sing})$ is such that $\tilde{u}^+_{reg|\Omega_{p,n}} \in H^1_{1/2}(\Omega_{p,n})$ and $\tilde{u}^+_{sing}(x,y) = \tilde{g}^+(x,y)\mathbb{S}(x)$ with $\tilde{g}^+ \in H^1_{per}(\Omega)$.

Appendix

5.A Study in another case

We consider in this section the problem (5.1) with $\alpha(x,y) = x$, $\omega > 0$, and $f_\Omega \in L^2(\Omega)$. As in § 5.2, we add some absorption $\nu > 0$, and we seek the solution of

$$\left|\begin{array}{ll} \text{find } u^\nu \in H^1(\Omega) \text{ such that} & \\ -\operatorname{div}\left((x+i\nu)\,\nabla u^\nu\right) - \omega^2 u^\nu = f_\Omega & \text{in } \Omega, \\ u^\nu = 0 & \text{on } \Gamma_n \cup \Gamma_p, \\ u^\nu(x,0) = u^\nu(x,L), \quad \Big((x+i\nu)\,\partial_y\Big)\,u^\nu(x,0) = \Big((x+i\nu)\,\partial_y\Big)\,u^\nu(x,L), & x \in (-1,1). \end{array}\right.$$

Since $u^\nu \in L^2(\Omega)$, we can decompose it onto $(\psi_k)_{k\in\mathbb{N}}$, see Definition 5.1.2, which leads to the following equation:

$$-\partial_x\left((x+i\nu)\partial_x u^\nu_k\right) + \left((x+i\nu)\lambda_k^2 - \omega^2\right)u^\nu_k = f_k, \quad \text{for } k \geq 0.$$

In order to get back to a known differential equation, we will use a substitution proposed in [51, (1.13(iv))]. Firstly, we set $z = x + i\nu$, which yields

$$-\partial_z\left(z\partial_z u^\nu_k\right) + \left(z\lambda_k^2 - \omega^2\right)u^\nu_k = f_k, \qquad z \in \mathbb{C}. \tag{5.27}$$

Next, we focus on the substitution [51, (1.13.13)], which gives[2] $w_k(z) = \sqrt{z}\, u_k^\nu(z)$. This leads to the equation

$$\partial_z^2 w_k + \left(\frac{1}{4z^2} + z\lambda_k^2 - \omega^2\right) w_k = \sqrt{z} f_k.$$

Finally, we obtain with the substitution $\eta = 2\lambda_k z$:

$$\partial_\eta^2 w_k + \left(\frac{1/4}{\eta^2} + \frac{\omega^2/2\lambda_k}{\eta} - \frac{1}{4}\right) w_k = -\frac{1}{\sqrt{2\lambda_k \eta}} f_k.$$

This is the Whittaker equation, cf. [51, (13.14)], with the parameters $\left(\frac{\omega^2}{2\lambda_k}, 0\right)$. The solution to the associated homogeneous equation is given by [51, (13.14.2), (13.14.3)]. Therefore, the general solution of (5.27) is written as

$$u_k^\nu(z) = u_k^{0,\nu}(z) + \hat{u}_k^\nu(z), \qquad \text{with} \quad u_k^{0,\nu}(z) = a_k^\nu u_k^{1,\nu}(z) + b_k^\nu u_k^{2,\nu}(z),$$

and

$$u_k^{1,\nu}(z) = \sqrt{2\lambda_k} e^{-\lambda_k z} M\left(\frac{1}{2} - \frac{\omega^2}{2\lambda_k}, 1, 2\lambda_k z\right), \qquad u_k^{2,\nu}(z) = \sqrt{2\lambda_k} e^{-\lambda_k z} U\left(\frac{1}{2} - \frac{\omega^2}{2\lambda_k}, 1, 2\lambda_k z\right),$$

where M, U are the standard solutions of the Kummer's equation, cf. [51, (13.2)]. The particular solution $\hat{u}_k^\nu$ can be computed like in the proof of 5.2.1.

The major point of the chapter is the behavior of the homogeneous solution, as in Lemma 5.2.6. This lemma can be easily adapted in the following proposition.

Lemma 5.A.1. *The homogeneous solutions $u_k^{1,\nu}$ and $u_k^{2,\nu}$ satisfy*

$$\frac{u_k^{1,\nu}(z)}{\sqrt{2\lambda_k}} \underset{z\to 0}{\sim} 1, \qquad \Gamma\left(\frac{1}{2} - \frac{\omega^2}{2\lambda_k}\right) u_k^{2,\nu}(z) \underset{z\to 0}{\sim} -\log(z),$$

$$\frac{u_k^{1,\nu}(z)}{\sqrt{2\lambda_k}} \underset{|z|\to+\infty}{\sim} \frac{(2\lambda_k z)^{-\omega^2/2\lambda_k}}{\sqrt{1 - \frac{\omega^2}{\lambda_k}}} \times \frac{e^{\lambda_k z}}{\sqrt{\lambda_k z}}, \qquad \Gamma\left(\frac{1}{2} - \frac{\omega^2}{2\lambda_k}\right) u_k^{2,\nu}(z) \underset{|z|\to+\infty}{\sim} (2\lambda_k z)^{-\omega^2/2\lambda_k} \frac{\Gamma\left(\frac{1}{2} - \frac{\omega^2}{2\lambda_k}\right)}{\sqrt{2\lambda_k z}} e^{-\lambda_k z},$$

for $|\arg z| < \pi/2 - \delta$, $\delta > 0$ arbitrary small. $z \mapsto \Gamma(z)$ is the usual Gamma function, cf. [51, §5].

Proof. For the asymptotics with small argument, we use [51, (13.2.13)] and [51, (13.2.19)]. For the asymptotics with large argument, we use [51, (13.2.4), (13.2.23)] and [51, (13.2.6)]. □

Let us make few remark about these asymptotics. Given $z \in \mathbb{C} \setminus (-\infty, 0]$, we have that

$$(2\lambda_k z)^{-\omega^2/2\lambda_k} \xrightarrow[k\to+\infty]{} 1, \qquad \Gamma\left(\frac{1}{2} - \frac{\omega^2}{2\lambda_k}\right) \xrightarrow[k\to+\infty]{} \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Therefore, we obtain a very similar behavior as observed in Lemma 5.2.6. Then, up to a renormalization, most of the development done in this chapter could be iterated and should lead toward the same results, in particular on the presence of a logarithmic singularity, the regularity of its amplitude and the limiting absorption principle.

[2]Within the notation of [51, (1.13)], we set $f(z) = z^{-1}$ and $g(z) = \lambda_k^2 - \omega^2/z$.

CHAPTER 6

Mixed variational formulation

Contents

6.1 Problem setting

Let $\Omega = (-a, a) \times (0, L)$, with the notations of Chapter 4, $\nu > 0$ be an absorption parameter. Consider the following family of problems:

$$\left| \begin{array}{ll} \text{find } u^\nu \in H^1(\Omega) \text{ such that} & \\ -\operatorname{div}((\alpha + i\nu)\nabla u^\nu) - \omega^2 u^\nu = f_\Omega & \text{in } \Omega, \\ (\alpha + i\nu)\,\partial_n u^\nu + i\lambda u^\nu = f_\Gamma & \text{on } \Gamma_n \cup \Gamma_p, \\ u^\nu(x,0) = u^\nu(x,L), \quad \big((\alpha + i\nu)\,\partial_y\big)\, u^\nu(x,0) = \big((\alpha + i\nu)\,\partial_y\big)\, u^\nu(x,L), & x \in (-a,\, a), \end{array} \right. \tag{6.1}$$

where $f_\Omega \in L^2(\Omega)$ and $f_\Gamma \in L^2(\Gamma_n \cup \Gamma_p)$. Notice the slight difference with the setting from Chapter 4 due to the introduction of the volume source term f_Ω. The function $\alpha(x,y) \in \mathscr{C}^2_{per,y}\left(\overline{\Omega};\mathbb{R}\right)$ is such that $\alpha(x,y) > 0$ on $\Omega_p = \{(x,y) : x > 0\}$ and $\alpha(x,y) < 0$ on $\Omega_n = \{(x,y) : x < 0\}$. Therefore, $\alpha(0,y) = 0$.

As discussed in chapter 4, we introduce the singularity $\mathrm{S}(x) := \log|x| + i\pi\mathbb{1}_{x<0}$, the space of "regular functions"

$$H^1_{1/2}(\Omega_j) = \left\{v \in L^2(\Omega_j) \,:\, |\alpha|^{1/2}\nabla v \in L^2(\Omega_j)\right\}, \quad j \in \{p,n\},$$

and $Q = H^1_{1/2}\left(\Omega_p\right) \times H^1_{1/2}\left(\Omega_p\right)$, see (4.6). For this chapter, we make the following assumption introduced in chapter 4.

Assumption 6.1.1. *The family of solutions $(u^\nu)_{\nu>0}$ of (6.1) converges in $L^2(\Omega)$ to the limiting absorption solution $u^+ \in L^2(\Omega)$*

$$u^\nu \xrightarrow[\nu\to 0+]{L^2(\Omega)} u^+. \tag{6.2}$$

Moreover, u^+ can be represented as

$$u^+ = u^+_{reg} + u^+_{sing},$$

where the pair (u^+_{reg}, u^+_{sing}) is such that the regular part $u^+_{reg}\big|_{\Omega_j} \in H^1_{1/2}(\Omega_j)$ for $j \in \{p,n\}$ and the singular part $u^+_{sing}(x,y) = g^+(y)\mathrm{S}(x)$ with $g^+ \in H^1_{per}(\Sigma)$.

Remark 6.1.2. This assumption is justified by the observations in Chapter 5, if the source term f_Ω satisfies $\operatorname{supp} f_\Omega \cap \Sigma = \emptyset$. Nevertheless, the content of this chapter seems valid even for source term which does not vanish near the interface.

Remark 6.1.3. Given $\alpha(x,y) = \mathtt{r}(y)x$ as in Chapter 5, we proved the limiting absorption principle when the absorption is set as $\mathtt{r}(y)\,(x+i\nu)$. Then, notice that the absorption principle is set slightly differently in Assumption 6.1.1, since it reads $\mathtt{r}(y)x + i\nu$.

We identify the function u^+_{reg} with a pair $\mathbf{u}^+ = \left(u^+_{reg}\big|_{\Omega_p}, u^+_{reg}\big|_{\Omega_n}\right) \in Q$. For generic $g(y)$, we use the notation $s_g(x,y) = g(y)\mathrm{S}(x)$ as introduced in (4.7). First, u^+ is one of the solutions of the problem

$$\left|\begin{array}{ll} \text{find } u \in L^2(\Omega) \text{ such that} & \\ -\operatorname{div}(\alpha\nabla u) - \omega^2 u = f_\Omega & \text{in } \Omega, \\ \alpha\partial_n u + i\lambda u = f_\Gamma & \text{on } \Gamma_n \cup \Gamma_p, \\ u(x,0) = u(x,L), \quad (\alpha\partial_y)u(x,0) = (\alpha\partial_y)u(x,L), & x \in (-a,\, a), \end{array}\right. \tag{6.3}$$

see proposition 6.2.1 below. The aim of this chapter is to find and analyze a variational problem for which $u^+ \equiv (\mathbf{u}^+, g^+)$, as defined in assumption 6.1.1, constitutes a solution.

Let $u \in H^1_{1/2}(\Omega_j)$ be such that $\operatorname{div}(\alpha\nabla u) \in L^2(\Omega_j)$ and periodic boundary conditions are imposed between $\{(x,y) : y = 0\}$ and $\{(x,y) : y = L\}$. Then, Green's identity gives for $v \in H^1_{1/2}(\Omega_j)$

$$-\int_{\Omega_j} \operatorname{div}(\alpha\nabla u)\,\overline{v}\mathrm{dx} = \int_{\Omega_j} \alpha\nabla u \cdot \overline{\nabla v}\mathrm{dx} - \int_{\Gamma_j} \alpha\partial_n u\overline{v}\mathrm{ds}.$$

Given the absorbing boundary conditions of the problem, the sesquilinear form associated to the problem (6.3) which operates on the regular part is

$$b^{(1)}_{reg}(\mathbf{u},\mathbf{v}) := \sum_{j\in\{p,n\}} \int_{\Omega_j} \left(\alpha\nabla u_j \cdot \overline{\nabla v_j} - \omega^2 u_j\overline{v_j}\right)\mathrm{dx} + i\lambda\int_{\Gamma_j} u_j\overline{v_j}\mathrm{ds}, \quad \mathbf{u},\mathbf{v} \in Q. \tag{6.4}$$

Next, let $g \in H^1_{per}(\Sigma)$ be such that $\operatorname{div}\left(\alpha \nabla s_g\right) \in L^2(\Omega)$[1]. Unfortunately, Green's identity cannot be applied since $\alpha\left|\partial_x s_g\right|^2 \notin L^1(\Omega)$. However, we have $\partial_x\left(\alpha \partial_x s_g\right) \in L^2(\Omega)$, $\alpha\left|\partial_y s_g\right|^2 \in L^1(\Omega)$, and these two quantities involve only g and its derivative $\partial_y g$. Therefore, *integrating by parts only along the y-direction* gives, for $v \in H^1_{1/2}(\Omega_j)$

$$-\int_{\Omega_j} \operatorname{div}\left(\alpha \nabla s_g\right) v \mathrm{d}\mathbf{x} = -\int_{\Omega_j} \partial_x\left(\alpha \partial_x s_g\right) v \mathrm{d}\mathbf{x} + \int_{\Omega_j} \alpha \partial_y s_g \partial_y v \mathrm{d}\mathbf{x}.$$

Notice that the two last integrals are well-defined for all $g \in H^1_{per}(\Sigma)$. With this observation, one can define a variational formulation of (6.3) with $g \in H^1_{per}(\Sigma)$ by *integrating by parts only along the y-direction*, to be compared with $g \in H^2_{per}(\Sigma)$ in [49]. Then, multiplying (6.3) by a test function $\mathbf{v} \in Q$, integrating by parts along the y-direction, and taking into account the absorbing boundary conditions of the problem, we obtain the sesquilinear form associated with the problem that operates on the singular part s_g. For $g \in H^1_{\mathrm{per}}(\Sigma)$ and $\mathbf{v} \in Q$, this sesquilinear form is given by:

$$b^{(1)}_{sing}(g, \mathbf{v}) := \sum_{j \in \{p,n\}} \int_{\Omega_j} \left(\alpha \partial_y s_g \overline{\partial_y v_j} + (-\partial_x(\alpha \partial_x s_g) - \omega^2 s_g)\overline{v_j}\right) \mathrm{d}\mathbf{x} + \int_{\Gamma_j} (\alpha \partial_n s_g + i\lambda s_g)\overline{v_j} \mathrm{d}s. \tag{6.5}$$

Hence, the variational problem associated to the problem (6.3) is

$$\left| \begin{aligned} &\text{find } (\mathbf{u}, g) \in Q \times H^1_{per}(\Sigma) \text{ such that} \\ &b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}) = \ell^{(1)}(\mathbf{v}), \qquad \forall \mathbf{v} \in Q, \end{aligned} \right. \tag{6.6}$$

where the left-hand side is for $\mathbf{v} \in Q$

$$\ell^{(1)}(\mathbf{v}) := \sum_{j \in \{p,n\}} \int_{\Omega_j} f_\Omega \overline{v_j} \mathrm{d}\mathbf{x} + \int_{\Gamma_j} f_\Gamma \overline{v_j} \mathrm{d}s. \tag{6.7}$$

Unfortunately, the previous problem is not well-posed. As a matter of fact, the operator $\mathrm{B}^{(1)}_{reg} : Q \to Q'$ associated to $b^{(1)}_{reg}$ is invertible, see [49, propositions 4]. As a consequence, for all $g \in H^1_{per}(\Omega)$, there is a unique $\mathbf{u}(g)$ such that for all $\mathbf{v} \in Q$

$$b^{(1)}_{reg}(\mathbf{u}(g), \mathbf{v}) = \ell^{(1)}(\mathbf{v}) - b^{(1)}_{sing}(g, \mathbf{v}).$$

A natural way to recover the well-posedness is to add a new condition to the problem. Therefore, following the approach of [49], the subsequent section is devoted to establishing this new condition, leading to the formulation of a mixed variational problem.

Remark 6.1.4. It does not seem possible to further reduce the regularity of g compared to [49] because its first derivative $\partial_y g$ appears in the first term of (6.5).

6.2 Construction of a mixed problem

6.2.1 Properties of the limiting absorption solution

In order to derive a new problem solved by the absorption solution u^+, we show in this paragraph some simple properties of u^+. Let us introduce the variational formulation for (6.1):

$$\left| \begin{aligned} &\text{find } u^\nu \in H^1(\Omega) \text{ such that} \\ &b^\nu(u, v) = \ell^{(1)}(v), \qquad \forall v \in H^1(\Omega) \end{aligned} \right. \tag{6.8}$$

[1]Notice that this implies $g \in H^2(\Sigma)$.

with

$$b^{\nu}(u,v) := \int_{\Omega} \left[(\alpha + i\nu)\nabla u \cdot \overline{\nabla v} - \omega^2 u\bar{v} \right] \mathrm{d}x + i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u\bar{v}\mathrm{d}s, \tag{6.9}$$

and

$$\ell^{(1)}(v) = \int_{\Omega} f_{\Omega}\bar{v}\mathrm{d}x + \int_{\Gamma_p\cup\Gamma_n} f_{\Gamma}\bar{v}\mathrm{d}s. \tag{6.10}$$

Proposition 6.2.1. *Let $u^+ \equiv (\mathbf{u}^+, g^+) \in Q \times H^1_{per}(\Sigma)$ be governed by the assumption 6.1.1. Then, for all $\mathbf{v} \in Q$,*

$$b^{(1)}_{reg}(\mathbf{u}^+, \mathbf{v}) + b^{(1)}_{sing}(g^+, \mathbf{v}) = \ell^{(1)}(\mathbf{v}). \tag{6.11}$$

Moreover, u^+ solves (6.3) in the sense of distributions.

The proof this proposition relies on two technical lemmas.

Lemma 6.2.2. *Let $\nu > 0$, and u^ν satisfy (6.1). Under assumption 6.1.1, there holds*

$$(\alpha + i\nu)\nabla u^\nu \xrightarrow[\nu\to 0+]{L^2(\Omega)} \alpha\nabla(u^+_{reg} + u^+_{sing}).$$

Proof. Until the end of the proof, we write $\|\cdot\|$ for the $L^2(\Omega)$-norm and $\|\cdot\|_\infty$ for the $L^\infty(\Omega)$-norm. We split the proof into two steps.

Step 1. Proof that $\nu\|\nabla u^\nu\| \to 0$ as $\nu \to 0+$. We test the equation (6.9) with $v^\nu = u^\nu$ and take the imaginary part of the resulting expression. This yields

$$\nu\|\nabla u^\nu\|^2 + \lambda\|u^\nu\|^2_{L^2(\Gamma_p\cup\Gamma_\nu)} \le \|f_\Omega\|\,\|u^\nu\| + \|f_\Gamma\|_{L^2(\Gamma_p\cup\Gamma_n)}\|u^\nu\|_{L^2(\Gamma_p\cup\Gamma_\nu)}.$$

Then, using Young's inequality, one obtains

$$\begin{aligned}
\nu\|\nabla u^\nu\|^2 &\le \|f_\Omega\|\,\|u^\nu\| + \|f_\Gamma\|_{L^2(\Gamma_p\cup\Gamma_n)}\,\|u^\nu\|_{L^2(\Gamma_p\cup\Gamma_\nu)} - \lambda\|u^\nu\|^2_{L^2(\Gamma_p\cup\Gamma_\nu)}\\
&\le \|f_\Omega\|\,\|u^\nu\| + \frac{1}{4\lambda}\|f_\Gamma\|^2_{L^2(\Gamma_p\cup\Gamma_n)} + \frac{2\lambda}{2}\|u^\nu\|^2_{L^2(\Gamma_p\cup\Gamma_\nu)} - \lambda\|u^\nu\|^2_{L^2(\Gamma_p\cup\Gamma_\nu)}\\
&\le \|f_\Omega\|\,\|u^\nu\| + \frac{1}{4\lambda}\|f_\Gamma\|^2_{L^2(\Gamma_p\cup\Gamma_n)}.
\end{aligned}$$

Finally, thanks to assumption 6.1.1, $\|u^\nu\| \to \|u^+\|$ as $\nu \to 0+$, and therefore $\nu\|\nabla u^\nu\| \to 0$.

Step 2. Proof that $\alpha\nabla u^\nu \to \alpha\nabla u^+$ in $L^2(\Omega)$. We will show that $(\alpha\nabla u^{\nu_n})_{n\in\mathbb{N}}$ is a Cauchy sequence for all $(\nu_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, s.t. $\lim_{n\to+\infty}\nu_n = 0$. Since we know that $(u^{\nu_n})_{n\in\mathbb{N}}$ and $(\nu_n\nabla u^{\nu_n})_{n\in\mathbb{N}}$ are Cauchy sequences in $L^2(\Omega)$, let us denote $e_{nm} = u^{\nu_n} - u^{\nu_m}$ and $\tilde{\mathbf{e}}_{nm} = \nu_n\nabla u^{\nu_n} - \nu_m\nabla u^{\nu_m}$. Then, we want to control $\alpha\nabla e_{nm}$.

For this we consider the difference of (6.9) written for $\nu = \nu_n$ and $\nu = \nu_m$, namely

$$b^{\nu_n}(u^{\nu_n}, v) - b^{\nu_m}(u^{\nu_m}, v) = \int_\Omega \left[\alpha\nabla e_{nm}\cdot\overline{\nabla v} + i\tilde{\mathbf{e}}_{nm}\cdot\overline{\nabla v} - \omega^2 e_{nm}\bar{v}\right]\mathrm{d}x + i\lambda\int_{\Gamma_p\cup\Gamma_n} e_{nm}\bar{v}\,\mathrm{d}s = 0 \tag{6.12}$$

We test the equation (6.12) with $v = \alpha e_{nm}$, which yields

$$\int_\Omega |\alpha\nabla e_{nm}|^2\mathrm{d}x + \int_\Omega \left[\alpha\nabla e_{nm}\cdot\overline{e_{nm}\nabla\alpha} + i\tilde{\mathbf{e}}_{nm}\cdot\overline{(\alpha\nabla e_{nm} + e_{nm}\nabla\alpha)} - \omega^2\alpha|e_{nm}|^2\right]\mathrm{d}x$$
$$+ i\lambda\int_{\Gamma_p\cup\Gamma_n}\alpha|e_{nm}|^2\,\mathrm{d}s = 0$$

Taking the real part of the above, and using the Cauchy-Schwarz inequality to bound sign-indefinite terms yields:

$$\begin{aligned}\int_\Omega |\alpha\nabla e_{nm}|^2 \mathrm{d}\mathbf{x} &\le \int_\Omega \left[\left|\alpha\nabla e_{nm}\cdot\overline{e_{nm}\nabla\alpha}\right| + \left|\tilde{\mathbf{e}}_{nm}\cdot\overline{(\alpha\nabla e_{nm}+e_{nm}\nabla\alpha)}\right| + \omega^2|\alpha||e_{nm}|^2\right]\mathrm{d}\mathbf{x}\\ &\le \|\nabla\alpha\|_\infty\,\|\alpha\nabla e_{nm}\|\,\|e_{nm}\| + \|\tilde{\mathbf{e}}_{nm}\|\left(\|\alpha\nabla e_{nm}\| + \|\nabla\alpha\|_\infty\|e_{nm}\|\right) + \omega^2\|\alpha\|_\infty\|e_{nm}\|^2.\end{aligned}$$

With the help of the Young inequality, we obtain the following bound:

$$\|\alpha\nabla e_{nm}\| \le C\left(\|e_{nm}\| + \|\tilde{\mathbf{e}}_{nm}\|\right),$$

where the constant C depends on $\|\alpha\|_\infty$, $\|\nabla\alpha\|_\infty$ and ω^2 only. Because $\nu\|\nabla u^\nu\| \to 0$ as $\nu \to 0+$ and u^ν converges in $L^2(\Omega)$ as $\nu \to 0+$, we conclude that $(\alpha\nabla u^{\nu_n})_{n\in\mathbb{N}}$ is an $L^2(\Omega)$-Cauchy sequence, and thus converges. Evidently, its limit is $\alpha\nabla u^+$; this follows from the following expression (which allows to define the distribution $\alpha\nabla v$ for $v \in L^2(\Omega)$ and $\alpha \in C^1(\overline{\Omega})$):

$$\alpha\nabla u^\nu = \nabla(\alpha u^\nu) - u^\nu\nabla\alpha.$$

We have that $u^\nu\nabla\alpha \to u^+\nabla\alpha$ in $L^2(\Omega)$; similarly, $\alpha u^\nu \to \alpha u^+$, thus, in the sense of distributions, $\nabla(\alpha u^\nu) \to \nabla(\alpha u^+)$. Finally, by the uniqueness of the distributional limit, we conclude with the desired result. □

Remark 6.2.3. Let $\mathcal{O} \subset \Omega$ be an open set such that $\overline{\mathcal{O}} \cap \Sigma = \emptyset$. Since there is a constant $C_{\mathcal{O}}$ such that $|\alpha| > C_{\mathcal{O}} > 0$, the norms $\|\cdot\|_{H^1_{1/2}(\mathcal{O})}$ and $\|\cdot\|_{H^1(\mathcal{O})}$ are equivalent. Therefore, previous lemma shows that $u^\nu \to u^+$ in $H^1(\mathcal{O})$, and in particular, by continuity of the trace application, $u^\nu|_{\Gamma_{p,n}} \to u^+|_{\Gamma_{p,n}}$ in $L^2\left(\Gamma_{p,n}\right)$.

Let $g \in H^1_{per}(\Sigma)$. Recall the “singularity with absorption”

$$s_g^\nu(x,y) = g(y)\log\left(x + \frac{i\nu}{\mathtt{r}(y)}\right) \tag{6.13}$$

defined in (4.8). We obviously have $s_g^\nu \in H^1(\Omega)$, i.e., s_g^ν is not singular, but it approaches the singularity s_g as ν goes to 0+, as can be seen in the following lemma.

Lemma 6.2.4. *Given $g \in H^1_{per}(\Sigma)$, the following limits hold in $L^2(\Omega)$ as $\nu \to 0+$:*

$$\begin{aligned} &s_g^\nu \to s_g, && \partial_y s_g^\nu \to \partial_y s_g,\\ &(\alpha+i\nu)\partial_x s_g^\nu \to \alpha\partial_x s_g, && \partial_x((\alpha+i\nu)\partial_x s_g^\nu) \to \partial_x(\alpha\partial_x s_g).\end{aligned}$$

Proof. By direct computation, we have

$$\begin{aligned}(\alpha+i\nu)\,\partial_x s_g^\nu &= g(y)\mathtt{r}(y)\frac{\alpha+i\nu}{\mathtt{r}(y)x+i\nu}\\ \partial_y s_g^\nu &= \partial_y g(y)\log\left(x+\frac{i\nu}{\mathtt{r}(y)}\right) - g(y)\frac{i\nu\partial_y\mathtt{r}(y)}{\mathtt{r}(y)(\mathtt{r}(y)x+i\nu)}\\ \partial_x\left((\alpha+i\nu)\partial_x s_g^\nu\right) &= g(y)\mathtt{r}(y)\frac{\mathtt{r}(y)(\partial_x\alpha\, x-\alpha)+i\nu(\partial_x\alpha-\mathtt{r}(y))}{(\mathtt{r}(y)x+i\nu)^2},\end{aligned}$$

so that the convergence holds almost everywhere. In order to use the Lebesgue’s dominated convergence theorem, it suffices to check that all functions above are L^2-integrable uniformly with respect to ν. Recall that we have $\alpha(x,y) = \mathtt{r}(y)x + \mathcal{O}(x^2)$ in a neighborhood of the interface. The following ratios are easily bounded:

$$\left|\frac{\alpha + i\nu}{\mathtt{r}x + i\nu}\right| \lesssim 1, \quad \left|\frac{\partial_x \alpha\, x - \alpha}{(\mathtt{r}x + i\nu)^2}\right|^2 \lesssim 1,$$

and the convergence holds for $(\alpha + i\nu)\partial_x s_g^\nu$. On the other hand, we have

$$\int_{|x|<a}\int_y \left|\nu g \frac{\partial_y \mathtt{r}}{\mathtt{r}(\mathtt{r}x + i\nu)}\right|^2 \mathrm{dx} = \int_y \left|\frac{g\partial_y \mathtt{r}}{\mathtt{r}}\right|^2 \int_{|x|<a} \frac{\mathrm{dx}}{1 + \left(\frac{\mathtt{r}x}{\nu}\right)^2} \mathrm{dy} \le \nu\pi \int_y \frac{|g|^2(\partial_y \mathtt{r})^2}{\mathtt{r}^3} \mathrm{dy} \xrightarrow[\nu\to 0]{} 0,$$

$$\int_{|x|<a}\int_y \left|\nu g \mathtt{r} \frac{\partial_x \alpha - \mathtt{r}}{(\mathtt{r}x + i\nu)^2}\right|^2 \mathrm{dx} \lesssim \int_y |g|^2 \int_{|x|<a} \frac{\left(\frac{\mathtt{r}x}{\nu}\right)^2 \mathrm{dx}}{\left(1 + \left(\frac{\mathtt{r}x}{\nu}\right)^2\right)^2} \mathrm{dy} \lesssim \nu \int_y \frac{|g|^2}{\mathtt{r}} \mathrm{dy} \xrightarrow[\nu\to 0]{} 0.$$

This allows us to conclude about $\partial_y s_g^\nu$ and $\partial_x\left((\alpha + i\nu)\partial_x s_g^\nu\right)$. ☐

A natural corollary of previous lemmas is the following.

Corollary 6.2.5. *Let $\nu > 0$ and u^ν satisfy (6.1). Under assumption 6.1.1, $u^\nu \in H^1(\Omega)$ can be decomposed as*

$$u^\nu = u_{reg}^\nu + s_{g^+}^\nu,$$

and it holds that

$$u_{reg}^\nu \xrightarrow[\nu\to 0+]{L^2(\Omega)} u_{reg}^+, \quad \text{and} \quad (\alpha + i\nu)\nabla u_{reg}^\nu \xrightarrow[\nu\to 0+]{L^2(\Omega)} \alpha\nabla u_{reg}^+.$$

Proof. By lemma 6.2.4, $s_{g^+}^\nu \to s_{g^+}$ and $\alpha\nabla s_{g^+}^\nu \to \alpha\nabla s_{g^+}$ in $L^2(\Omega)$-norm as $\nu \to 0+$. Therefore, with the assumption 6.1.1 and lemma 6.2.2, the result holds. ☐

Proof of proposition 6.2.1. First, because of assumption 6.1.1, the limiting absorption solution $u^+ = u_{reg}^+ + s_g^+$ satisfies

$$-\operatorname{div}(\alpha\nabla u^+) - \omega^2 u^+ = f_\Omega \text{ in } \mathscr{D}'(\Omega). \tag{6.14}$$

Moreover, using the decomposition of lemma 6.2.5 and the fact that $\partial_x(\alpha\partial_x s_g^+) \in L^2(\Omega)$, we conclude that

$$\begin{aligned} \mathrm{d}^+ &:= -\operatorname{div}(\alpha\nabla u^+) + \partial_x(\alpha\partial_x s_g^+) \\ &= -\operatorname{div}\left(\alpha\nabla u_{reg}^+\right) - \partial_y\left(\alpha\partial_y s_g^+\right) \in L^2(\Omega). \end{aligned} \tag{6.15}$$

Notice that $\operatorname{div}\left(\alpha\nabla u_{reg}^+\right)$ does not necessarily belong to $L^2(\Omega)$, so that the integration by parts must be done carefully. Testing on Ω_p the equation (6.14) with $v_p \in C^\infty_{per,y}\left(\overline{\Omega_p}\right)$, using the boundary conditions of (6.3) and integrating by parts yields (cf. (6.15), corollary 6.2.5 and the periodicity of g^+):

$$\begin{aligned} &\int_{\Omega_p} \left\{\alpha\nabla u_{reg}^+ \cdot \overline{\nabla v_p} + \alpha\partial_y s_{g^+}\overline{\partial_y v_p} - \partial_x\left(\alpha\partial_x s_{g^+}\right)\overline{v_p} - \omega^2\left(u_{reg}^+ + s_{g^+}\right)\overline{v_p}\right\} \mathrm{dx} \\ &\quad - \int_{\Gamma_p} \alpha\partial_{\mathbf{n}} u_{reg}^+ \overline{v_p} \mathrm{ds} - \left\langle \alpha\partial_{\mathbf{n}} u_{reg}^+, \gamma_0 v_p\right\rangle_{\left(H^{1/2}_{per}(\Sigma)\right)', H^{1/2}_{per}(\Sigma)} = \int_{\Omega_p} f_\Omega \overline{v_p} \mathrm{dx} + \int_{\Gamma_p} f_\Gamma \overline{v_p} \mathrm{ds}. \end{aligned} \tag{6.16}$$

Here $\mathbf{n} = (n_x, n_y)$ is the outgoing unit normal from Ω_p. Finally, it remains to show that $\alpha\partial_{\mathbf{n}} u^+_{reg}\big|_\Sigma = 0$ and $\alpha\partial_{\mathbf{n}} u^+_{reg} = f_\Gamma - \alpha\partial_{\mathbf{n}} - i\lambda\left(u^+_{reg} + s_{g^+}\right)$ on Γ_p, see the boundary conditions of the problem (6.3).

Let $\varphi_1 \in \mathscr{C}^1_0\left((-a, a); \mathbb{R}\right)$, $\varphi_1 = 1$ in the vicinity of 0 and $\varphi(x, y) = \varphi_1(x)$ be a truncation function as in Definition 4.2.2 and its scaled version $\varphi_\varepsilon(x, y) = \varphi\left(\frac{x}{\varepsilon}, y\right)$. Integrating by parts as above with $v_p\varphi_\varepsilon$, we then have for all $\varepsilon > 0$,

$$\left\langle \alpha\partial_{\mathbf{n}} u^+_{reg}, \gamma_0 v_p \right\rangle_{\left(H^{1/2}_{per}(\Sigma)\right)', H^{1/2}_{per}(\Sigma)} = -\int_{\Omega_p} \left\{ \mathrm{d}^+\, \overline{v_p\varphi_\varepsilon} + \alpha\partial_y(u^+_p + s^+_g)\, \overline{\partial_y(\varphi_\varepsilon v_p)} \right\} \mathbf{dx} + \int_{\Omega_p} \alpha\partial_x u^+_p\, \overline{\partial_x(v_p\varphi_\varepsilon)} \mathbf{dx}.$$

It suffices to consider the case $\varepsilon \to 0$. The convergence of the first integral to 0 follows from Lebesgue's dominated convergence theorem and the fact that $\partial_y\varphi_\varepsilon = 0$. Finally, as for the second integral, we can estimate it as follows (where $\Omega^\varepsilon_p = \Omega_p \cap \operatorname{supp}\varphi_\varepsilon$):

$$\begin{aligned} \left| \int_{\Omega_p} \alpha\partial_x u^+_p\, \overline{\partial_x(v_p\varphi_\varepsilon)} \mathbf{dx} \right| &\leq \left| \int_{\Omega^\varepsilon_p} \alpha\partial_x u^+_p\, \partial_x\overline{v_p}\, \varphi_\varepsilon \mathbf{dx} \right| + \left| \int_{\Omega^\varepsilon_p} \alpha\partial_x u^+_p\, \overline{v_p}\, \partial_x\varphi_\varepsilon\, \mathbf{dx} \right| \\ &\leq C\left(\left\| u^+_p \right\|_{H^1_{1/2}(\Omega^\varepsilon_p)} \left\| v_p \right\|_{H^1_{1/2}(\Omega^\varepsilon_p)} + \left\| u^+_p \right\|_{H^1_{1/2}(\Omega^\varepsilon_p)} \left\| v_p \right\|_{L^\infty(\Omega^\varepsilon_p)} \|\varphi_\varepsilon\|_{H^1_{1/2}(\Omega^\varepsilon_p)} \right). \end{aligned}$$

Remark that there exists $C_1 > 0$, s.t. for all $\varepsilon > 0$, $\|\varphi_\varepsilon\|_{H^1_{1/2}(\Omega^\varepsilon_p)} \leq C_1$. Since, additionally, $u^+_p \in H^1_{1/2}(\Omega_p)$, we conclude that, as $\varepsilon \to 0+$,

$$\left| \int_{\Omega_p} \alpha\partial_x u^+_p\, \overline{\partial_x(v_p\varphi_\varepsilon)} \mathbf{dx} \right| \longrightarrow 0.$$

Therefore, $\alpha\partial_{\mathbf{n}} u^+_{reg}\big|_\Sigma = 0$.

Let $v_p \in \mathscr{C}^\infty_{per,y}\left(\overline{\Omega_p}\right)$ such that $\operatorname{supp} v_p \cap \Sigma = \emptyset$. Since $u^\nu = u^\nu_{reg} + s^\nu_{g^+}$ is the solution of the problem with absorption 6.8, we have

$$\begin{aligned} &\int_{\Omega_p} f_\Omega \overline{v_p} \mathbf{dx} + \int_{\Gamma_p} f_\Gamma \overline{v_p} \mathrm{ds} \\ &\quad = \int_{\Omega_p} \left\{ (\alpha + i\nu)\, \nabla u^\nu \cdot \overline{\nabla v_p} - \omega^2 u^\nu \overline{v_p} \right\} \mathbf{dx} + i\lambda \int_{\Gamma_p} u^\nu \overline{v_p} \mathrm{ds} \\ &\quad = \int_{\Omega_p} \left\{ (\alpha + i\nu)\, \nabla u^\nu_{reg} \cdot \overline{\nabla v_p} + (\alpha + i\nu)\, \partial_y s^\nu_{g^+} \cdot \overline{\partial_y v_p} - \partial_x\left((\alpha + i\nu)\, \partial_x s^\nu_{g^+} \right) \overline{v_p} \right\} \mathbf{dx} \\ &\quad\quad - \int_{\Omega_p} \omega^2 \left(u^\nu_{reg} + s^\nu_{g^+} \right) \overline{v_p} \mathbf{dx} + \int_{\Gamma_p} \left[(\alpha + i\nu)\, \partial_{\mathbf{n}} s^\nu_{g^+} - i\lambda s^\nu_{g^+} \right] \overline{v_p} \mathrm{ds} + i\lambda \int_{\Gamma_p} u^\nu_{reg} \overline{v_p} \mathrm{ds}. \end{aligned}$$

Combining the last identity with (6.16) yields

$$\begin{aligned} \int_{\Gamma_p} \alpha\partial_{\mathbf{n}} u^+_{reg} \overline{v_p} \mathrm{ds} = &\int_{\Omega_p} \Big\{ \left[\alpha\nabla u^+_{reg} - (\alpha + i\nu)\, \nabla u^\nu_{reg} \right] \cdot \overline{\nabla v_p} + \left[\alpha\partial_y s_{g^+} - (\alpha + i\nu)\, \partial_y s^\nu_{g^+} \right] \overline{\partial_y v_p} \\ &- \left[\partial_x\left(\alpha\partial_x s_{g^+} \right) - \partial_x\left((\alpha + i\nu)\, \partial_x s^\nu_{g^+} \right) \right] \overline{v_p} - \omega^2 \left(u^+_{reg} + s_{g^+} - u^\nu_{reg} + s^\nu_{g^+} \right) \overline{v_p} \Big\} \mathbf{dx} \\ &+ \int_{\Gamma_p} \left[f_\Gamma - i\lambda u^\nu_{reg} - (\alpha + i\nu)\, \partial_{\mathbf{n}} s^\nu_{g^+} - i\lambda s^\nu_{g^+} \right] \overline{v_p} \mathrm{ds}. \end{aligned}$$

As $\nu \to 0+$, the integral on Ω_p vanishes thanks to lemmas 6.2.2, 6.2.4, and assumption 6.1.1. Obviously,

$$(\alpha + i\nu)\,\partial_{\mathbf{n}} s^{\nu}_{g^+} - i\lambda s^{\nu}_{g^+} \xrightarrow[\nu\to 0+]{L^2(\Gamma_p)} \alpha\partial_{\mathbf{n}} s_{g^+} - i\lambda s_{g^+},$$

and $u^{\nu}_{reg}\big|_{\Gamma_p} \to u^{+}_{reg}\big|_{\Gamma_p}$ by continuity of the trace, see remark 6.2.3. As a result, we have

$$\alpha\partial_{\mathbf{n}}\left(u^+_{reg} + s_{g^+}\right) + i\lambda\left(u^+_{reg} + s_{g^+}\right) = f_\Gamma, \text{ on } \Gamma_p.$$

Hence, for all $v_p \in C^{\infty}_{per,y}\left(\overline{\Omega_p}\right)$, it holds that

$$\int_{\Omega_p} \left\{\alpha\nabla u^+_{reg}\cdot\overline{\nabla v_p} + \alpha\partial_y s_{g^+}\overline{\partial_y v_p} - \partial_x\left(\alpha\partial_x s_{g^+}\right)\overline{v_p} - \omega^2\left(u^+_{reg} + s_{g^+}\right)\overline{v_p}\right\}\mathrm{d}\mathbf{x}$$
$$+ i\lambda\int_{\Gamma_p}(u^+_p + s^+_g)\,\overline{v_p}\mathrm{d}s + \int_{\Gamma_p}\alpha\partial_n s_{g^+}\overline{v_p}\mathrm{d}s = \int_{\Omega_p} f_\Omega\overline{v_p}\mathrm{d}\mathbf{x} + \int_{\Gamma_p} f_\Gamma\overline{v_p}\mathrm{d}s.$$

Repeating the argument for $v_n \in C^{\infty}_{per,y}\left(\overline{\Omega_n}\right)$, we conclude that a similar identity holds true in Ω_n. By density of the functions $C^{\infty}_{per,y}\left(\overline{\Omega_n}\right)\times C^{\infty}_{per,y}\left(\overline{\Omega_p}\right)$ in Q, we arrive at the formulation (6.11). □

Remark 6.2.6. The proof of proposition 6.2.1 illustrates that, for $u \in H^1_{1/2}(\Omega_j)$ such that $\mathrm{div}(\alpha\nabla u) \in L^2(\Omega)$, $\alpha\partial_x u|_\Sigma = 0$ is a natural consequence. On another hand, for $g \in L^2(\Sigma)$, we have $\alpha\partial_x s_g\big| = g\mathtt{r}$.

The "energy" localized near the interface Σ of the singular part is finite. Recall the $L^2(\Sigma)$-weighted norm $\|g\|_{\mathtt{r}} = \left(\int_\Sigma |g|^2\mathtt{r}\mathrm{d}s\right)^{1/2}$ and the associated inner product $(\cdot,\cdot)_{\mathtt{r}}$.

Lemma 6.2.7. *Let $g \in H^1(\Sigma)$ and let φ be a truncation function as in definition 4.2.2. Then the following limit holds:*

$$\lim_{\nu\to 0+}\int_\Omega \nu|\nabla s^{\nu}_g|^2\varphi\,\mathrm{d}\mathbf{x} = \pi\|g\|^2_{\mathtt{r}}\,. \tag{6.17}$$

Proof. By direct computation, we have

$$\partial_x s^{\nu}_g(x,y) = \frac{g(y)\,\mathtt{r}(y)}{\mathtt{r}(y)x + i\nu},\quad \partial_y s^{\nu}_g(x,y) = \partial_y g(y)\log\left(x + \frac{i\nu}{\mathtt{r}(y)}\right) - \frac{i\nu g(y)\mathtt{r}'(y)}{\mathtt{r}(y)\,(\mathtt{r}(y)x + i\nu)}.$$

Then, one can check that

$$\int_\Omega \nu|\partial_x s^{\nu}_g|^2\varphi\,\mathrm{d}\mathbf{x} = \int_\Sigma |g|^2\mathtt{r}\left(\int_{-a}^{a}\frac{\varphi_1(x)}{(\mathtt{r}x/\nu)^2 + 1}\frac{\mathtt{r}\mathrm{d}x}{\nu}\right)\mathrm{d}y \xrightarrow[\nu\to 0+]{} \pi\|g\|^2_{\mathtt{r}}\,,$$

whereas, using Young's inequality and that $\left|\frac{\nu}{\mathtt{r}(y)x + i\nu}\right|^2 \leq 1$,

$$\int_\Omega \nu|\partial_y s^{\nu}_g|^2\varphi\,\mathrm{d}\mathbf{x} \leq 2\nu\int_\Omega\left(\left|\partial_y g(y)\log\left(x + \frac{i\nu}{\mathtt{r}(y)}\right)\right|^2 + \left|\frac{g(y)\mathtt{r}'(y)}{\mathtt{r}(y)}\right|^2\right)\varphi(x,y)\,\mathrm{d}\mathbf{x} \xrightarrow[\nu\to 0+]{} 0.$$

□

One can also characterize the "energy" localized near the interface Σ of the regular part.

Proposition 6.2.8. *Let $(u^{\nu})_{\nu} > 0$ be a family governed by (6.1) fulfilling assumption 6.1.1, and $\varphi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega};\mathbb{R}\right)$ be such that $\mathrm{supp}\,\varphi \cap \overline{(\Gamma_p \cup \Gamma_n)} = \emptyset$. Then the following limit holds:*

$$\lim_{\nu\to 0+}\int_\Omega \nu\left|\nabla u^{\nu}_{reg}\right|^2\varphi\,\mathrm{d}\mathbf{x} = 0.$$

Proof. Firstly, remark that $\int_\Omega \nu \left|\nabla u_{reg}^\nu\right|^2 \varphi \, \mathbf{dx} = \operatorname{Im} \mathscr{E}_{reg}^\nu$ with

$$\mathscr{E}_{reg}^\nu = \int_\Omega \left\{(\alpha + i\nu) \left|\nabla u_{reg}^\nu\right|^2 \varphi - \omega^2 |u_{reg}^\nu|^2 \varphi\right\} \mathbf{dx}.$$

Therefore, using that $\nabla u_{reg}^\nu \varphi = \nabla(u_{reg}^\nu \varphi) - u_{reg}^\nu \nabla\varphi$ and that $u^\nu = u_{reg}^\nu + s_g^\nu$ is a solution of the problem with absorption (6.8), one has

$$\begin{aligned}
\mathscr{E}_{reg}^\nu &= \int_\Omega \left\{(\alpha + i\nu)\nabla u_{reg}^\nu \cdot \overline{\nabla(u_{reg}^\nu \varphi)} - \omega^2 u_{reg}^\nu \overline{u_{reg}^\nu \varphi}\right\} \mathbf{dx} - \int_\Omega (\alpha + i\nu)\nabla u_{reg}^\nu \cdot \overline{u_{reg}^\nu \nabla\varphi} \mathbf{dx} \\
&= \int_\Omega f_\Omega u_{reg}^\nu \varphi \mathbf{dx} - \int_\Omega \left\{(\alpha + i\nu)\nabla s_g^\nu \cdot \overline{\nabla(u_{reg}^\nu \varphi)} - \omega^2 s_g^\nu \overline{u_{reg}^\nu \varphi}\right\} \mathbf{dx} - \int_\Omega (\alpha + i\nu)\nabla u_{reg}^\nu \cdot \overline{u_{reg}^\nu \nabla\varphi} \mathbf{dx} \\
&= \int_\Omega f_\Omega u_{reg}^\nu \varphi \mathbf{dx} - \int_\Omega \left\{(\alpha + i\nu)\partial_y s_g^\nu \overline{\partial_y(u_{reg}^\nu \varphi)} - \partial_x \left((\alpha + i\nu)\partial_x s_g^\nu\right) \overline{u_{reg}^\nu \varphi} - \omega^2 s_g^\nu \overline{u_{reg}^\nu \varphi}\right\} \mathbf{dx} \\
&\qquad - \int_\Omega (\alpha + i\nu)\nabla u_{reg}^\nu \cdot \overline{u_{reg}^\nu \nabla\varphi} \mathbf{dx},
\end{aligned}$$

where an integration by parts in the x-direction is made in the last equality. According to lemma 6.2.4 and lemma 6.2.5, and the definitions (6.4), (6.5), (6.7) of the forms $b_{reg}^{(1)}$, $b_{sing}^{(1)}$ and $\ell^{(1)}$, $\mathscr{E}_{reg}^\nu$ converges as $\nu \to 0+$ to

$$\begin{aligned}
\mathscr{E}_{reg}^+ &= \ell^{(1)}(\mathbf{u}^+ \varphi) - b_{sing}^{(1)}(g^+, \mathbf{u}^+ \varphi) - \sum_{j \in \{p,n\}} \int_{\Omega_j} \alpha \nabla u_j^+ \cdot \overline{u_j^+ \nabla\varphi} \mathbf{dx} \\
&= b_{reg}^{(1)}(\mathbf{u}^+, \mathbf{u}^+ \varphi) - \sum_{j \in \{p,n\}} \int_{\Omega_j} \alpha \nabla u_j^+ \cdot \overline{u_j^+ \nabla\varphi} \mathbf{dx} \\
&= \int_\Omega \left\{\alpha \left|\nabla u_j^+\right|^2 \varphi - \omega^2 |u_j^+|^2 \varphi\right\} \mathbf{dx},
\end{aligned}$$

where we used the proposition 6.2.1, the identity $\nabla(u_j^+ \varphi) - u_j^+ \nabla\varphi = \nabla u_j^+ \varphi$ and the condition on the disjoint supports. Finally, considering $\operatorname{Im} \mathscr{E}_{reg}^+$ gives the desired result. □

6.2.2 From the energy functional to the mixed formulation

The aim of this section is to find a well-posed problem that is satisfied by the limiting absorption solution u^+ defined in Assumption 6.1.1. We start by rewriting proposition 6.2.8 for a given $\varphi \in \mathscr{C}_{per,y}^1(\overline{\Omega}; \mathbb{R})$, following Assumption 4.2.2, i.e., $\partial_y \varphi = 0$ and $\varphi = 1$ in the vicinity of the interface Σ, as

$$\lim_{\nu \to 0+} \int_\Omega \nu \left|\nabla \left(u^\nu - s_{g^+}^\nu\right)\right|^2 \varphi \, \mathbf{dx} = 0.$$

Then, the idea developed below involves introducing an unknown $h \in H_{per}^1(\Sigma)$ with the aim of constructing an "energy" functional. The minimum of this functional should be achieved by u^+, characterized by $\left(u_{reg}^+, g^+\right)$, where $h = g^+$. Next, the functional will be differentiated, which will result in a mixed problem.

Let $(u^\nu)_{\nu>0}$ be such that $u^\nu \in H_{per,y}^1(\Omega)$ and $-\operatorname{div}((\alpha + i\nu)\nabla u^\nu) - \omega^2 u^\nu = f_\Omega$ in $\operatorname{supp}\varphi$, and $h \in H_{per}^1(\Sigma)$. Moreover, we suppose $u^\nu \to \mathbf{u} + s_g$ in $L^2(\Omega)$ as $\nu \to 0+$, like in assumption 6.1.1. Notice that we do not impose u^ν to be necessarily the solution of the problem with absorption (6.1). Though it may seem strange, but $-\operatorname{div}((\alpha + i\nu)\nabla u^\nu) - \omega^2 u^\nu = f_\Omega$ must be seen more as a constraint under which the argument below is developed. Of course, $(u^\nu)_{\nu>0}$ as in assumption 6.1.1 obey the constraint. We define the following the "energy" functional as in Proposition 6.2.8:

$$\mathcal{J}^\nu(u^\nu,h) = \int_\Omega \nu\left|\nabla\left(u^\nu - s_h^\nu\right)\right|^2 \varphi\,\mathbf{dx} = \operatorname{Im}\mathcal{E}^\nu,\ \text{ where}$$
$$\mathcal{E}^\nu = \int_\Omega \left[(\alpha(x,y)+i\nu)\left|\nabla\left(u^\nu - s_h^\nu\right)\right|^2 - \omega^2\left|u^\nu - s_h^\nu\right|^2\right]\varphi\,\mathbf{dx}.$$

Lemma 6.2.9. *Let $(u^\nu)_{\nu>0}$ be as in assumption 6.1.1 and $h\in H^1_{per}(\Sigma)$. Then,*

$$0 \le \lim_{\nu\to 0+} \mathcal{J}^\nu(u^\nu,h) \le \pi\left\|g^+ - h\right\|_\Gamma^2.$$

Proof. The positiveness comes by definition. Beginning with the following triangular inequality

$$\left(\int_\Omega \nu\left|\nabla\left(u^\nu - s_h^\nu\right)\right|^2\varphi\,\mathbf{dx}\right)^{1/2} = \left(\int_\Omega \nu\left|\nabla\left(u^\nu_{reg} + s^\nu_{g^+-h}\right)\right|^2\varphi\,\mathbf{dx}\right)^{1/2}$$
$$\le \left(\int_\Omega \nu\left|\nabla u^\nu_{reg}\right|^2\varphi\,\mathbf{dx}\right)^{1/2} + \left(\int_\Omega \nu\left|\nabla s^\nu_{g^+-h}\right|^2\varphi\,\mathbf{dx}\right)^{1/2},$$

Lemma 6.2.7 and Proposition 6.2.8 enable us to conclude regarding the upper bound. □

In order to compute the limit of $\mathcal{J}^\nu(u^\nu,h)$, we will integrate by parts the expression for $\mathcal{E}^\nu$ as in the proof of proposition 6.2.8. First, using the fact that $\partial_y\varphi = 0$, and the identity $(\nabla U)\varphi = \nabla(U\varphi) - U\nabla\varphi$,one can rewrite $\mathcal{E}^\nu$ as

$$\mathcal{E}^\nu = \int_\Omega \left[(\alpha(x,y)+i\nu)\nabla\left(u^\nu - s_h^\nu\right)\cdot\overline{\nabla\left(\left(u^\nu - s_h^\nu\right)\varphi\right)} - \omega^2\left(u^\nu - s_h^\nu\right)\overline{\left(\left(u^\nu - s_h^\nu\right)\varphi\right)}\right]\mathbf{dx}$$
$$- \int_\Omega \left[(\alpha(x,y)+i\nu)\partial_x\left(u^\nu - s_h^\nu\right)\overline{\left(u^\nu - s_h^\nu\right)}\partial_x\varphi\right]\mathbf{dx}.$$

Then we separate

$$\mathcal{E}^\nu = b^\nu(u^\nu,(u^\nu - s_h^\nu)\varphi) - b^\nu(s_h^\nu,(u^\nu - s_h^\nu)\varphi) - c^\nu(u^\nu - s_h^\nu, u^\nu - s_h^\nu),$$

where

$$c^\nu(u,v) = \int_\Omega (\alpha(x,y)+i\nu)\partial_x u\bar{v}\,\partial_x\varphi\mathbf{dx}.$$

Since $-\operatorname{div}((\alpha+i\nu)\nabla u^\nu) - \omega^2 u^\nu = f_\Omega$ on $\operatorname{supp}\varphi$, $b^\nu(u^\nu,(u^\nu - s_h^\nu)\varphi) = \ell^{(1)}\left((u^\nu - s_h^\nu)\varphi\right)$. Then, in the view of the definition (6.7) of $\ell^{(1)}$, Assumption 6.1.1 and Lemma 6.2.4, we have

$$b^\nu(u^\nu,(u^\nu - s_h^\nu)\varphi) \xrightarrow[\nu\to 0+]{} \ell^{(1)}\left(\left(\mathbf{u} + s_{g-h}\right)\varphi\right). \tag{6.18}$$

It remains to integrate by parts the term $b^\nu(s_h^\nu,(u^\nu - s_h^\nu)\varphi)$, which is allowed since $h\in H^1_{per}(\Sigma)$:

$$b^\nu(s_h^\nu,\varphi(u^\nu - s_h^\nu))$$
$$= \int_\Omega \left\{(\alpha(x,y)+i\nu)\partial_y s_h^\nu \partial_y\overline{\left(\left(u^\nu - s_h^\nu\right)\varphi\right)} - \partial_x\left((\alpha(x,y)+i\nu)\partial_x s_h^\nu\right)\overline{\left(u^\nu - s_h^\nu\right)}\varphi - \omega^2 s_h^\nu\overline{\left(u^\nu - s_h^\nu\right)}\varphi\right\}\mathbf{dx}.$$

Therefore, since $u^\nu \to \mathbf{u} + s_g$ in $L^2(\Omega)$ as $\nu\to 0+$, and using lemma 6.2.4, we have that

$$b^\nu(s_h^\nu,(u^\nu - s_h^\nu)\varphi) \xrightarrow[\nu\to 0+]{} b^{(1)}_{sing}\left(h,\left(\mathbf{u} + s_{g-h}\right)\varphi\right),$$
$$c^\nu(u^\nu - s_h^\nu, u^\nu - s_h^\nu) \xrightarrow[\nu\to 0+]{} c\left(\mathbf{u} + s_{g-h}, \mathbf{u} + s_{g-h}\right),$$

where the last sesquilinear forms is given by

$$c(u,v) = \sum_{j\in\{p,n\}} \int_{\Omega_j} \alpha(x,y)\partial_x u\bar{v}\partial_x\varphi \mathrm{d}\mathbf{x}.$$

We observe that the limits depend on the triple $(\mathbf{u}, g, h)$. Then we define

$$\mathcal{J}^+(\mathbf{u},g,h) := -\operatorname{Im}\left[b^{(1)}_{sing}\left(h,\left(\mathbf{u}+s_{g-h}\right)\varphi\right) + c\left(\mathbf{u}+s_{g-h},\mathbf{u}+s_{g-h}\right) - \ell^{(1)}\left(\left(\mathbf{u}+s_{g-h}\right)\varphi\right)\right], \quad (6.19)$$

From this point on, one finds by integrating by parts that $\mathcal{J}^+\left(\mathbf{u}^+,g^+,g^+\right) = 0$ (this is reminiscent of the proof of lemma 6.2.8). Since we know that the limit of $\mathcal{J}^\nu(u^\nu,h)$ is non-negative, we conclude that $\left(\mathbf{u}^+,g^+,g^+\right)$ is a minimizer of $\mathcal{J}^+$. It should be noted that the existence of other minimizers is uncertain.

The next step of the construction is computing the differential of $\mathcal{J}^+$. Rather than directly computing the differential with the functional (6.19), let's consider the following simplified example:

$$\mathcal{J}(u,g) = \operatorname{Im}\left(b(u,g) - \ell(u)\right).$$

This is a dummy functional with dummy variables u, g, a dummy sesquilinear form b, and a dummy antilinear form. Let v, k be the dual variables associated to u, g, and $t > 0$. By definition,

$$\mathrm{d}\mathcal{J}((u,g),(v,k)) = \lim_{t\to 0} \mathcal{J}((u,g)+t(v,k)) - \mathcal{J}(u,g).$$

Then, one can easily expand

$$\mathcal{J}((u,g)+t(v,k)) = \mathcal{J}(u,g) + t\operatorname{Im}\left[b(u,k) - \overline{b(v,g)} - \ell(v)\right] + t^2 b(v,k),$$

so that $\mathrm{d}\mathcal{J}((u,g),(v,k)) = \operatorname{Im}\left[b(u,k) - \overline{b(v,g)} - \ell(v)\right]$. Applying the computation above to (6.19) gives

$$\mathrm{d}\mathcal{J}^+((\mathbf{u},g,h),(\mathbf{v},k,l)) = -\operatorname{Im}\left[a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l)) - \ell^{(1)}((\mathbf{v}+s_{k-l})\varphi)\right],$$

where $a^{(1)}$ is a sesquilinear form defined on $V^{(1)} \times V^{(1)}$, with $V^{(1)} := Q \times H^1_{per}(\Sigma) \times H^1_{per}(\Sigma)$ by

$$\begin{aligned}
&a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l)) \\
&\quad = b^{(1)}_{sing}(h,(\mathbf{v}+s_{k-l})\varphi) - \overline{b^{(1)}_{sing}\left(l,\left(\mathbf{u}+s_{g-h}\right)\varphi\right)} + C_\varphi\left(\mathbf{u}+s_{g-h},\mathbf{v}+s_{k-l}\right) \\
&\quad = \sum_{j\in\{p,n\}} \int_{\Omega_j} \left(\alpha\partial_y s_h \overline{\partial_y((v_j+s_{k-l})\varphi)} - \partial_x(\alpha\partial_x s_h)\overline{(v_j+s_{k-l})}\varphi - \omega^2 s_h\overline{(v_j+s_{k-l})\varphi}\right)\mathrm{d}\mathbf{x} \\
&\qquad - \int_{\Omega_j} \left(\alpha\partial_y\left(\left(u_j+s_{g-h}\right)\varphi\right)\overline{\partial_y s_l} - \left(u_j+s_{g-h}\right)\overline{\partial_x(\alpha\partial_x s_l)}\varphi - \omega^2\left(u_j+s_{g-h}\right)\overline{s_l}\varphi\right)\mathrm{d}\mathbf{x} \quad (6.20) \\
&\qquad + \int_{\Omega_j} \alpha\left[\partial_x\left(u_j+s_{g-h}\right)\overline{(v_j+s_{k-l})} - \left(u_j+s_{g-h}\right)\overline{\partial_x(v_j+s_{k-l})}\right]\partial_x\varphi\mathrm{d}\mathbf{x},
\end{aligned}$$

where

$$C_\varphi(U,V) := \int_\Omega \alpha\left[\partial_x U\overline{V} - U\overline{\partial_x V}\right]\partial_x\varphi\mathrm{d}\mathbf{x}.$$

Remark 6.2.10. Notice that $\mathcal{J}^+(\mathbf{u},g,h) = \operatorname{Im}\left[\ell^{(1)}\left(\left(\mathbf{u}+s_{g-h}\right)\varphi\right)\right] - \frac{1}{2i}a^{(1)}((\mathbf{u},g,h),(\mathbf{u},g,h))$.

Finally, since we want $(\mathbf{u}, g)$ to be a weak solution of (6.3), we define

$$b^{(1)}((\mathbf{u}, g, h), \mathbf{v}) := b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}), \tag{6.21}$$

and we introduce the mixed formulation, as in [49]:

$$\left|\begin{aligned} &\text{Find } (\mathbf{u}, g, h) \in V^{(1)}, \boldsymbol{\lambda} \in Q \text{ such that} \\ &a^{(1)}((\mathbf{u}, g, h), (\mathbf{v}, k, l)) + \overline{b^{(1)}((\mathbf{v}, k, l), \boldsymbol{\lambda})} = \ell^{(1)}((\mathbf{v} + s_{k-l})\varphi), \quad \forall (\mathbf{v}, k, l) \in V, \\ &\qquad\qquad b^{(1)}((\mathbf{u}, g, h), \boldsymbol{\mu}) = \ell^{(1)}(\boldsymbol{\mu}), \qquad\qquad \forall \mathbf{v} \in Q. \end{aligned}\right. \tag{6.22}$$

Notice the introduction of a Lagrange multiplier $\boldsymbol{\lambda} \in Q$. At some point, the operator $\mathrm{B}^{(1)} : V^{(1)} \mapsto Q'$ associated to the form $b^{(1)}$ will be used. The mixed problem will be studied in details, in particular its existence and uniqueness, see section 6.4. The tools used for this study are developed in the next section 6.3.

Remark 6.2.11. Notice that the right-hand side of the first equation vanishes if $f_\Omega = 0$, as in [49]. An important point in the development done in this part is the existence of the limit $\ell^{(1)}\left(s^{\nu}_{g}\varphi\right)$ in (6.18). On the other hand, this approach cannot be carried out if we only had $\ell^{(1)} \in Q'$. Indeed, in that case $\ell^{(1)}(s_k)$ would not be defined since $s_g \notin Q$.

Remark 6.2.12. A similar development above can be conducted for all $\varphi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega}; \mathbb{R}\right)$ such that $\operatorname{supp} \varphi \cap \left(\Gamma_p \cup \Gamma_n\right) = \emptyset$. On the other hand, a particular and convenient choice of φ satisfying Assumption 4.2.2 is

$$\varphi(x, y) = \begin{cases} \dfrac{1 + \cos(2\pi x)}{2}, & |x| \leq \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

6.3 Jump and technical results

6.3.1 Weak jump

It is possible to define a notion of a jump in a weak sense for functions $\mathbf{u} \in Q$ which satisfy the constraint in the mixed formulation (6.22). We introduce the following space of regular functions on the interface

$$H^1_{per}(\Sigma, \mathrm{r}) := \left\{g \in L^2(\Sigma) \,:\, \partial_y g \in L^2(\Sigma), g(0) = g(L)\right\},$$

paired with the norm

$$\|g\|_{H^1(\Sigma,\mathrm{r})} = \left(\int_\Sigma \left(|\partial_y g(y)|^2 + |g(y)|^2\right) \mathrm{r}(y) \mathrm{d}y\right)^{1/2}.$$

For the sake of conciseness, $\langle \cdot, \cdot \rangle_\Sigma$ denotes $\langle \cdot, \cdot \rangle_{\left(H^1_{per}(\Sigma,\mathrm{r})\right)', H^1_{per}(\Sigma,\mathrm{r})}$ until the end of this chapter. Notice that, given $g \in L^2(\Sigma)$ and $k \in H^1(\Sigma, \mathrm{r})$, we also set $\langle g, k \rangle_\Sigma = (g, k)_\mathrm{r}$.

Let us define the following class of sequences of cutoff functions.

Definition 6.3.1. Given a cutoff function φ following Definition 4.2.2, let $(\varphi_m)_{m \in \mathbb{N}}$ be defined by $\varphi_m(x, y) = \varphi(mx, y)$.

Then, given $\ell \in Q'$, $\psi \in \mathscr{C}^1\left(\overline{\Omega}\right)$ and $k \in H^1_{per}(\Sigma)$, we denote

$$\ell_\infty(\psi s_k) := \lim_{m\to+\infty} \ell(\psi s_k(1-\varphi_m)) \tag{6.23}$$

if this limit exists and is independent of the choice of the sequence of cutoff functions $(\varphi_m)_{m\in\mathbb{N}}$ satisfying the above definition. Obviously, ℓ_∞ will not necessarily exist for all $\ell \in Q'$. But notice that $\ell^{(1)}_\infty$, associated to $\ell^{(1)}$ exists, see Remark 6.2.11.

Therefore, assuming that ℓ_∞ exists, the definition of the jump is formalized in the statement of the following lemma.

Lemma 6.3.2. *Let $\mathbf{u} \in Q$, $g \in H^1_{per}(\Sigma)$ and $\ell \in Q'$ related by*

$$b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}) = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in Q. \tag{6.24}$$

Let $(\varphi_m)_{m\in\mathbb{N}} \subset \mathscr{C}^2_{per,y}(\Omega)$ be as in definition 6.3.1. If $\ell_\infty(s_k) = \lim_{m\to+\infty} \ell\left(s_g(1-\varphi_m)\right)$ exists and does not depend on the choice of $(\varphi_m)_m$, then jump $[\mathbf{u}]_\Sigma$ of the regular part is defined as

$$\forall k \in H^1_{per}(\Sigma), \quad \langle [\mathbf{u}]_\Sigma, k\rangle_\Sigma := \lim_{m\to+\infty} \sum_{j\in\{p,n\}} \int_{\Omega_j} u_j \overline{(-\partial_x(\alpha\partial_x(s_k\varphi_m)))} d\mathbf{x}, \tag{6.25}$$

and it is finite for all k. The limit is independent of the choice of $(\varphi_m)_m$, and it holds for all $k \in H^1_{per}(\Sigma)$

$$\langle [\mathbf{u}], k\rangle_{\left(H^1_{per}(\Sigma,\mathrm{r})\right)', H^1_{per}(\Sigma,\mathrm{r})} = \overline{b^{(1)}_{sing}(k, \mathbf{u})} + b^{(1)}_{sing}(g, s_k) + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j \overline{s_k} \mathrm{d}s - \ell_\infty(s_k). \tag{6.26}$$

Before proving the lemma, let us make a few comments. For piecewise regular $\mathbf{u} \in H^1(\Omega_p) \times H^1(\Omega_n)$, the above definition of the jump coincides with the classical definition $[\mathbf{u}]_\Sigma = \gamma_0(u_p) - \gamma_0(u_n)$, where γ_0 denotes the trace on Σ, seen either as a part of Ω_p, or of Ω_n. Indeed, in this case (6.25) yields

$$\begin{aligned}\sum_{j\in\{p,n\}} \int_{\Omega_j} & u_j \overline{(-\partial_x(\alpha\partial_x(s_k\varphi_m)))} d\mathbf{x} \\ &= \int_\Sigma \left[\gamma_0(u_p) - \gamma_0(u_n)\right] \overline{(\alpha\partial_x(s_k\varphi_m))|_\Sigma} dy + \sum_{j\in\{p,n\}} \int_{\Omega_j} \alpha\partial_x u_j \overline{\partial_x(s_k\varphi_m)} d\mathbf{x}.\end{aligned}$$

The first term in the right-hand side of the above can be made more explicit. Indeed, by the regularity assumption on α and using an explicit form of s_k, we have $\alpha\partial_x(s_k\varphi_m) \in H^1(\Omega)$ and $\alpha\partial_x(s_k\varphi_m)|_\Sigma = k(y)\mathrm{r}(y)$. Observing that $\|\alpha\partial_x(s_k\varphi_n)\|_{L^2(\Omega_j)} \lesssim \|k\|_{L^2(\Sigma)}$, the second term is bounded with the help of Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\int_{\Omega_j} \alpha\partial_x u_j \overline{\partial_x(s_k\varphi_m)} d\mathbf{x}\right| &\leq \|u_j\|_{H^1(\Omega_j\cap \mathrm{supp}\,\varphi_m)} \|\alpha\partial_x(s_k\varphi_m)\|_{L^2(\Omega_j)} \\ &\lesssim \|u_j\|_{H^1(\Omega_j\cap \mathrm{supp}\,\varphi_m)} \|k\|_{L^2(\Sigma)} \xrightarrow[m\to+\infty]{} 0,\end{aligned}$$

so that

$$\sum_{j\in\{p,n\}} \int_{\Omega_j} u_j \overline{(-\partial_x(\alpha\partial_x(s_k\varphi_m)))} d\mathbf{x} = \int_\Sigma [\mathbf{u}]\overline{k}\,\mathrm{r}\,dy + o_{m\to+\infty}(1).$$

On the other hand, the jump defined by (6.25) is not finite for all $\mathbf{u} \in Q$; take e.g., $\mathbf{u} = (\log|\log|x||, 0)$. Finally, we note that, even for $\mathbf{u}$ satisfying (6.24), the jump $[\mathbf{u}]_\Sigma$ is defined in a very weak sense, since it is taken in the dual space of $H^1_{per}(\Sigma, r)$.

Proof of lemma 6.3.2. We test (6.24) with $\mathbf{v} = s_k(1-\varphi_m) \in Q$. On one hand, we have

$$b^{(1)}_{reg}(\mathbf{u}, s_k(1-\varphi_m)) \xrightarrow[m\to+\infty]{} \ell_\infty(s_k) - b^{(1)}_{sing}(g, s_k), \tag{6.27}$$

where ℓ_∞ is defined in (6.23). On the other hand, integrating by parts in the x-direction the term $b^{(1)}_{reg}(\mathbf{u}, s_k(1-\varphi_m))$ gives:

$$\begin{aligned} & b^{(1)}_{reg}(\mathbf{u}, s_k(1-\varphi_m)) \\ &= \sum_{j\in\{p,n\}} \int_{\Omega_j} \left\{u_j\overline{(-\partial_x(\alpha\partial_x(s_k(1-\varphi_m))))} + \alpha\partial_y u_j \overline{\partial_y(s_k(1-\varphi_m))} - \omega^2 u_j\overline{s_k}(1-\varphi_m)\right\} \mathrm{d}\mathbf{x} \\ &\quad + \sum_{j\in\{p,n\}} \int_{\Gamma_j} \left(u_j\overline{(\alpha\partial_n s_k)} + i\lambda u_j\overline{s_k}\right) \mathrm{d}s \\ &= \overline{b^{(1)}_{sing}(k,\mathbf{u})} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\overline{s_k}\mathrm{d}s - \sum_{j\in\{p,n\}} \int_{\Omega_j} \left\{\alpha\partial_y u_j\overline{\partial_y s_k} - \omega^2 u_j\overline{s_k}\right\}\varphi_m \mathrm{d}\mathbf{x} - J_m, \end{aligned}$$

where we used $\partial_y\varphi_m = 0$ and with

$$J_m = \sum_{j\in\{p,n\}} \int_{\Omega_j} u_j\overline{(-\partial_x\alpha\partial_x(s_k\varphi_m))}\mathrm{d}\mathbf{x},$$

cf. the definition of the jump (6.25). Next, using Lebesgue's dominated convergence theorem yields

$$-\sum_{j\in\{p,n\}} \int_{\Omega_j} \left\{\alpha\partial_y u_j\overline{\partial_y s_k} - \omega^2 u_j\overline{s_k}\right\}\varphi_m \mathrm{d}\mathbf{x} \xrightarrow[n\to+\infty]{} 0,$$

so that

$$\lim_{m\to+\infty} b^{(1)}_{reg}(\mathbf{u}, s_k(1-\varphi_m)) = \overline{b^{(1)}_{sing}(k,\mathbf{u})} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\overline{s_k}\mathrm{d}s - \lim_{n\to+\infty} J_m.$$

Replacing the left-hand side of the above by (6.27) shows that $\lim_{m\to+\infty} J_m$ is finite and, with the definition (6.25), the jump of $\mathbf{u}$ is expressed as

$$\langle[\mathbf{u}], k\rangle_{\left(H^1_{per}(\Sigma,\mathrm{r})\right)', H^1_{per}(\Sigma,\mathrm{r})} = \overline{b^{(1)}_{sing}(k,\mathbf{u})} + b^{(1)}_{sing}(g, s_k) + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\overline{s_k}\mathrm{d}s - \ell_\infty(s_k).$$

As claimed, the last expression does not depend on the chosen function sequence $(\varphi_m)_m$. ☐

Remark 6.3.3. The definition of the jump (6.25) does not depend on the jump part of the singularity $i\pi k(y)\mathbb{1}_{x<0}$. Indeed, in (6.25), s_k can be replaced by $k(y)\log|x|$ or $k(y)(\log|x| - i\pi\mathbb{1}_{x<0})$. This holds because, given $(\varphi_m)_m$ as in lemma 6.3.2, we have, by the Cauchy-Schwarz inequality and after integration by parts in the x-direction,

$$\begin{aligned} \left|\int_\Omega u\partial_x(\alpha\partial_x(\mathbb{1}_{x<0}\varphi_m k(y)))\mathrm{d}\mathbf{x}\right| &= \left|\int_{\Omega_n\cap\operatorname{supp}\varphi_m} u_n\partial_x(\alpha\partial_x(\varphi_m k(y)))\mathrm{d}\mathbf{x}\right| \\ &= \left|\int_{\Omega_n\cap\operatorname{supp}\varphi_m} \alpha\partial_x u_n\partial_x\varphi_m k(y)\mathrm{d}\mathbf{x}\right| \lesssim \|u_n\|_{H^1_{1/2}(\Omega_n\cap\operatorname{supp}\varphi_m)}\|k\|_{L^2(\Sigma)} \xrightarrow[m\to+\infty]{} 0. \end{aligned}$$

Therefore, the jump can also be computed as

$$\langle[\mathbf{u}], k\rangle_\Sigma = \lim_{n\to+\infty} \sum_{j\in\{p,n\}} \int_{\Omega_j} u_j\overline{(-\partial_x(\alpha\partial_x(k(y)(\log|x| - i\pi\mathbb{1}_{x<0})\varphi_m)))}d\mathbf{x}. \tag{6.28}$$

This identity will be useful later, see section 6.3.2.

6.3.2 Jump of the limiting absorption solution

Given $g \in H^1_{per}(\Sigma)$, let us introduce some "artificial singularities" with non-zero absorption λ

$$s_g^\lambda(x,y) = g(y)\log\left(x + \frac{i\lambda}{\mathrm{r}(y)}\right).$$

For positive λ, one recovers the "singularities with absorption" of (6.13). We remark that one has convergence almost everywhere as $\nu \to 0+$:

$$s_g^\nu \xrightarrow[\nu\to 0+]{a.e.} s_g^+ := g\left(\log|x| + i\pi\mathbb{1}_{x<0}\right), \quad \text{and} \quad s_g^{-\nu} \xrightarrow[\nu\to 0+]{a.e.} s_g^- := g\left(\log|x| - i\pi\mathbb{1}_{x<0}\right).$$

We then have the following lemma (which generalizes lemma 6.2.4 to the case of artificial singularities), whose proof is left to the reader.

Lemma 6.3.4. *Given $g \in H^1_{per}(\Sigma)$, the following limits hold in $L^2(\Omega)$ as $\nu \to 0+$:*

$$\begin{aligned} &s_g^{\pm\nu} \to s_g^\pm, && \partial_y s_g^{\pm\nu} \to \partial_y s_g^\pm, \\ &(\alpha \pm i\nu)\partial_x s_g^{\pm\nu} \to \alpha\partial_x s_g^\pm, && \partial_x((\alpha \pm i\nu)\partial_x s_g^{\pm\nu}) \to \partial_x(\alpha\partial_x s_g^\pm). \end{aligned}$$

Note that $s_g = s_g^+$. We adopt this convention from now on. From the above lemma, it follows in particular that for all $\psi \in C^\infty\left(\overline{\Omega};\mathbb{R}\right)$,

$$\partial_x((\alpha \pm i\nu)\partial_x(s_g^{\pm\nu}\psi)) \xrightarrow[\nu\to 0+]{} \partial_x(\alpha\partial_x(s_g^\pm\psi)) \text{ in } L^2(\Omega). \tag{6.29}$$

Next, we show that the limiting absorption solution $(\mathbf{u}^+, g^+)$ from assumption 6.1.1 has a vanishing jump.

Proposition 6.3.5. *Let $\left(\mathbf{u}^+, g^+\right) \in Q \times H^1_{per}(\Sigma)$ be as in assumption 6.1.1. Then $\left[\mathbf{u}^+\right]_\Sigma = 0$.*

Proof. Let $\mathbf{u}^+$, g^+ be like in assumption 6.1.1. To prove that $[\mathbf{u}^+]_\Sigma = 0$, we will use the identity (6.28) defining the jump with s_k^-, for a given $k \in H^1_{per}(\Sigma)$. More precisely, let φ be a truncation function as in the definition 4.2.2 and, for $\varepsilon > 0$, $\varphi_\varepsilon(x,y) = \varphi\left(\frac{x}{\varepsilon}, y\right)$. We will show that the quantity below is well-defined and converges to 0 as $\varepsilon \to 0+$:

$$J_\varepsilon(k) = \sum_{j\in\{p,n\}} \int_{\Omega_j} u_j^+ \overline{\left(-\partial_x\left(\alpha\partial_x\left(s_k^-\varphi_\varepsilon\right)\right)\right)}\mathrm{d}\mathbf{x}.$$

We reexpress $J_\varepsilon(k)$ with the help of (6.29) and the convergence $(u^\nu_{reg})_\nu$ to $\mathbf{u}^+$ of Lemma 6.2.5:

$$J_\varepsilon(k) = \lim_{\nu\to 0+} J_\varepsilon^\nu(k), \quad \text{with} \quad J_\varepsilon^\nu(k) = \sum_{j\in\{p,n\}} \int_{\Omega_j} u^\nu_{reg}\overline{\left(-\partial_x\left((\alpha - i\nu)\,\partial_x\left(s_k^{-\nu}\varphi_\varepsilon\right)\right)\right)}\mathrm{d}\mathbf{x} \quad \text{for } \varepsilon > 0.$$

The main idea of the proof consists in reexpressing J_ε^ν via $b^\nu(u^\nu, s_k^{-\nu}\varphi_\varepsilon)$. Since u^ν verifies (6.1) and is decomposed as $u^\nu = u^\nu_{reg} + s^\nu_{g^+}$, defined in lemma 6.2.5,

$$b^\nu(u^\nu_{reg}, s_k^{-\nu}\varphi_\varepsilon) + b^\nu(s^\nu_{g^+}, s_k^{-\nu}\varphi_\varepsilon) = \ell^{(1)}\left(s_k^{-\nu}\varphi_\varepsilon\right). \tag{6.30}$$

One has by, integrating by parts in the x-direction,

$$\begin{aligned} b^\nu(u^\nu_{reg}, s_k^{-\nu}\varphi_\varepsilon) &= \int_\Omega \partial_x u^\nu_{reg}\overline{(\alpha - i\nu)\,\partial_x\left(s_k^{-\nu}\varphi_\varepsilon\right)}\mathrm{d}\mathbf{x} + \int_\Omega \left[(\alpha + i\nu)\partial_y u^\nu_{reg}\overline{\partial_y(s_k^{-\nu}\varphi_\varepsilon)} - \omega^2 u^\nu_{reg}\overline{s_k^{-\nu}\varphi_\varepsilon}\right]\mathrm{d}\mathbf{x} \\ &= J_\varepsilon^\nu(k) + \int_\Omega \left[(\alpha + i\nu)\partial_y u^\nu_{reg}\overline{\partial_y s_k^{-\nu}} - \omega^2 u^\nu_{reg}\overline{s_k^{-\nu}}\right]\varphi_\varepsilon\mathrm{d}\mathbf{x}. \end{aligned}$$

Indeed, the boundary terms vanish due to the choice of φ in the vicinity of Γ_n and Γ_p. As $\nu \to 0+$, by lemmas 6.2.5, 6.3.4 and the limit (6.29), it holds that

$$b^\nu(u^\nu_{reg}, s_k^{-\nu}\varphi_\varepsilon) \to J_\varepsilon(k) + I_\varepsilon(k), \quad I_\varepsilon(k) = \sum_{j\in\{p,n\}} \int_{\Omega_j} \left[\alpha \partial_y u_j^+ \overline{\partial_y s_k^-} - \omega^2 u_j^+ \overline{s_k^-}\right] \varphi_\varepsilon \mathrm{d}\mathbf{x}. \tag{6.31}$$

Next let us consider the second term in (6.30). Performing once again integration by parts in the x-direction, one finds

$$b^\nu(s^\nu_{g^+}, s_k^{-\nu}\varphi_\varepsilon) = \int_\Omega \left[(\alpha + i\nu)\, \partial_y s^\nu_{g^+}\, \overline{\partial_y \left(s_k^{-\nu}\varphi_\varepsilon\right)} + \left(\left(-\partial_x\left((\alpha+i\nu)\,\partial_x s^\nu_{g^+}\right)\right) - \omega^2 s^\nu_{g^+}\right)\overline{\left(s_k^{-\nu}\varphi_\varepsilon\right)}\right] \mathrm{d}\mathbf{x},$$

and by lemma 6.3.4, as $\nu \to 0+$,

$$b^\nu(s^\nu_{g^+}, s_k^{-\nu}\varphi_\varepsilon) \to b^{(1)}_{sing}\left(g^+, s_k^-\varphi_\varepsilon\right) = \int_\Omega \left[\alpha\partial_y s^+_{g^+}\, \overline{\partial_y\left(s_k^-\varphi_\varepsilon\right)} - \partial_x\left(\alpha\partial_x s^+_{g^+}\right)\overline{\left(s_k^-\varphi_\varepsilon\right)} - \omega^2 s^+_{g^+}\overline{\left(s_k^-\varphi_\varepsilon\right)}\right] \mathrm{d}\mathbf{x}. \tag{6.32}$$

Since $s_k^{-\nu}$ converges to s_k^- in $L^2(\Omega)$ and by definition (6.7) of $\ell^{(1)}$, $\lim_{\nu\to0+} \ell^{(1)}\left(s_k^{-\nu}\varphi_\varepsilon\right) = \ell^{(1)}\left(s_k^-\varphi_\varepsilon\right)$. Finally, by Lebesgue's dominated convergence theorem, as $\varepsilon \to 0+$, $I_\varepsilon(k)$, $b^{(1)}_{sing}\left(g^+, s_k^-\varphi_\varepsilon\right)$ and $\ell^{(1)}\left(s_k^-\varphi_\varepsilon\right)$ both go to 0. Therefore, combining (6.31) and (6.32) in (6.30), and taking $\varepsilon \to 0+$, we obtain that

$$\lim_{\varepsilon\to0+} J_\varepsilon(k) = 0,$$

which leads to the conclusion thanks to the alternate definition of the jump (6.28). □

6.3.3 Green's identities

Once the notion of jump defined, one next step is to extend the Green's identities. Recall the expression (6.20) of $a^{(1)}$ in which appears the following sesquilinear antihermitian form

$$C_\psi(U,V) = \int_\Omega \alpha\left[(\nabla U)\overline{V} - U\overline{(\nabla V)}\right]\cdot\nabla\psi \mathrm{d}\mathbf{x}, \tag{6.33}$$

with $U = \mathbf{u} + s_{g-h}$ and $V = \mathbf{v} + s_{k-l}$. The goal of this section is to express $C_\psi(U,V)$ using the sesquilinear forms $b^{(1)}_{reg}$ and $b^{(1)}_{sing}$. The first step is the following manipulation, which will be used elsewhere:

$$\begin{aligned}\left[\nabla u\bar{v} - u\,\overline{\nabla v}\right]\cdot\nabla\psi &= \nabla u\cdot\overline{\nabla(v\psi)} - \nabla u\cdot\overline{\nabla v}\psi - \nabla(u\psi)\cdot\overline{\nabla v} + \nabla u\cdot\overline{\nabla v}\psi \\ &= \nabla u\cdot\overline{\nabla(v\psi)} - \nabla(u\psi)\cdot\overline{\nabla v}.\end{aligned} \tag{6.34}$$

Therefore, given a Lipschitz domain $\mathcal{O}$ and u, v smooth in $\overline{\mathcal{O}}$, we have

$$\int_{\mathcal{O}} \alpha\left[\nabla u\bar{v} - u\overline{\nabla v}\right]\cdot\nabla\psi \mathrm{d}\mathbf{x} = \int_{\mathcal{O}} \alpha\left[\nabla u\cdot\overline{\nabla(v\psi)} - \nabla(u\psi)\cdot\overline{\nabla v}\right]\mathrm{d}\mathbf{x}. \tag{6.35}$$

We observe that, depending on whether U, V are regular, i.e., belonging to $H^1_{1/2}(\Omega_p)\times H^1_{1/2}(\Omega_n)$, or singular, i.e., of the form s_g, with $g \in H^1_{per}(\Sigma)$, the expression (6.33) of $C_\psi(U,V)$ will obviously change. There are three different cases:

- U, V are both regular, in Q, see proposition 6.3.6,

- U, V are both singular, i.e., $U = s_g$ and $V = s_k$, see proposition 6.3.8,
- U is regular and V is singular, see proposition 6.3.10.

The simplest case is when $U, V \in Q$. According to the above, we reexpress the right-hand side of the identity (6.35) with $\mathcal{O} = \operatorname{int}\left(\overline{\Omega_p} \cup \overline{\Omega_n}\right)$. Namely,

$$\int_{\mathcal{O}} \alpha\left[\nabla U \overline{V} - U \overline{\nabla V}\right] \cdot \nabla \psi \mathbf{dx} = \int_{\mathcal{O}} \left[\alpha \nabla U \cdot \overline{\nabla (V\psi)} - \omega^2 U \overline{(V\psi)}\right] \mathbf{dx}. - \int_{\mathcal{O}} \left[\alpha \nabla (U\psi) \cdot \overline{\nabla V} - \omega^2 (U\psi) \overline{V}\right] \mathbf{dx}. \tag{6.36}$$

Recalling the definition (6.4) of $b^{(1)}_{reg}$:

$$b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) = \sum_{j \in \{p,n\}} \int_{\Omega_j} \left(\alpha \nabla u_j \cdot \overline{\nabla v_j} - \omega^2 u_j \overline{v_j}\right) \mathbf{dx} + i\lambda \int_{\Gamma_j} u_j \overline{v_j} \mathrm{ds},$$

one has the following proposition.

Proposition 6.3.6. *Let $\mathbf{u}, \mathbf{v} \in Q$ and $\psi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega}\right)$. Then*

$$\sum_{j \in \{p,n\}} \int_{\Omega_j} \alpha \left(\nabla u_j \overline{v_j} - u_j \overline{\nabla v_j}\right) \cdot \nabla \psi \mathbf{dx} = b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}\psi) - \overline{b^{(1)}_{reg}(\mathbf{v}, \mathbf{u}\psi)} - 2i\lambda \sum_{j \in \{p,n\}} \int_{\Gamma_j} u_j \overline{v_j} \psi \mathrm{ds}.$$

Let us now consider the second case, when U and V are both singular, that is $U = s_g$ and $V = s_k$. Evidently, we cannot apply (6.36) for $\mathcal{O} = \Omega_{p,n}$ and ψ non-vanishing on the interface, since the terms $\int_{\Omega_{p,n}} \alpha \nabla (s_g \psi) \cdot \overline{\nabla s_k} dx$ and $\int_{\Omega_{p,n}} \alpha \nabla s_g \cdot \overline{\nabla (s_k \psi)} \mathbf{dx}$ are not defined. This difficulty can be overcome by integrating by parts in the x-direction. Let u, v be sufficiently smooth in $\overline{\mathcal{O}}$, then

$$\begin{aligned}
&\int_{\mathcal{O}} \alpha\left[\nabla u \overline{v} - u \overline{\nabla v}\right] \cdot \nabla \psi \mathbf{dx} \\
&= \int_{\mathcal{O}} \alpha \left[\partial_y u \cdot \overline{\partial_y (v\psi)} - \partial_y (u\psi) \cdot \overline{\partial_y v}\right] \mathbf{dx} + \int_{\mathcal{O}} \left[(-\partial_x (\alpha \partial_x u)) \overline{v\psi} - (u\psi) \overline{(-\partial_x (\alpha \partial_x v))}\right] \mathbf{dx} \\
&+ \int_{\partial \mathcal{O}} \left[(\alpha \partial_n u) \overline{v} - u \overline{(\alpha \partial_n v)}\right] \psi \mathrm{ds}
\end{aligned} \tag{6.37}$$

$$\begin{aligned}
&= \int_{\mathcal{O}} \left[\alpha \partial_y u \overline{\partial_y (v\psi)} + \left(-\partial_x (\alpha \partial_x u) - \omega^2 u\right) \overline{v\psi}\right] \mathbf{dx} + \int_{\partial \mathcal{O}} (\alpha \partial_n u) \overline{v} \psi \mathrm{ds} \\
&- \int_{\mathcal{O}} \left[\alpha \partial_y (u\psi) \overline{\partial_y v} + (u\psi) \overline{\left(-\partial_x (\alpha \partial_x v) - \omega^2 v\right)}\right] \mathbf{dx} - \int_{\partial \mathcal{O}} u \overline{(\alpha \partial_n v)} \psi \mathrm{ds}.
\end{aligned} \tag{6.38}$$

Compare with the definition of $b^{(1)}_{sing}$:

$$b^{(1)}_{sing}(g, \mathbf{v}) = \sum_{j \in \{p,n\}} \int_{\Omega_j} \left[\alpha \partial_y s_g \overline{\partial_y v_j} + (-\partial_x (\alpha \partial_x s_g) - \omega^2 s_g) \overline{v_j}\right] \mathbf{dx} + \int_{\Gamma_j} (\alpha \partial_n s_g + i\lambda s_g) \overline{v_j} \mathrm{ds}.$$

Lemma 6.3.7. *For $g, k \in H^1_{per}(\Sigma)$ and $\psi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega}; \mathbb{R}\right)$, for $j \in \{p, n\}$, it holds that*

$$\begin{aligned}
\int_{\Omega_j} \alpha \left[\partial_x s_g \overline{s_k} - s_g \overline{\partial_x s_k}\right] \partial_x \psi \mathbf{dx} &= \int_{\Omega_j} \left[\left(-\partial_x (\alpha \partial_x s_g)\right) \overline{s_k} - s_g \overline{(-\partial_x (\alpha \partial_x s_k))}\right] \psi \mathbf{dx} \\
&+ \int_{\Gamma_j} \left[(\alpha \partial_n s_g) \overline{s_k} - s_g \overline{(\alpha \partial_n s_k)}\right] \psi \mathrm{ds} - \sigma_j \int_{\Sigma} g(y) \overline{k(y)} \mathbf{r}(y) \psi(0, y)\, dy,
\end{aligned}$$

where $\sigma_p = 0$ and $\sigma_n = 2i\pi$.

Proof. Applying (6.37) in $\mathcal{O} = \Omega_j^\varepsilon = \{\mathbf{x} \in \Omega_j \ : \ \operatorname{dist}(\mathbf{x}, \Sigma) > \varepsilon\}$, $\varepsilon > 0$, $j \in \{p, n\}$, with $u = s_g$ and $v = s_k$ yields

$$\int_{\Omega_j^\varepsilon} \alpha \left[s_g \overline{\partial_x s_k} - \partial_x s_g \overline{s_k} \right] \partial_x \psi \mathrm{d}\mathbf{x} = \int_{\Omega_j^\varepsilon} \left[s_g \overline{(-\partial_x(\alpha \partial_x s_k))} - \overline{s_k}(-\partial_x(\alpha \partial_x s_g)) \right] \psi \mathrm{d}\mathbf{x} + \int_{\Gamma_j} \left[s_g \overline{(\alpha \partial_n s_k)} - (\alpha \partial_n s_g) \overline{s_k} \right] \psi \mathrm{d}s - a_j I_j^\varepsilon,$$

with $a_p = 1$ and $a_n = -1$ and

$$I_j^\varepsilon = \int_{\{x = a_j \varepsilon\}} \left[s_g \overline{(\alpha \partial_x s_k)} - \left(\alpha \partial_x s_g \right) \overline{s_k} \right] \psi dy.$$

As $\varepsilon \to 0+$, the volume integrals over Ω_j^ε converge to the volume integrals over Ω_j, since the integrands are obviously in $L^1(\Omega_j)$. Let us compute the remaining limit $\lim_{\varepsilon \to 0+} I_j^\varepsilon$. Recall that $s_g(x, y) = g(y) \mathrm{S}(x)$ with $\mathrm{S}(x) = \log|x| + i\pi \mathbb{1}_{x<0}$. As $\varepsilon \to 0+$,

$$I_j^\varepsilon = \int_\Sigma g(y) \overline{k(y)} \frac{\alpha(a_j \varepsilon, y)}{a_j \varepsilon} \left[\mathrm{S}(a_j \varepsilon) - \overline{\mathrm{S}(a_j \varepsilon)} \right] \psi(a_j \varepsilon, y) dy \to \sigma_j \int_\Sigma g(y) \overline{k(y)} \mathrm{r}(y) \psi(0, y) dy,$$

where $\sigma_p = 0$, $\sigma_n = 2i\pi$. □

The proposition below is a rewriting of formula (6.38) using the above lemma.

Proposition 6.3.8. *Let $g, k \in H^1_{per}(\Sigma)$ and $\psi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega}; \mathbb{R}\right)$. It holds that*

$$\int_\Omega \alpha \left[\nabla s_g \overline{s_k} - s_g \overline{\nabla s_k} \right] \cdot \nabla \psi \mathrm{d}\mathbf{x} = b_{sing}^{(1)}(g, s_k \psi) - \overline{b_{sing}^{(1)}\left(k, s_g \psi\right)} - 2i\lambda \sum_{j \in \{p,n\}} \int_{\Gamma_j} s_g \overline{s_k} \psi \mathrm{d}s - 2i\pi \int_\Sigma g(y) \overline{k(y)} \mathrm{r}(y) \psi(0, y) dy.$$

Applying the last proposition with $\psi = 1$ yields immediately the following counterpart of Green's third formula.

Corollary 6.3.9. *For each g, $k \in H^1_{per}(\Sigma)$,*

$$b_{sing}^{(1)}(g, s_k) - \overline{b_{sing}^{(1)}(k, s_g)} = 2i\pi (g, k)_{\mathrm{r}} + 2i\lambda \sum_{j \in \{p,n\}} \int_{\Gamma_j} s_g \overline{s_k} \mathrm{d}s. \tag{6.39}$$

The third and last case consists in taking U regular and V singular, namely U belonging to a certain subspace of $H^1_{1/2}(\Omega_{p,n})$ and $V = s_k$. Let us introduce $\psi_\Sigma(x, y) = \psi(0, y)$ for $(x, y) \in \overline{\Omega}$. Notice that ψ_Σ is actually a constant if $\partial_y \psi = 0$.

Proposition 6.3.10. *Let $\mathbf{u} \in Q$, $g \in H^1_{per}(\Sigma)$ and $\ell \in Q'$, be such that*

$$b_{reg}^{(1)}(\mathbf{u}, \mathbf{v}) + b_{sing}^{(1)}(g, \mathbf{v}) = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in Q. \tag{6.40}$$

Moreover, assume that we can define ℓ_∞ as in (6.23). Let $k \in H^1_{per}(\Sigma)$ and $\psi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega}; \mathbb{R}\right)$ satisfying $\partial_y \psi = 0$. Then

$$\sum_{j \in \{p,n\}} \int_{\Omega_j} \alpha \left[\nabla u_j \overline{s_k} - u_j \overline{\nabla s_k} \right] \cdot \nabla \psi \mathrm{d}\mathbf{x} = \ell_\infty(s_k \psi) - b_{sing}^{(1)}(g, s_k \psi) - \overline{b_{sing}^{(1)}(k, \mathbf{u}\psi)} - 2i\lambda \sum_{j \in \{p,n\}} \int_{\Gamma_j} u_j \overline{s_k} \psi \mathrm{d}s + \psi_\Sigma \langle [\mathbf{u}], k \rangle_\Sigma, \tag{6.41}$$

where the jump $[\mathbf{u}]$ is defined in the statement of lemma 6.3.2 and $\ell_\infty(s_k \psi)$ as in (6.23).

The proof of the previous proposition relies on the following technical lemma, most of the proof of which is left to the reader.

Lemma 6.3.11. *Let* $\mathbf{u} \in Q$, $g \in H^1_{per}(\Sigma)$, *and* $\psi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega}; \mathbb{R}\right)$. *Then,* $(\psi - \psi_\Sigma)\nabla u_j \in L^2\left(\Omega_j\right)$, *and* $(\psi - \psi_\Sigma)\,\partial_x s_g \in L^2\left(\Omega\right)$. *As a consequence,* $u \in L^2(\Omega)$, *s.t.* $u|_{\Omega_{p,n}} = u_{p,n}$, *satisfies* $(\psi - \psi_\Sigma)\,u \in H^1(\Omega)$. *Moreover, the trace* $\gamma_0\left[(\psi - \psi_\Sigma)u\right] = (\psi - \psi_\Sigma)u|_\Sigma$ *vanishes.*

Proof. Firstly, there is a neighborhood of Σ such that $|\psi - \psi_\Sigma| \le C|x| \le C|\alpha|^{1/2}$. Consequently, inside this neighborhood, $\left|(\psi - \psi_\Sigma)\nabla u_j\right| \le C|\alpha|^{1/2}\left|\nabla u_j\right|$ and $\left|(\psi - \psi_\Sigma)\,\partial_x s_g\right| \le C|g|$ which prove the first assertion. This also proves that $(\psi - \psi_\Sigma)\,u_j \in H^1(\Omega_j)$ since $\nabla\left((\psi - \psi_\Sigma)\,u_j\right) = \nabla(\psi - \psi_\Sigma)\,u_j + (\psi - \psi_\Sigma)\nabla u_j$, for $j \in \{p, n\}$.

Secondly, we prove the statement about the trace of $h_p := (\psi - \psi_\Sigma)u_p \in H^1(\Omega_p)$ only. With the standard density argument, it suffices to prove the result for $u_p \in C^\infty(\overline{\Omega_p})$. We start with the expression

$$h_p(0, y) = h_p(x, y) - \int_0^x \partial_s h_p(s, y)ds, \quad (x, y) \in \Omega_p.$$

Applying the Cauchy-Schwarz inequality in $\mathbb{R}^2$ and $L^2(\Omega_p)$ yields

$$\begin{aligned}|h_p(0, y)|^2 &\le 2|h_p(x, y)|^2 + 2\left|\int_0^x \partial_s h_p(s, y)ds\right|^2 \\ &\le 2|h_p(x, y)|^2 + 2x\int_0^x |\partial_s h_p(s, y)|^2 \mathrm{d}s.\end{aligned}$$

Remark that a priori $(x, y) \mapsto h_p(x, y)/x \in L^2(\Omega_p)$. Integrating both sides of the above inequality in the strip $\Omega_p^\varepsilon = \{(x, y) \in \Omega_p \;:\; |x| < \varepsilon\}$, $\varepsilon \in (0, 1)$, allows to obtain the following inequality, where all terms in the right-hand side are finite:

$$\begin{aligned}\int_\Sigma |h_p(0, y)|^2 dy &\le \varepsilon^{-1}\left(2\int_{\Omega_p^\varepsilon} |h_p(x, y)|^2 \mathrm{d}\mathbf{x} + \varepsilon^2\|\nabla h_p\|^2_{L^2(\Omega_p^\varepsilon)}\right) \\ &\le \varepsilon\left(2\left\|\frac{h_p(x, y)}{x}\right\|^2_{L^2(\Omega_p^\varepsilon)} + \|\nabla h_p\|^2_{L^2(\Omega_p^\varepsilon)}\right).\end{aligned}$$

The above is valid for all $\varepsilon > 0$, hence taking $\varepsilon \to 0$ in the above shows that $\|\gamma_0 h_p\|_{L^2(\Sigma)} = 0$. Repeating the argument for $h_n = (\psi - \psi_\Sigma)\,u_n$ leads to $(\psi - \psi_\Sigma)\,u \in H^1(\Omega)$ with $u|_{\Omega_{p,n}} = u_{p,n}$. □

Proof of proposition 6.3.10. We start by using (6.35), with $\mathscr{O} = \Omega_{p,n}$:

$$\sum_{j\in\{p,n\}} \int_{\Omega_j} \alpha\left[\nabla u_j \overline{s_k} - u_j \nabla\overline{s_k}\right] \cdot \nabla\psi \mathrm{d}\mathbf{x} = I_1 - I_2, \quad \text{with}$$

$$I_1 = \sum_{j\in\{p,n\}} \int_{\Omega_j} \alpha\partial_x u_j \overline{\partial_x\left(s_k\left(\psi - \psi_\Sigma\right)\right)}\mathrm{d}\mathbf{x}. \quad \text{and} \quad I_2 = \sum_{j\in\{p,n\}} \int_{\Omega_j} \alpha\partial_x\left(u_j\left(\psi - \psi_\Sigma\right)\right)\overline{\partial_x s_k}\mathrm{d}\mathbf{x},$$

Remark that the above two integrals are well-defined by lemma 6.3.11. On one hand,

$$\begin{aligned}I_1 = b^{(1)}_{reg}\left(\mathbf{u}, s_k(\psi - \psi_\Sigma)\right) - \sum_{j\in\{p,n\}} \int_{\Omega_j} \left(\alpha\partial_y u_j \overline{\partial_y\left(s_k\left(\psi - \psi_\Sigma\right)\right)} - \omega^2 u_j \overline{\left(s_k\left(\psi - \psi_\Sigma\right)\right)}\right)\mathrm{d}\mathbf{x} \\ - i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\left(\psi - \psi_\Sigma\right)\overline{s_k}\mathrm{d}s.\end{aligned}$$

On the other hand, integrating by parts in Ω_j, $j = p, n$, and noting that, according to proposition 6.3.10, $(\psi - \psi_\Sigma)\, u \in H^1(\Omega)$ with vanishing trace on Σ, and $\alpha\partial_n s_k|_\Sigma = k(y)\,\mathrm{r}(y)$ yields

$$
\begin{aligned}
I_2 &= \sum_{j\in\{p,n\}} \int_{\Omega_j} u_j\,(\psi-\psi_\Sigma)\,\overline{(-\partial_x\,(\alpha\partial_x s_k))}\mathrm{d}\mathbf{x} + \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\,(\psi-\psi_\Sigma)\,\overline{\alpha\partial_n s_k}\mathrm{d}s \\
&= \overline{b^{(1)}_{sing}\,(k,\mathbf{u}(\psi-\psi_\Sigma))} - \sum_{j\in\{p,n\}} \int_{\Omega_j} \left(\alpha\partial_y\left(u_j\,(\psi-\psi_\Sigma)\right)\overline{\partial_y s_k} - \omega^2\left(u_j\,(\psi-\psi_\Sigma)\right)\overline{s_k}\right)\mathrm{d}\mathbf{x} \\
&\qquad\qquad + i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\,(\psi-\psi_\Sigma)\,\overline{s_k}\mathrm{d}s.
\end{aligned}
$$

Hence, using $\partial_y\psi = 0$,

$$
\begin{aligned}
\sum_{j\in\{p,n\}} \int_{\Omega_j} \alpha\left[\nabla u_j\overline{s_k} - u_j\overline{\nabla s_k}\right]\cdot\nabla\psi\mathrm{d}\mathbf{x} &= b^{(1)}_{reg}\,(\mathbf{u}, s_k(\psi-\psi_\Sigma)) - \overline{b^{(1)}_{sing}\,(k,\mathbf{u}(\psi-\psi_\Sigma))} \\
&\quad - 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\overline{s_k}\,(\psi-\psi_\Sigma)\,\mathrm{d}s.
\end{aligned}
\tag{6.42}
$$

Comparing the above with the statement of the proposition, it remains to rewrite the term $b^{(1)}_{reg}\,(\mathbf{u}, s_k(\psi-\psi_\Sigma))$, using the identity (6.40) and the fact that ψ_Σ is constant:

$$
\begin{aligned}
b^{(1)}_{reg}\,(\mathbf{u}, s_k(\psi-\psi_\Sigma)) &= \ell\,(s_k(\psi-\psi_\Sigma)) - b^{(1)}_{sing}\,(g, s_k(\psi-\psi_\Sigma)) \\
&= \ell_\infty\,(s_k\psi) - b^{(1)}_{sing}\,(g, s_k\psi) - \psi_\Sigma\left(\ell_\infty\,(s_k) - b^{(1)}_{sing}\,(g, s_k)\right),
\end{aligned}
\tag{6.43}
$$

where $\ell_\infty\,(s_k\psi)$ and $\ell_\infty\,(s_k)$ are well-defined because $\operatorname{supp}\ell\cap\Sigma = \emptyset$. Notice that $b^{(1)}_{sing}\,(g, s_k\psi)$ and $b^{(1)}_{sing}\,(g, s_k)$ are also well-defined since $b^{(1)}_{sing}\,(g,\mathbf{v})$ is well-defined as soon as $\mathbf{v}\in L^2\,(\Omega)$, $\partial_y\mathbf{v}\in L^2\,(\Omega)$ and the trace of $\mathbf{u}$ on Γ_j belongs to $L^2(\Gamma_j)$ for $j = p, n$. Recall that the jump $[\mathbf{u}]$ satisfies (6.26), namely

$$
\ell_\infty\,(s_k) - b^{(1)}_{sing}\,(g, s_k) - \overline{b^{(1)}_{sing}\,(k,\mathbf{u})} = 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\overline{s_k}\mathrm{d}s - \langle[\mathbf{u}], k\rangle_\Sigma.
$$

Combining (6.42), (6.43) and the above identity results in the desired expression. □

Remark 6.3.12. Let φ be as in definition 4.2.2. With this particular regular function, the previous propositions are respectively summarized as, with $\mathbf{u}$, $\mathbf{v}$, g and k satisfying the assumptions of the corresponding propositions,

$$
\text{(prop. 6.3.6)} \qquad C_\varphi\,(\mathbf{u},\mathbf{v}) = b^{(1)}_{reg}(\mathbf{u},\mathbf{v}\varphi) - \overline{b^{(1)}_{reg}(\mathbf{v},\mathbf{u}\varphi)}, \tag{6.44}
$$

$$
\text{(prop. 6.3.8)} \qquad C_\varphi\left(s_g, s_k\right) = b^{(1)}_{sing}(g, s_k\varphi) - \overline{b^{(1)}_{sing}(k, s_g\varphi)} - 2i\pi(g,k)_{\mathrm{r}}, \tag{6.45}
$$

$$
\text{(identity 6.42)} \qquad C_\varphi\,(\mathbf{u}, s_k) = b^{(1)}_{reg}\,(\mathbf{u}, s_k\,(\varphi-1)) - \overline{b^{(1)}_{sing}\,(k,\mathbf{u}\,(\varphi-1))} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j\overline{s_k}\mathrm{d}s \tag{6.46}
$$

$$
\text{(prop. 6.3.10)} \qquad = \ell_\infty\,(s_k\varphi) - b^{(1)}_{sing}\,(g, s_k\varphi) - \overline{b^{(1)}_{sing}\,(k,\mathbf{u}\varphi)} + \langle[\mathbf{u}], k\rangle_\Sigma. \tag{6.47}
$$

6.3.4 Expressions of $a^{(1)}$

Since we obtained a mixed formulation in section 6.2.2, it is important to study $a^{(1)}$ on the kernel of $\mathrm{B}^{(1)} : V^{(1)} \to Q'$, the operator associated to $b^{(1)}_{reg}$:

$$
\operatorname{Ker}\mathrm{B}^{(1)} := \left\{(\mathbf{u}, g, h)\in V^{(1)} \,:\, b^{(1)}\left((\mathbf{u}, g, h),\mathbf{v}\right) = b^{(1)}_{reg}(\mathbf{u},\mathbf{v}) + b^{(1)}_{sing}(g,\mathbf{v}) = 0\ \forall\mathbf{v}\in Q\right\}.
$$

For the convenience of the reader, we recall the expression (6.20) of the form $a^{(1)}$:

$$a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l)) = b^{(1)}_{sing}(h,(\mathbf{v}+s_{k-l})\varphi) - \overline{b^{(1)}_{sing}\left(l,(\mathbf{u}+s_{g-h})\varphi\right)} + C_\varphi\left(\mathbf{u}+s_{g-h},\mathbf{v}+s_{k-l}\right).$$

From now on, we assume that $\varphi \in \mathscr{C}^1_{per,y}\left(\overline{\Omega};\mathbb{R}\right)$, supp $\varphi \cap \overline{\Gamma_{n,p}} = \emptyset$, $\partial_y\varphi = 0$ and $\varphi|_\Sigma = 1$, as in remark 6.2.12, so that φ satisfies the assumptions of proposition 6.3.10. The following technical lemma allows to reexpress the form $a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l))$ for $(\mathbf{u},g)$ satisfying a variational equation like (6.11), $h \in H^1_{per}(\Sigma)$ and $(\mathbf{v},k,l) \in V^{(1)}$.

Lemma 6.3.13. *Let $\mathbf{u} \in Q$, $g \in H^1_{per}(\Sigma)$ and $\ell \in Q'$, be such that*

$$b^{(1)}_{reg}(\mathbf{u},\mathbf{v}) + b^{(1)}_{sing}(g,\mathbf{v}) = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in Q, \tag{6.48}$$

and ℓ_∞ exists as in lemma 6.3.2. for all $(\mathbf{v},k,l) \in V^{(1)}$, it holds

$$\begin{aligned} a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l)) + \overline{b^{(1)}((\mathbf{v},k,l),\mathbf{u}\varphi)} &= \ell(\mathbf{v}\varphi) + \ell_\infty(s_{k-l}\varphi) \\ &\quad + \overline{b^{(1)}\left((\mathbf{v},k,l), s_{g-h}(1-\varphi)\right)} \\ &\quad - 2i\pi(g-h,k-l)_{L^2_\mathfrak{r}(\Sigma)} + \langle[\mathbf{u}],k-l\rangle_\Sigma \\ &\quad - b^{(1)}_{sing}(g-h,\mathbf{v}) - \overline{b^{(1)}_{sing}\left(k,s_{g-h}\right)} + 2i\lambda\sum_{j\in\{p,n\}}\int_{\Gamma_j} s_{g-h}\overline{v_j}ds. \end{aligned} \tag{6.49}$$

Moreover, assume that $\mathbf{v}$ has a jump as in lemma 6.3.2, i.e., there is $\tilde{\ell} \in Q'$ such that $b^{(1)}_{reg}(\mathbf{v},\boldsymbol{\mu}) + b^{(1)}_{sing}(k,\boldsymbol{\mu}) = \tilde{\ell}(\boldsymbol{\mu})$ for all $k \in H^1_{per}(\Sigma)$,$\boldsymbol{\mu} \in Q$, and $\tilde{\ell}_\infty$ has a sense. Then, we have for all $k,l \in H^1_{per}(\Sigma)$

$$\begin{aligned} a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l)) &= (\ell(\mathbf{v}\varphi) + \ell_\infty(s_{k-l}\varphi)) - \overline{\left(\tilde{\ell}(\mathbf{u}\varphi) + \tilde{\ell}_\infty\left(s_{g-h}\varphi\right)\right)} \\ &\quad + \langle[\mathbf{u}],k-l\rangle_\Sigma - \overline{\langle[\mathbf{v}],g-h\rangle_\Sigma} \\ &\quad - 2i\pi(g-h,k-l)_{L^2_\mathfrak{r}(\Sigma)}. \end{aligned} \tag{6.50}$$

In particular, if $(\mathbf{v},k,l) \in \operatorname{Ker} \mathrm{B}^{(1)}$, i.e., $\tilde{\ell} = 0$, then

$$a^{(1)}((\mathbf{u},g,h),(\mathbf{v},k,l)) = \ell(\mathbf{v}\varphi) + \ell_\infty(s_{k-l}\varphi) + \langle[\mathbf{u}],k-l\rangle_\Sigma - \overline{\langle[\mathbf{v}],g-h\rangle_\Sigma} - 2i\pi(g-h,k-l)_{L^2_\mathfrak{r}(\Sigma)}. \tag{6.51}$$

Proof. We start by developing the first term in the definition of $a^{(1)}$ given by (6.20). Our goal is to rewrite it in terms of the forms $b^{(1)}$, $b^{(1)}_{sing}$ and $b^{(1)}_{sing}$, and then rearrange the terms as in the proposition. Using (6.44), (6.45), (6.46) and (6.47) gives

$$\begin{aligned} &C_\varphi\left(\mathbf{u}+s_{g-h},\mathbf{v}+s_{k-l}\right) \\ &\quad = C_\varphi(\mathbf{u},\mathbf{v}) + C_\varphi(\mathbf{u},s_{k-l}) + C_\varphi\left(s_{g-h},\mathbf{v}\right) + C_\varphi\left(s_{g-h},s_{k-l}\right) \\ &\quad = b^{(1)}_{reg}(\mathbf{u},\mathbf{v}\varphi) - \overline{b^{(1)}_{reg}(\mathbf{v},\mathbf{u}\varphi)} && (6.44) \\ &\qquad + \ell_\infty(s_{k-l}\varphi) - b^{(1)}_{sing}(g,s_{k-l}\varphi) - \overline{b^{(1)}_{sing}(k-l,\mathbf{u}\varphi)} + \langle[\mathbf{u}]_\Sigma,k-l\rangle_\Sigma && (6.47) \\ &\qquad + b^{(1)}_{sing}(g-h,\mathbf{v}(\varphi-1)) - \overline{b^{(1)}_{reg}\left(\mathbf{v},s_{g-h}(\varphi-1)\right)} + 2i\lambda\sum_{j\in\{p,n\}}\int_{\Gamma_j} s_{g-h}\overline{v_j}ds && (6.46) \\ &\qquad + b^{(1)}_{sing}(g-h,s_{k-l}\varphi) - \overline{b^{(1)}_{sing}(k-l,s_{g-h}\varphi)} - 2i\pi(g-h,k-l)_\mathfrak{r}. && (6.45) \end{aligned}$$

Remark that in the above the term $\ell_\infty(s_{k-l}\varphi)$ is well-defined. Rearranging the terms in the above yields

$$\begin{aligned} C_\varphi\left(\mathbf{u}+s_{g-h}, \mathbf{v}+s_{k-l}\right) = {} & \ell_\infty\left(s_{k-l}\varphi\right) + \left[b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}\varphi) + b^{(1)}_{sing}\left(g, \mathbf{v}\varphi\right)\right] - \overline{\left[b^{(1)}_{reg}(\mathbf{v}, \mathbf{u}\varphi) + b^{(1)}_{sing}\left(k, \mathbf{u}\varphi\right)\right]} \\ & - \overline{\left[b^{(1)}_{reg}\left(\mathbf{v}, s_{g-h}\left(\varphi-1\right)\right) + b^{(1)}_{sing}\left(k, s_{g-h}\left(\varphi-1\right)\right)\right]} \\ & - 2i\pi\left(g-h, k-l\right)_{\mathbf{r}} + \langle[\mathbf{u}]_\Sigma, k-l\rangle_\Sigma \\ & - b^{(1)}_{sing}(g-h, \mathbf{v}) - \overline{b^{(1)}_{sing}\left(k, s_{g-h}\right)} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} s_{g-h}\overline{v_j}\mathrm{d}s \\ & + \overline{b^{(1)}_{sing}(l, \left(\mathbf{u}+s_{g-h}\right)\varphi)} - b^{(1)}_{sing}\left(h, \left(\mathbf{v}+s_{k-l}\right)\varphi\right) \end{aligned}$$

Using the definition (6.21) of the form $b^{(1)}$, namely $b^{(1)} = b^{(1)}_{reg} + b^{(1)}_{sing}$, and the assumptions of the lemma on $(\mathbf{v}, k, l)$ and $(\mathbf{u}, g, h)$ we rewrite the above as follows:

$$\begin{aligned} C_\varphi\left(\mathbf{u}+s_{g-h}, \mathbf{v}+s_{k-l}\right) = {} & \ell\left(\mathbf{v}\varphi\right) + \ell_\infty\left(s_{k-l}\varphi\right) - \overline{b^{(1)}\left(\left(\mathbf{v}, k, l\right), \mathbf{u}\varphi\right)} \\ & - \overline{b^{(1)}\left(\left(\mathbf{v}, k, l\right), s_{g-h}\left(\varphi-1\right)\right)} \\ & - 2i\pi\left(g-h, k-l\right)_{\mathbf{r}} + \langle[\mathbf{u}]_\Sigma, k-l\rangle_\Sigma \\ & - b^{(1)}_{sing}(g-h, \mathbf{v}) - \overline{b^{(1)}_{sing}\left(k, s_{g-h}\right)} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} s_{g-h}\overline{v_j}\mathrm{d}s \\ & + \overline{b^{(1)}_{sing}(l, \left(\mathbf{u}+s_{g-h}\right)\varphi)} - b^{(1)}_{sing}\left(h, \left(\mathbf{v}+s_{k-l}\right)\varphi\right) \end{aligned} \tag{6.52}$$

Plugging in the resulting expression into the definition (6.20) of $a^{(1)}$ yields the first expression in the statement of the lemma:

$$\begin{aligned} a^{(1)}\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) + \overline{b^{(1)}\left(\left(\mathbf{v}, k, l\right), \mathbf{u}\varphi\right)} = {} & \ell\left(\mathbf{v}\varphi\right) + \ell_\infty\left(s_{k-l}\varphi\right) + \overline{b^{(1)}\left(\left(\mathbf{v}, k, l\right), s_{g-h}\left(1-\varphi\right)\right)} \\ & - 2i\pi\left(g-h, k-l\right)_{L^2_{\mathbf{r}}(\Sigma)} + \langle[\mathbf{u}], k-l\rangle_\Sigma \\ & - b^{(1)}_{sing}(g-h, \mathbf{v}) - \overline{b^{(1)}_{sing}\left(k, s_{g-h}\right)} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} s_{g-h}\overline{v_j}\mathrm{d}s. \end{aligned}$$

To obtain the second identity, given $(\mathbf{v}, k, l) \in \operatorname{Ker} V^{(1)}$ such that the jump $[\mathbf{v}]_\Sigma$ is well-defined, it verifies identity (6.26) with $\ell_\infty = \tilde{\ell}_\infty$:

$$\langle[\mathbf{v}]_\Sigma, g-h\rangle_\Sigma = \overline{b^{(1)}_{sing}(g-h, \mathbf{v})} + b^{(1)}_{sing}\left(k, s_{g-h}\right) + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} v_j\overline{s_{g-h}}\mathrm{d}s - \tilde{\ell}_\infty\left(s_{g-h}\right).$$

Therefore, replacing the last terms of the first identity of the proposition gives precisely the second one. Finally, given $(\mathbf{v}, k, l) \in \operatorname{Ker} \mathrm{B}^{(1)}$, the third identity is obtained naturally from the second since $\tilde{\ell}_\infty = 0$. □

The results that follow lead to an alternative expression to $a^{(1)}$ on $\operatorname{Ker} \mathrm{B}^{(1)} \times \operatorname{Ker} \mathrm{B}^{(1)}$.

Lemma 6.3.14. *Let* $(\mathbf{u}, g, h), (\mathbf{v}, k, l) \in \operatorname{Ker} \mathrm{B}^{(1)}$. *Then we have the following identity:*

$$\langle[\mathbf{u}]_\Sigma, k\rangle_\Sigma - \overline{\langle[\mathbf{v}]_\Sigma, g\rangle_\Sigma} = 2i\pi\left(g, k\right)_{\mathbf{r}} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} \left(u_j + s_g\right)\overline{\left(v_j + s_k\right)}\mathrm{d}s. \tag{6.53}$$

Proof. Let $J := \langle [\mathbf{u}], k\rangle_\Sigma - \overline{\langle [\mathbf{v}], g\rangle_\Sigma}$. According to the jump formula (6.26), we have

$$J = \overline{b^{(1)}_{sing}(k, \mathbf{u})} + b^{(1)}_{sing}(g, s_k) + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j \overline{s_k} ds - b^{(1)}_{sing}(g, \mathbf{v}) - \overline{b^{(1)}_{sing}\left(k, s_g\right)} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} s_g \overline{v_j} ds.$$

Making use of the fact that $b^{(1)}((\mathbf{u}, g, h), \mathbf{v}) = 0$, $b^{(1)}((\mathbf{v}, k, l), \mathbf{u}) = 0$ and using the definition (6.21) of $b^{(1)}$ yields

$$J = \underbrace{b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) - \overline{b^{(1)}_{reg}(\mathbf{v}, \mathbf{u})}}_{J_1} + \underbrace{b^{(1)}_{sing}(g, s_k) - \overline{b^{(1)}_{sing}\left(k, s_g\right)}}_{J_2} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} \left(u_j \overline{s_k} + s_g \overline{v_j}\right) ds.$$

From the definition (6.4) of $b^{(1)}_{reg}$, it follows that $J_1 = 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j \overline{v_j} ds$. Applying (6.39) to reformulate J_2, we readily arrive at (6.53). □

The above lemma yields immediately the following property.

Corollary 6.3.15. *Let* $(\mathbf{u}, g, h) \in \operatorname{Ker} \mathrm{B}^{(1)}$ *with* $g \neq 0$. *Then* $\operatorname{Im} \langle [\mathbf{u}]_\Sigma, g\rangle_\Sigma > 0$.

Proof. It is a direct application of previous proposition with $\mathbf{v} = \mathbf{u}$ and $k = g$, so that

$$\operatorname{Im} \langle [\mathbf{u}]_\Sigma, g\rangle_\Sigma = \pi \|g\|^2_{L^2_{\mathfrak{r}}(\Sigma)} + \lambda \sum_{j\in\{p,n\}} \left\|\mathbf{u} + s_g\right\|^2_{L^2(\Gamma_j)}.$$

□

Finally, lemmas 6.3.13, 6.3.14 allow us to prove the following result, the second part of which is proposition 23 from [49].

Corollary 6.3.16. *Let* $(\mathbf{u}, g, h)$, $(\mathbf{v}, k, l) \in \operatorname{Ker} \mathrm{B}^{(1)}$. *Then*

$$a^{(1)}((\mathbf{u}, g, h), (\mathbf{v}, k, l)) = 2i\pi (g, k)_{\mathfrak{r}} - 2i\pi (g - h, k - l)_{\mathfrak{r}} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} (\mathbf{u} + s_g)\overline{(\mathbf{v} + s_k)} \mathrm{d}s.$$

In particular, it holds that

$$a^{(1)}((\mathbf{u}, g, 0), (\mathbf{u}, g, 0)) = 2i\lambda \sum_{j\in\{p,n\}} \left\|u_j + s_g\right\|^2_{L^2(\Gamma_j)}, \text{ and } a^{(1)}((\mathbf{0}, 0, h), (\mathbf{0}, 0, h)) = -2i\pi \|h\|^2_{L^2_{\mathfrak{r}}(\Sigma)}. \tag{6.54}$$

6.4 Well-posedness of the mixed problem

We now study the mixed problem (6.22) in more details. Namely, we are interested in the uniqueness and existence of its solution. First, we study a stabilized version of the mixed problem (6.22) Next, we prove its uniqueness. Finally, we address the solution provided by assumption 6.1.1.

6.4.1 Stabilized problem

The study of $a^{(1)}$ on $\operatorname{Ker} B^{(1)}$, in particular lemma 6.3.16 shows a lack of control in the norm $\|\cdot\|_{H^1(\Sigma)}$:

$$\operatorname{Im} a^{(1)}\left((\mathbf{u},g,h),(\mathbf{u},g,-h)\right) = 2\pi\|h\|^2_{L^2_{\mathfrak{r}}(\Sigma)} + 2\lambda \sum_{j\in\{p,n\}} \left\|u_j + s_g\right\|^2_{L^2(\Gamma_j)}. \tag{6.55}$$

Therefore, let the stabilized counterpart of (6.22) be:

$$\left|\begin{aligned} &\text{Find } \left((\mathbf{u},g,h),\boldsymbol{\lambda}\right) \in V^{(1)} \times Q \text{ such that} \\ &a_\rho^{(1)}\left((\mathbf{u},g,h),(\mathbf{v},k,l)\right) + \overline{b^{(1)}\left((\mathbf{v},k,l),\boldsymbol{\lambda}\right)} = \ell^{(1)}(\mathbf{v}\varphi) + \ell_\infty^{(1)}(s_{k-l}\varphi), \quad \forall (\mathbf{v},k,l) \in V, \\ &\qquad\qquad b^{(1)}\left((\mathbf{u},g,h),\boldsymbol{\mu}\right) = \ell^{(1)}(\boldsymbol{\mu}), \qquad\qquad \forall \boldsymbol{\mu} \in Q \end{aligned}\right. \tag{6.56}$$

where

$$a_\rho^{(1)}\left((\mathbf{u},g,h),(\mathbf{v},k,l)\right) = a^{(1)}\left((\mathbf{u},g,h),(\mathbf{v},k,l)\right) - i\rho\left((g,k)_{H^1(\Sigma)} - (h,l)_{H^1(\Sigma)}\right), \quad \text{with } \rho > 0.$$

The stabilization terms involve the $H^1_{per}(\Sigma)$ inner product, which corresponds to the fact that g must belong to $H^1_{per}(\Sigma)$. So one cannot choose an $H^s_{per}(\Sigma)$ inner product for $s < 1$.

Retracing the steps of [49], we can prove the well-posedness result below regarding the stabilized variational formulation. We use a classical approach to the well-posedness of the mixed formulations. According to the Babuška-Brezzi theory, it is sufficient to prove a surjectivity property of the operator $B^{(1)} : V^{(1)} \mapsto Q'$ associated to the form $b^{(1)}$ and an inf-sup condition for the form $a_\rho^{(1)}$ on the kernel of $B^{(1)}$. The kernel of the operator $B^{(1)}$ is characterized as

$$\operatorname{Ker} B^{(1)} = \{(\mathbf{u},g,h) \in V^{(1)} : b^{(1)}_{reg}(\mathbf{u},\mathbf{v}) + b^{(1)}_{sing}(g,\mathbf{v}) = 0 \ \forall \mathbf{v} \in Q\}.$$

We observe that, since the third variable h does not appear in the characterization of the kernel, it can take any value. We use this property on several occasions throughout the manuscript. Introduce the following problem: given $\ell \in Q'$,

$$\left|\begin{aligned} &\text{Find } \mathbf{v} \in Q \text{ s. t.} \\ &b^{(1)}_{reg}(\mathbf{v},\boldsymbol{\mu}) = \ell(\mu), \quad \forall \boldsymbol{\mu} \in Q. \end{aligned}\right.$$

The problem is well-posed, see [49, proposition 4]. As a straightforward consequence, one finds that the operator $B^{(1)}$ is onto Q'. We can now state the well-posedness result of the stabilized variational formulation.

Theorem 6.4.1. *Let $\rho > 0$. For all $f \in L^2(\Gamma_p \cup \Gamma_n)$, the stabilized mixed formulation (6.56) admits a unique solution.*

Proof. One needs to verify an inf-sup condition for the form $a_\rho^{(1)}$ on the kernel of $B^{(1)}$. The proof mimics the one in [49], and is based on identities (6.54) for $(\mathbf{u},g,h) \in \operatorname{Ker} B^{(1)}$. Recall that $a^{(1)}$ is anti-hermitian, due to its structure (6.20). Regarding the inf-sup condition, we note that for all $(\mathbf{u},g,h) \in \operatorname{Ker} B^{(1)}$,

$$\begin{aligned} \operatorname{Im} a_\rho^{(1)}((\mathbf{u},g,h),(\mathbf{u},g,-h)) &= 2\pi\|h\|^2_{L^2_{\mathfrak{r}}(\Sigma)} + 2\lambda \sum_{j\in\{p,n\}} \left\|u_j + s_g\right\|^2_{L^2(\Gamma_j)} \\ &\quad + \rho\|g\|^2_{H^1(\Sigma)} + \rho\|h\|^2_{H^1(\Sigma)} \\ &\geq \tilde{C}\left(\|\mathbf{u}\|^2_Q + \|g\|^2_{H^1(\Sigma)} + \|h\|^2_{H^1(\Sigma)}\right), \end{aligned}$$

since the norm of $\mathbf{u}$ is controlled by the norm of g. In the statement of [49, proposition 13], with $f = 0$, one shows that $\|\mathbf{u}\|_Q \leq \left\|b^{(2)}_{sing}(g,\cdot)\right\|_{Q'} \lesssim \|g\|_{H^2(\Sigma)}$, while in our case the same argument yields a stability bound $\|\mathbf{u}\|_Q \lesssim \left\|b^{(1)}_{sing}(g,\cdot)\right\|_{Q'} \lesssim \|g\|_{H^1(\Sigma)}$. □

6.4.2 Uniqueness of the solution

The goal of this section is to prove the following result, which shows that the mixed variational formulation (6.22) is injective. Let $\mathtt{A}^{(1)} : V^{(1)} \to \left(V^{(1)}\right)'$ be the operator associated to $a^{(1)}$.

Proposition 6.4.2. $\operatorname{Ker}\mathtt{B}^{(1)} \cap \operatorname{Ker}\mathtt{A}^{(1)} = \{(\mathbf{0},0,0)\}$.

Remark 6.4.3. Let us remark that the injectivity of the non-stabilized mixed formulation does not follow from the identities (6.54) previously obtained in the article [49], which later on served to construct the stabilized mixed formulation. Indeed, applying these identities allows to conclude that $h = 0$, and $(u_j + s_g)\big|_{\Gamma_j} = 0$; the latter, however, does not imply that $\mathbf{u} = 0$ and $g = 0$.

As a matter of fact, given a solution $(\mathbf{u}, g, h) \in V^{(1)}$ of the mixed variational formulation (6.22), the conditions of lemma 6.3.2 are obviously satisfied with g and $\ell = \ell^{(1)}$, and consequently the jump $[\mathbf{u}]_\Sigma$ is well-defined. Therefore, we can reexpress $a^{(1)}$ using the jump $[\mathbf{u}]_\Sigma$.

Remark 6.4.4. In the context of lemma 6.3.13 and in the light of remark 6.2.10, one can rewrite the minimization functional $\mathcal{J}^+(\mathbf{u}, g, h)$ on the kernel of $\mathtt{B}^{(1)}$ with the help of the jump $[\mathbf{u}]_\Sigma$:

$$\mathcal{J}^+(\mathbf{u}, g, h) = \pi\|g-h\|^2_{L^2_\mathtt{r}(\Sigma)} - \operatorname{Im}\left[\langle[\mathbf{u}]_\Sigma, g-h\rangle_\Sigma\right], \quad \forall(\mathbf{u}, g, h) \in \operatorname{Ker}\mathtt{B}^{(1)}.$$

Proof of proposition 6.4.2. Let $(\mathbf{u}, g, h) \in \operatorname{Ker}\mathtt{B}^{(1)} \cap \operatorname{Ker}\mathtt{A}^{(1)}$. Using the identities (6.54), it yields

$$\operatorname{Im} a^{(1)}\left((\mathbf{u}, g, h), (\mathbf{u}, g, -h)\right) = 2\pi\|h\|^2_{L^2_\mathtt{r}(\Sigma)} + 2\lambda \sum_{j\in\{p,n\}} \left\|u_j + s_g\right\|^2_{L^2(\Gamma_j)} = 0,$$

so that $h = 0$. Next, since $(\mathbf{0}, 0, g), (\mathbf{u}, g, 0) \in \operatorname{Ker}\mathtt{B}^{(1)}$, from (6.51) it follows that

$$\begin{aligned} a^{(1)}((\mathbf{u}, g, 0), (0, 0, g)) &= 2i\pi\|g\|^2_{L^2_\mathtt{r}(\Sigma)} - \langle[\mathbf{u}]_\Sigma, g\rangle_\Sigma, \\ a^{(1)}((\mathbf{u}, g, 0), (\mathbf{u}, g, 0)) &= -2i\pi\|g\|^2_{L^2_\mathtt{r}(\Sigma)} + 2i\operatorname{Im}\langle[\mathbf{u}]_\Sigma, g\rangle_\Sigma. \end{aligned}$$

Combining the two identities above yields

$$\operatorname{Im} a^{(1)}\left((\mathbf{u}, g, 0), (\mathbf{u}, g, 2g)\right) = 2\pi\|g\|^2_{L^2_\mathtt{r}(\Sigma)}.$$

Because $(\mathbf{u}, g, 0) \in \operatorname{Ker}\mathtt{A}^{(1)}$, the above implies that $g = 0$. Finally, $(\mathbf{u}, 0, 0) \in \operatorname{Ker}\mathtt{B}^{(1)}$ implies that $b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v} \in Q$. Together with proposition 4 of [49] about the well-posedness of the problem (6.4.1), this implies that $\mathbf{u} = 0$. □

Theorem 6.4.5. *The solution to (6.22), if it exists, is unique.*

Proof. By linearity, it is sufficient that to check that if $(\mathbf{u}, g, h, \boldsymbol{\lambda})$ is a solution of (6.22) with $\ell^{(1)}$, then $(\mathbf{u}, g, h, \boldsymbol{\lambda}) = (\mathbf{0}, 0, 0, \mathbf{0})$. By proposition 6.4.2, $\mathbf{u} = 0$ and $g = h = 0$. Finally, we have for all $(\mathbf{v}, k, l) \in V^{(1)}$

$$0 = \overline{b^{(1)}_{reg}\left((\mathbf{v}, k, l), \boldsymbol{\lambda}\right)} = \left\langle {\mathtt{B}^{(1)}}^\dagger \boldsymbol{\lambda}, (\mathbf{v}, k, l)\right\rangle_{V^{(1)\prime}, V^{(1)}} = \overline{\left\langle \mathtt{B}^{(1)}(\mathbf{v}, k, l), \boldsymbol{\lambda}\right\rangle_{Q', Q}}.$$

We conclude that $\boldsymbol{\lambda} = 0$ because $\mathtt{B}^{(1)}$ is onto Q'. □

6.4.3 Existence of the solution

Proposition 6.4.2 shows that the mixed formulation (6.22) has at most one solution. Therefore, if we construct a solution to this formulation, it will be unique. It is thus reasonable to look for $(\mathbf{u}, g)$ as the limiting absorption solution of the original formulation (6.1), as $\nu \to 0+$. From the content of section 6.2.2, we should expect that $h = g$. Moreover, again, by explicit computations, in this case one can show that the Lagrange multiplier $\boldsymbol{\lambda} = \mathbf{u}\varphi$, see lemma 6.3.13.

However, the above said is not straightforward when comparing the section 6.2.2 and the remark 6.4.4, because of the presence of the extra term involving the jump $[\mathbf{u}]_\Sigma$ in the functional $\mathcal{J}^+$, which did not seem to occur in the original functional $\mathcal{J}^\nu$. This term becomes more apparent in the following proposition.

Proposition 6.4.6. *Let $(\mathbf{u}, g) \in Q \times H^1_{per}(\Sigma)$ be such that $b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}) = \ell^{(1)}(\mathbf{v})$ for all $\mathbf{v} \in Q$. Then, for all $(\mathbf{v}, k, l) \in V^{(1)}$,*

$$a^{(1)}\left((\mathbf{u}, g, g), (\mathbf{v}, k, l)\right) + \overline{b^{(1)}\left((\mathbf{v}, k, l), \mathbf{u}\varphi\right)} = \ell^{(1)}\left((\mathbf{v} + s_{k-l})\,\varphi\right) + \langle[\mathbf{u}]_\Sigma, k - l\rangle_\Sigma. \tag{6.57}$$

As a consequence, if $((\mathbf{u}, g, g), \mathbf{u}\varphi)$ is the solution of (6.22) then $[\mathbf{u}]_\Sigma = 0$. Conversely, if $((\mathbf{u}, g, h), \boldsymbol{\lambda})$ is the solution of (6.22) and $[\mathbf{u}]_\Sigma = 0$, then $g = h$ and $\boldsymbol{\lambda} = \mathbf{u}\varphi$.

Proof. Apply (6.49) to $(\mathbf{u}, g, g)$ yields the identity of the proposition. Then, if $((\mathbf{u}, g, g), \mathbf{u}\varphi)$ is the solution of (6.22), we have $\langle[\mathbf{u}]_\Sigma, k - l\rangle_\Sigma = 0$ for all $k, l \in H^1_{per}(\Sigma)$, i.e., $[\mathbf{u}]_\Sigma = 0$. Conversely, if $((\mathbf{u}, g, h), \boldsymbol{\lambda})$ is the solution of (6.22) and $[\mathbf{u}]_\Sigma = 0$, then $((\mathbf{u}, g, g), \mathbf{u}\varphi)$ is also a solution, so that $g = h$ and $\boldsymbol{\lambda} = \mathbf{u}\varphi$ by uniqueness of the solution, see theorem 6.4.5. □

Notice that the above proposition does not ensure the existence of a solution to (6.22), nor that $(\mathbf{u}, g, g)$ is the solution, since it may happen that the solution of the mixed variational formulation satisfies $[\mathbf{u}]_\Sigma \neq 0$.

Nonetheless, the above shows that the question of the consistency of the mixed variational formulation (6.22) with the original limiting absorption problem (6.1) reduces to the question of the jump of the regular part $[\mathbf{u}]_\Sigma$, where we seek $(\mathbf{u}, g)$ to be the limiting absorption solution.

Theorem 6.4.7. *Let $\left(\mathbf{u}^+, g^+\right)$ be like in assumption 6.1.1. Then $\left(\mathbf{u}^+, g^+, g^+, \mathbf{u}^+\varphi\right)$ is the unique solution of (6.22).*

Proof. It suffices to verify that the limiting absorption solution $\left(\mathbf{u}^+, g^+\right)$ as defined in assumption 6.1.1 satisfies the assumptions of proposition 6.4.6, with $[\mathbf{u}^+]_\Sigma = 0$. This follows from lemma 6.2.1 and proposition 6.3.5. Finally, this is the unique solution by theorem 6.4.5. □

Remark 6.4.8. The mixed formulation takes its origin in the minimization of a functional $\lim\limits_{\nu\to 0+} \mathcal{J}^\nu$. The minimum of this functional is achieved in particular when $h = g^+$, and thus it is unsurprising that the Lagrange multiplier h is chosen as g^+ in the above. As for an explicit form of $\boldsymbol{\lambda} = \mathbf{u}^+\varphi$, it follows from the computations, see also [49, section 5.2] and proposition 6.4.6.

6.4.4 Discussion on the stability of the solution

Since Theorem 6.4.7 ensures the existence of a unique solution of the mixed variational formulation (6.22), the last question to be addressed concerns the stability of this solution with respect to the data $f_\Omega \in L^2(\Omega)$ and $f_\Gamma \in L^2(\Gamma_p \cup \Gamma_n)$.

Before proceeding, let us recall the following proposition. It is a consequence of [4, Theorem 1], which uses mainly pseudo-differential operators.

Proposition 6.4.9. *Let $f_\Omega \in L^2(\Omega)$, $f_\Gamma \in L^2(\Gamma_p \cup \Gamma_n)$ and $\mathbf{u} \in Q$ be the unique solution of*

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in Q \text{ s.t.} \\ b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) = \ell^{(1)}(\mathbf{v}), \text{ for all } \mathbf{v} \in Q. \end{array} \right. \tag{6.58}$$

Then, $\partial_y\left(\alpha \partial_y \mathbf{u}\right) \in L^2(\Omega)$ and there is a constant $C > 0$ such that $\left\|\partial_y\left(\alpha\partial_y \mathbf{u}\right)\right\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Gamma_p \cup \Gamma_n)}$.

Let us state the main result of this section.

Proposition 6.4.10. *Let $(\mathbf{u}, g, h) \in V^{(1)}$ be the solution of (6.22). There exist $C > 0$ independent of $(\mathbf{u}, g, h)$, $f_\Omega \in L^2(\Omega)$ and $f_\Gamma \in L^2(\Gamma_p \cup \Gamma_n)$ be such that*

$$\|h\|^2_{L^2_\tau(\Sigma)} \leq C\left(\|f_\Omega\|_{L^2(\Omega)} + \|f_\Gamma\|_{L^2(\Gamma_p\cup\Gamma_n)}\right)\left(\|g\|_{L^2_\tau(\Sigma)} + \|h\|_{L^2_\tau(\Sigma)}\right).$$

The inequality above derive usually from the coercivity of the problem. However, in the case of our mixed problem, the norm associated with the space $V^{(1)}$ does not appear in the inequality (6.55), and the proposition is not obvious. Then, a natural corollary of the proposition above is the following.

Corollary 6.4.11. *Let $\left(\mathbf{u}^+, g^+\right) \in Q \times H^1_{per}(\Sigma)$ be as in theorem 6.4.7. There is $C > 0$ independent of $\left(\mathbf{u}^+, g^+\right)$, $f_\Omega \in L^2(\Omega)$ and $f_\Gamma \in L^2(\Gamma_p \cup \Gamma_n)$ be such that*

$$\|g^+\|_{L^2_\tau(\Sigma)} \leq C\left(\|f_\Omega\|_{L^2(\Omega)} + \|f_\Gamma\|_{L^2(\Gamma_p\cup\Gamma_n)}\right).$$

The proof of the proposition 6.4.10 relies on the following lemma.

Lemma 6.4.12. *Let $\mathbf{u} \in Q$ be such that $b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) = \ell^{(1)}(\mathbf{v})$. Then, there exist a constant $C > 0$ such that for all $k \in H^1_{per}(\Sigma)$*

$$\left|\langle[\mathbf{u}]_\Sigma, k\rangle_\Sigma\right| \leq C\left(\|f_\Omega\|_{L^2(\Omega)} + \|f_\Gamma\|_{L^2(\Gamma_p\cup\Gamma_n)}\right)\|k\|_{L^2_\tau(\Sigma)}.$$

Proof. The expression of the jump (6.26) gives here:

$$\langle[\mathbf{u}]_\Sigma, k\rangle_\Sigma = \overline{b^{(1)}_{sing}(k, \mathbf{u})} + 2i\lambda \sum_{j\in\{p,n\}} \int_{\Gamma_j} u_j \overline{s_k} \mathrm{ds} - \ell^{(1)}\left(s_k\right).$$

Since $\partial_y\left(\alpha\partial_y \mathbf{u}\right) \in L^2(\Omega)$ by Proposition 6.4.9, integrating by parts $\overline{b^{(1)}_{sing}(k, \mathbf{u})}$ in the y-direction yields:

$$\overline{b^{(1)}_{sing}(k, \mathbf{u})} = \sum_{j\in\{p,n\}} \int_{\Omega_j} \left[\left(-\partial_y\left(\alpha\partial_y u_j\right)\right)\overline{s_k} + u_j\overline{\left(-\partial_x\left(\alpha\partial_x s_k\right) - \omega^2 s_k\right)}\right]\mathrm{d}\mathbf{x} + \int_{\Gamma_j} u_j\overline{(\alpha\partial_n s_k + i\lambda s_k)}\mathrm{ds}.$$

The inequality of the lemma then follows easily. ☐

Proof of proposition 6.4.10. Let $\mathbf{u}_\ell \in Q$ be such that $b^{(1)}_{reg}(\mathbf{u}_\ell, \mathbf{v}) = \ell^{(1)}(\mathbf{v})$ for all $\mathbf{v} \in Q$ (see [49, Proposition 4]). Then, consider $\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_\ell$. We have $(\mathbf{u}_0, g, -h) \in \operatorname{Ker} \mathrm{B}^{(1)}$. Therefore, using identity (6.55) yields

$$\begin{aligned} 2\pi \|h\|^2_{L^2_\Upsilon(\Sigma)} &\leq \left| a^{(1)}\left((\mathbf{u}_0, g, h), (\mathbf{u}_0, g, -h)\right)\right| \\ &= \left| \ell^{(1)}\left(\left(\mathbf{u}_0 + s_{g+h}\right)\varphi\right) - a^{(1)}\left((\mathbf{u}_\ell, 0, 0), (\mathbf{u}_0, g, -h)\right)\right|. \end{aligned}$$

Then, Lemma 6.3.13 links the last quantity with the jump of $\mathbf{u}_\ell$:

$$a^{(1)}\left((\mathbf{u}_\ell, 0, 0), (\mathbf{u}_0, g, -h)\right) = \ell^{(1)}\left(\left(\mathbf{u}_0 + s_{g+h}\right)\varphi\right) + \langle [\mathbf{u}_\ell], g + h\rangle_\Sigma,$$

so that

$$2\pi \|h\|^2_{L^2_\Upsilon(\Sigma)} \leq \left| \langle [\mathbf{u}_\ell], g + h\rangle_\Sigma \right|.$$

Finally, using Proposition 6.4.9 with Lemma 6.4.12 gives the result. □

Remark 6.4.13. Given the solution $\mathbf{u}$ of (6.58), Lemma 6.4.12 shows in particular that its jump belongs to $L^2_\Upsilon(\Sigma)$.

CHAPTER 7

Simplified variational formulation and numerical experiments

Contents

7.1 Simplified variational formulation

Let $\Omega = (-a, a) \times (0, L)$, with the notations of Chapter 4, and u^+ be the limiting absorption solution of

$$\left| \begin{array}{ll} \text{find } u \in L^2(\Omega) \text{ such that} & \\ -\operatorname{div}(\alpha \nabla u) - \omega^2 u = f_\Omega & \text{in } \Omega, \\ \alpha \partial_n u + i\lambda u = f_\Gamma & \text{on } \Gamma_n \cup \Gamma_p, \\ u(x,0) = u(x,L), \quad (\alpha \partial_y) u(x,0) = (\alpha \partial_y) u(x,L), & x \in (-a,\ a), \end{array} \right.$$

where supp $f \cap \Sigma = \emptyset$. In the view of the results from Chapters 5 and 6, u^+ satisfies the decomposition of Assumption 6.1.1, i.e., $u^+ = u^+_{reg} + u^+_{sing}$. Moreover, Propositions 5.3.5 and 6.3.5 show that the regular part has a vanishing jump through the interface, and Lemma 6.24 provides an explicit formula of the action of this jump as an element of $\left(H^1_{per}(\Sigma)\right)'$:

$$\langle [\mathbf{u}], k \rangle_{\left(H^1_{per}(\Sigma,\mathbf{r})\right)', H^1_{per}(\Sigma,\mathbf{r})} = \overline{b^{(1)}_{sing}(k, \mathbf{u})} + b^{(1)}_{sing}(g, s_k) + 2i\lambda \sum_{j \in \{p,n\}} \int_{\Gamma_j} u_j \overline{s_k} \mathrm{d}s - \ell_\infty(s_k). \tag{7.1}$$

Therefore, this naturally leads to the following problem:

$$\left| \begin{array}{l} \text{find } (\mathbf{u}, g) \in Q \times H^1_{per}(\Sigma) \text{ such that, for any } (\mathbf{v}, k) \in Q \times H^1_{per}(\Sigma), \\ b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}) + \langle [\mathbf{u}]_\Sigma, k \rangle_\Sigma = \ell^{(1)}(\mathbf{v}). \end{array} \right. \tag{7.2}$$

Refer to (6.4) (respectively (6.5)) for the definition of $b^{(1)}_{reg}$ (resp. $b^{(1)}_{sing}$). Then, using the expression (7.1) of the jump, this leads to the following simplified variational formulation:

$$\left| \begin{array}{l} \text{find } (\mathbf{u}, g) \in Q \times H^1_{per}(\Sigma) \text{ such that, for any } (\mathbf{v}, k) \in Q \times H^1_{per}(\Sigma), \\ b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}) + \widetilde{b^{(1)}_{sing}}(\mathbf{u}, k) + b^{(1)}_{sing}(g, s_k) = \ell^{(1)}(\mathbf{v} + s_k), \end{array} \right. \tag{7.3}$$

where we introduced

$$\widetilde{b^{(1)}_{sing}}(\mathbf{u}, k) := \overline{b^{(1)}_{sing}(k, \mathbf{u})} + 2i\lambda \sum_{j \in \{p,n\}} \int_{\Gamma_j} u_j \overline{s_k} \mathrm{ds}. \tag{7.4}$$

The mixed variational formulation proposed in Chapter 6 and the above variational formulation are equivalent in the following sense.

Theorem 7.1.1. *Let $(\mathbf{u}, g) \in Q \times H^1_{per}(\Sigma)$ be such that $((\mathbf{u}, g, g), \mathbf{u}\varphi) \in V^{(1)} \times Q$ is the solution of the mixed problem (6.22). Then, $(\mathbf{u}, g)$ is a solution of the problem (7.3).*

Reciprocally, let $(\mathbf{u}, g) \in Q \times H^1_{per}(\Sigma)$ be a solution of the simplified problem (7.3). Then, $((\mathbf{u}, g, g), \mathbf{u}\varphi) \in V^{(1)} \times Q$ is the solution of the mixed problem (6.22).

Proof. Let $((\mathbf{u}, g, g), \mathbf{u}\varphi) \in V^{(1)} \times Q$ be a solution to (6.22), and $k \in H^1_{per}(\Sigma)$. Then, according to proposition 6.4.6, we have

$$\ell^{(1)}((\mathbf{0} + s_{0-k})\varphi) = a^{(1)}((\mathbf{u}, g, g), (\mathbf{0}, 0, k)) + \overline{b((0, 0, k), \mathbf{u}\varphi)} = \langle [\mathbf{u}]_\Sigma, k \rangle_\Sigma - \ell^{(1)}(s_k \varphi).$$

Hence, $\langle [\mathbf{u}]_\Sigma, k \rangle_\Sigma = 0$ for all $k \in H^1_{per}(\Sigma)$ and $(\mathbf{u}, g)$ is a solution of (7.3).

Reciprocally, let $(\mathbf{u}, g) \in Q \times H^1_{per}(Q)$ be a solution to (7.3). Considering $k = 0$, we have for all $\mathbf{v} \in Q$

$$b^{(1)}_{reg}(\mathbf{u}, \mathbf{v}) + b^{(1)}_{sing}(g, \mathbf{v}) = \ell^{(1)}(\mathbf{v}).$$

Then, one can apply Proposition 6.4.6,

$$a^{(1)}((\mathbf{u}, g, g), (\mathbf{v}, k, l)) + \overline{b^{(1)}((\mathbf{v}, k, l), \mathbf{u}\varphi)} = \ell^{(1)}((\mathbf{v} + s_{k-l})\varphi) + \langle [\mathbf{u}]_\Sigma, k - l \rangle_\Sigma, \qquad \forall (\mathbf{v}, k, l) \in V^{(1)}. \tag{7.5}$$

On the other hand, taking $\mathbf{v} = 0$ in (7.3) proves that $[\mathbf{u}]_\Sigma = 0$. Finally, plugging it in (7.5) ends the proof. □

The last theorem, complemented by Theorem 6.4.5, has the following corollary.

Corollary 7.1.2. *The solution to (7.3), if it exists, is unique.*

We discretize the two problems (7.2) and (7.3) in Section 7.3. However, at this point no proof of the well-posedness are available.

7.2 Numerical experiments for the mixed variational formulations

The numerical experiments consist in checking the convergence rate of the quantity of interest, namely the regular part $\mathbf{u} \in Q$, the amplitude of the singular part $g \in H^1_{per}(\Sigma)$ or the jump of the regular part $[\mathbf{u}]_\Sigma$, for several test cases. We set $\Omega = (-1, 1)^2$. Simple test cases can be computed

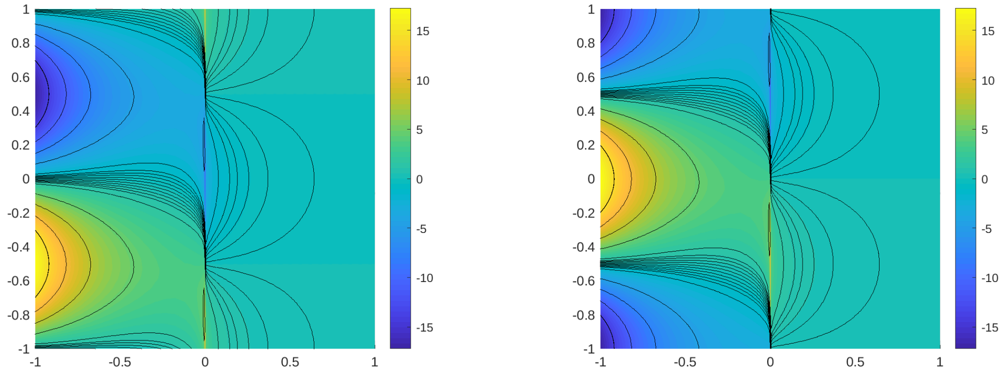


Figure 7.1: Real part (left) and imaginary part (right) of $u(x, y) = -e^{i\pi y} K_0(\pi x)$. One may observe that all level set lines start and finish on the interface at $x = 0$, because of the presence of the logarithmic plus jump singularity.

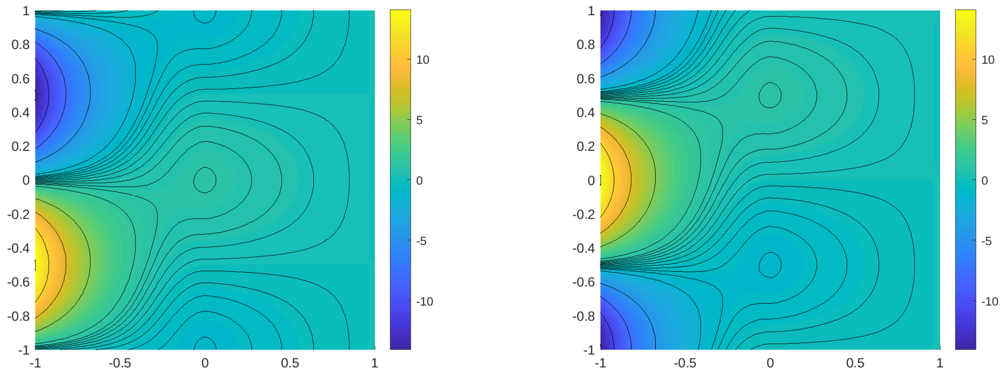


Figure 7.2: Real part (left) and imaginary part (right) of the regular part of $u(x, y) = -e^{i\pi y} K_0(\pi x)$. One may observe that all level set lines are continuous through the interface at $x = 0$, which indicates that it obviously has a vanishing jump.

from Chapter 5, with $\alpha(x, y) = x$. In that case, $\psi_k(y) = e^{i\pi y}$, $\lambda_k = k\pi$. Then, typical solutions, for $k \geq 1$, read:

u^+	g	u^+_{reg}
1	0	1
$\mathsf{S}(x)$	1	0
$e^{ik\pi y} I_0(k\pi x)$	0	$e^{ik\pi y} I_0(k\pi x)$
$-e^{ik\pi y} K_0(k\pi x)$	$e^{ik\pi y}$	$e^{ik\pi y}(-K_0(k\pi x) - \mathsf{S}(x))$

(7.6)

Notice that these solutions solve $\operatorname{div}(x\nabla u) = 0$ in Ω. We may also use the solutions above with $\omega \neq 0$, by computing the associated source term f_Ω.

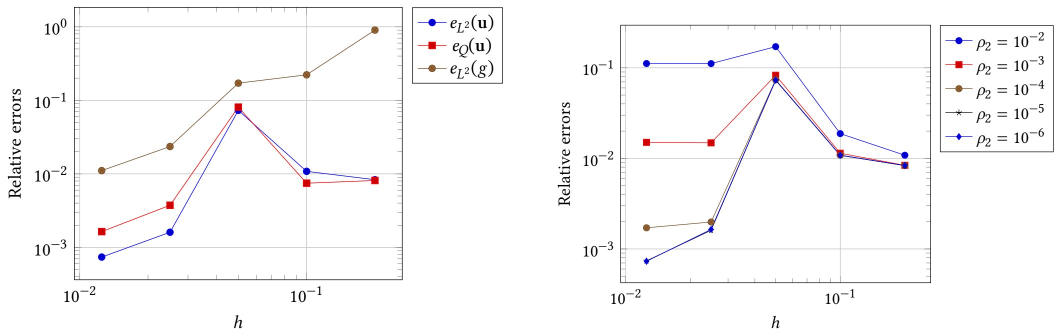
We first provide a few additional comments on the discretization proposed in [49]. Then, we discretize the method proposed in Chapter 6. Finally, we experiment the discretization of the simplified problem (7.3).

7.2.1 Mixed variational formulation in $H^2_{per}(\Sigma)$

Recall the conforming discretization of (4.17), $V^{(2)}_{h_1,h_2} = Q_{h_1} \times H^2_{h_2} \times H^2_{h_2}$, with

$$Q_{h_1} = \{v_{h_1} \in Q \,:\; v_{h_1}|_K \in P_1(K), \text{ for all } K \in \mathcal{T}^{\Omega}_{h_1}\},$$
$$H^2_{h_2} = \{p_{h_2} \in H^2_{per}(\Sigma) \,:\; p_{h_2}|_K \in H_m(K), \text{ for all } K \in \mathcal{T}^{\Sigma}_{h_2}\},$$

see Section 4.2.2 for more details. Recall that the approximation of the solutions $u^+ = 1$ and $u^+ = \mathtt{S}(x)$ does not converge. The situation is slightly different for the singular solution $u = -e^{i\pi y}K_0(\pi x)$, where one observes a monotonic decrease of the relative error in $L^2(\Sigma)$-norm for the singular coefficient g, see Figure 7.3a. However, convergence for the regular part is not obvious in $\|\cdot\|_Q$ norm and $L^2(\Omega)$-norm, see again figure 7.3a. In Figure 7.3b, we provide error curves depending on the choice of the parameters $\rho_2 = \mu_2$. As expected, the convergence stagnates for larger values of ρ_2, and we also see that decreasing ρ_2 from 10^{-5} to 10^{-6} has no visible effect on the error curves.



(a) $\rho_2 = \mu_2 = 10^{-5}$.

(b) Relative L^2 errors on the regular part with different stabilization parameters $\rho_2 = \mu_2$.

Figure 7.3: Relative errors for $u = -K_0(\pi x)e^{i\pi y}$.

7.2.2 Mixed variational formulation in $H^1_{per}(\Sigma)$

Below, we study the numerical approximation of (6.22), or of its stabilized version (6.56). In order to test the accuracy of the method described in Chapter 6, we reproduce the experiments conducted in section 4.2.2, replacing the discrete space $V^{(2)}_{h,h}$ with the discrete space $V^{(1)}_h = Q_h \times H^1_h \times H^1_h$,

$$Q_h = \left\{v_h \in Q \,:\; v_h|_K \in P_1(K), \text{ for all } K \in \mathcal{T}^{\Omega}_h\right\},$$
$$H^1_h = \left\{g_h \in H^1_{per}(\Sigma) \,:\; g_h|_K \in P_1(K), \text{ for all } K \in \mathcal{T}^{\Sigma}_h\right\}.$$

Above, $\mathcal{T}^{\Omega}_h$ is a triangulation of Ω that is conforming with respect to the interface Σ, and $\mathcal{T}^{\Sigma}_h$ is a triangulation of Σ, both with meshsize h, however we do not impose that $\mathcal{T}^{\Sigma}_h$ is the trace of $\mathcal{T}^{\Omega}_h$ on Σ. Different triangulations $\mathcal{T}^{\Omega}_h$ are used, which are all symmetric with respect to the interface Σ, and we choose uniform triangulations $\mathcal{T}^{\Sigma}_h$. Like in section 4.2.2, elements of Q_h have no matching condition at the interface. The relative errors e_{L^2} and e_Q are the same as those defined in section 4.2.2.

The code is written in `FreeFem++` [32]. Whereas 2D HCT finite elements were used to discretize the singular part g_h in [49], we now use P_1 Lagrange finite elements on the interface Σ. [1]

We consider the same setting as in section 4.2.2, where $\alpha(x,y) = x$ and $\omega = 0$, with a purely regular solution $u^+(x,y) = 1$, and with a singular solution $u^+(x,y) = -K_0(\pi x)e^{i\pi y}$ where the singular coefficient is equal to $g(y) = e^{i\pi y}$.

The cutoff function we use is the $\mathscr{C}^1(\Omega)$ function $\varphi(x,y) = \frac{1}{2}(1+\cos(2\pi x))\,\mathbb{1}_{|x|<0.5}$. Notice that φ is prescribed equal to 1 only on the interface in the experiments (compare with definition 4.2.2). The approach in section 6.2 and its theoretical justification in section 6.4 remain valid also for this choice of φ. It has been also checked numerically that the results presented below do not depend on the choice of φ provided that $\varphi \in \mathscr{C}^1(\Omega)$, $\partial_y \varphi = 0$, $\varphi|_\Sigma = 1$ and is compactly supported in $x \in (-a, a)$.

Influence of the triangulation. The design of $\mathscr{T}_h^\Omega$ has a noticeable influence on the numerical stability of the method. In particular, we observe that the method is unstable with an unstructured triangulation $\mathscr{T}_h^{\Omega,unstr}$, see Figure 7.4c. This instability occurs even though $\mathscr{T}_h^{\Omega,unstr}$ is symmetric. On the same figure, we see that one can stabilize the method by using a structured triangulation $\mathscr{T}_h^{\Omega,str}$, as long as structuring occurs on the geometrical support of φ (see Figure 7.4a).

Numerical convergence and stabilization parameter. We observe on Figure 7.5 that the method using H_h^1 performs significantly better than the one using $H_{h_2}^2$, compare with Figure 7.3. Let us remark that, as before, when computing $e_{L^2}(\mathbf{u})$ and $e_Q(\mathbf{u})$, we exclude cells that are adjacent to the interface.

In Figure 7.5a, we notice that the errors increase when decreasing h: this is likely due to the fact that already at the most coarse discretization the machine precision had been reached, and for finer discretizations we can observe the effects of the round-off errors in cells close to the interface. On the other hand, in Figure 7.5b, we observe that the approximation converges numerically towards the solution $u = -e^{i\pi y}K_0(\pi x)$ with decent rates with respect to the meshsize.

We observe on Figure 7.6 that the relative error on $\mathbf{u}_{\rho_1,h}$ decreases proportionally to the stabilization parameter ρ_1. Moreover, one can still compute the discrete solution for $\rho_1 = 0$, and it gives the same results as those obtained for $\rho_1 = 10^{-6}$. The latter is due to the fact that, for the chosen mesh sizes, the error due to stabilization is negligible for "small" values of ρ_1. On the other hand, we observe that one can compute solutions in absence of stabilization. This is because the non-stabilized problem (6.22) is injective, see proposition 6.4.2, and so is its conforming discretization and hence the discrete solution exists.

Experiments with more complicated version of α. Now, we take the same geometry, with $\alpha(x,y) = x(1+\frac{1}{2}\cos(\pi y)) + \frac{x^2}{2}\cos(\pi y)$, $\omega = 0$ and data $f_\Gamma = i\mathbb{1}_{\Gamma_p} - i\mathbb{1}_{\Gamma_n}$. Remark that α depends on y non-trivially, and thus the exact solution is not known. According to section 6.3.2, given the limiting absorption solution $(\mathbf{u}, g)$, which has a vanishing jump according to proposition 6.3.5, $(\mathbf{u}, g, g, \mathbf{u}\varphi)$ is equal to the solution $(\mathbf{u}, g, h, \boldsymbol{\lambda})$ of (6.22). Therefore, we expect that $\mathbf{u}_h\varphi - \boldsymbol{\lambda}_h$, $g_h - h_h$

[1] To our knowledge, in `FreeFem++`, it is not possible to combine 1D and 2D discretizations. So, in practice, to represent elements of H_h^1, we use P_1 Lagrange finite elements on a single elongated cell in the x-direction, and as many cells in the y-direction as there are in $\mathscr{T}_h^\Sigma$, with periodic conditions in x.

and finally $[\mathbf{u}_h]_\Sigma$ go to zero in the appropriate norms $\|\cdot\|_\bullet$ when the triangulations $\mathcal{T}_h^\Sigma$ and $\mathcal{T}_h^{\Omega,str}$ are refined. We will refer to the norms of these quantities as indicators. First, we observe that value of each norm $\|\boldsymbol{\lambda}_h\|_Q$, $\|g_h\|_{H^1(\Sigma)}$ and $\|h_h\|_{H^1(\Sigma)}$, stabilizes quickly with respect to the mesh size h. Hence, in Figure 7.7, we can report relative errors defined by

$$d_\bullet(\mathbf{u}\varphi, \boldsymbol{\lambda}) = \frac{\|\mathbf{u}_h\varphi - \boldsymbol{\lambda}_h\|_\bullet}{\|\ \boldsymbol{\lambda}_h\|_\bullet}, \text{ or } d_\bullet(g, h) = \frac{\|g_h - h_h\|_\bullet}{\|\ g_h\|_\bullet}.$$

In figures 7.7a, we see that the first two indicators converge nicely to 0. Regarding the last indicator (the norm of the jump $[\mathbf{u}]_\Sigma$), we observe in Figure 7.7b that it converges in $L^2(\Sigma)$-norm very slowly. This indicates that the jump must be handled carefully with this discretization.

7.3 Numerical experiments for the simplified variational formulations

We are now interested in the discretization of the problems form (7.2), (7.3). The discrete space is $Q_h \times H_h^1$ where Q_h and H_h^1 are defined in the Section 7.2.2. As before, the triangulations $\mathcal{T}_h^\Omega$ is conforming with respect to the interface Σ. Moreover, we impose that $\mathcal{T}_h^\Sigma$ is the trace of the triangulation $\mathcal{T}_h^\Omega$, and $\mathcal{T}_h^\Omega$ is symmetric with respect to Σ. We do not impose continuity conditions at the interface between $u_{p,h}$ and $u_{n,h}$. Notice that we do not take into accounts the cells which touch the interface when we measure the volume errors.

Since the well-posedness of the continuous counterparts of the numerical experiments is not proved, we focus on few cases. In particular, we take $\alpha(x, y) = x$. Two functions are approximated, as $u(x, y) = -e^{i\pi y} K_0(\pi x)$ and $u(x, y) = e^{i\pi y}(1 - x^2)$. The former is singular, and the latter is regular. Finally, several values of ω are taken.

We first discretize the sesquilinear form (7.3). Then, we discretize the same problem, but with the jump handled numerically, i.e., the problem (7.2).

7.3.1 First simplified variational formulation

The first experiment consists in discretizing directly the problem (7.3). This leads to linear system $\boldsymbol{B}U_{h,h} = L$ where:

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_p & 0 & \boldsymbol{B}_{\Sigma_g,p} \\ 0 & \boldsymbol{B}_n & \boldsymbol{B}_{\Sigma_g,n} \\ \boldsymbol{B}_{p,\Sigma_g} & \boldsymbol{B}_{n,\Sigma_g} & \boldsymbol{B}_{\Sigma_g} \end{pmatrix}, \qquad U_{h,h} = \begin{pmatrix} U_{p,h} \\ U_{n,h} \\ G_h \end{pmatrix}, \qquad L = \begin{pmatrix} L_p \\ L_n \\ L_{\Sigma_g} \end{pmatrix}.$$

Once again, the mesh has a great influence on the convergence : we observe that the discretizations do not converge with unstructured symmetric meshes, whereas they do with symmetric structured meshes, see Figure 7.8. Then, the relative errors with a structured mesh decrease with the same rate as in previous part, see Figure 7.9. Moreover, the jump converges toward zero in spite of the absence of explicit constraint during the discretization process.

7.3.2 Second simplified variational formulation

The second discretization consists in discretizing the problem (7.2) without expanding the definition of the jump. This now leads to a linear system $\boldsymbol{B}U_{h,h} = L$ where:

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_p & 0 & \boldsymbol{B}_{\Sigma_g,p} \\ 0 & \boldsymbol{B}_n & \boldsymbol{B}_{\Sigma_g,n} \\ \gamma_{p,\Sigma} & -\gamma_{n,\Sigma} & 0 \end{pmatrix}, \qquad U_{h,h} = \begin{pmatrix} U_{p,h} \\ U_{n,h} \\ G_h \end{pmatrix}, \qquad L = \begin{pmatrix} L_p \\ L_n \\ 0 \end{pmatrix}.$$

This method ensures a zero jump of the regular part with a magnitude to the machine precision by the discretization of the following sesquilinear form:

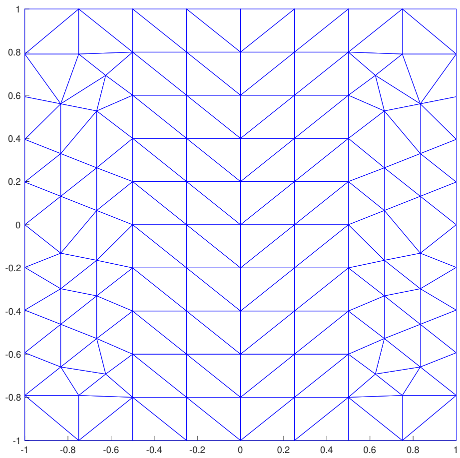
$$\int_{\Sigma} \left(\gamma_{p,\Sigma} u_p - \gamma_{n,\Sigma} u_n\right) \bar{k} \mathrm{d}s, \qquad \text{with } (u_p, u_n) \in Q, k \in H^1_{per}(\Sigma).$$

It is possible since we use $H^1(\Omega_p) \times H^1(\Omega_n)$ conforming finite elements. Then, one may observe in Figure 7.10 the convergence of the numerical solutions, for both the structured mesh $\mathscr{T}_h^{\Omega,str}$ and the unstructured mesh $\mathscr{T}_h^{\Omega,unstr}$. However, the convergence rates are clearly deteriorated in the case of the unstructured mesh. Finally, we compare in Figure 7.11 the approximations of $u(x,y) = -e^{i\pi y} K_0(\pi x)$, which has a singular part (7.6), and $u(x,y) = (1-x^2)e^{i\pi y}$. We observe that taking $\omega \neq 0$ seems to have a negligible influence on the convergence. Moreover, we also see that the absence of a singular part in the solution seems to improve the convergence.

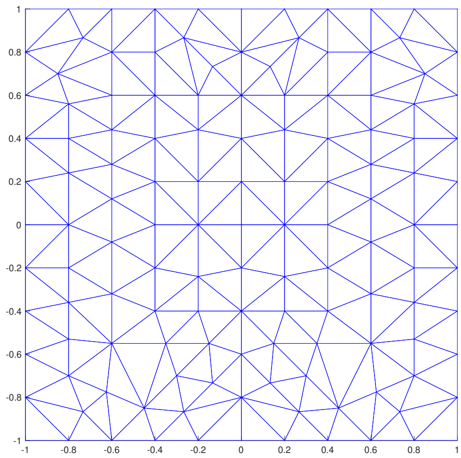
7.4 Conclusions

From the standpoint of Chapter 6, we proposed in this chapter a new variational formulation (7.3) to solve the degenerate PDE introduced in Chapter 4. Numerical experiments have been conducted on the variational formulations proposed in this thesis. The first observation, compared to the existing literature, is that the discretization spaces Q_h and H^1_h seem to be well-fitted to the problem. Then, the second observation is that the choice of meshes has an important influence on the convergence of the approximations : the main two issues are whether they are structured or not, and matching at the interface. The mixed variational formulation studied in Chapter 6 requires structured meshes on the support of a cutoff function φ. On the other hand, the matching condition of the meshes at the interface has an influence on how the vanishing jump condition is taken into account for the second simplified formulation (7.2). Importantly, taking into account the jump condition as in §7.3.2 gives the best results, and moreover, the approximation even with unstructured meshes converges numerically. Finally, the computations are faster for the second simplified formulation, compared to the other ones.

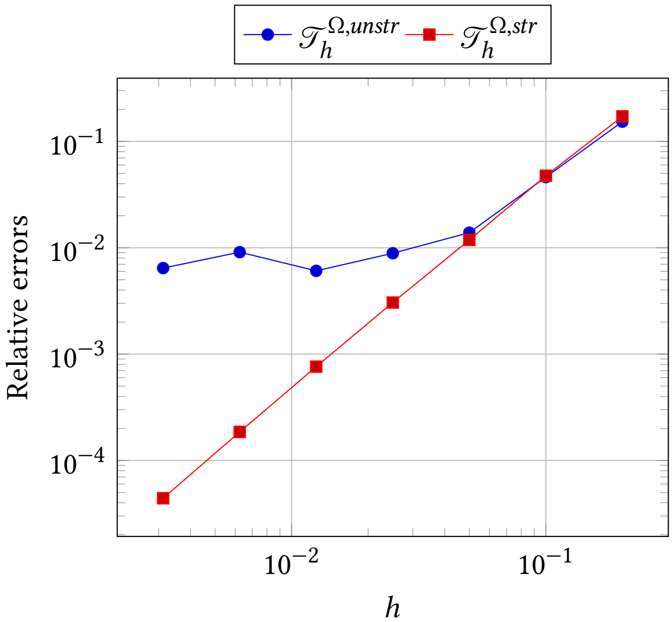
We thank Anouk Nicolopoulos for providing the code she initially developed in her PhD thesis [50], which was a great help for the numerical experiments.



(a) $\mathcal{T}_h^{\Omega,str}$ with $h = 0.25$.

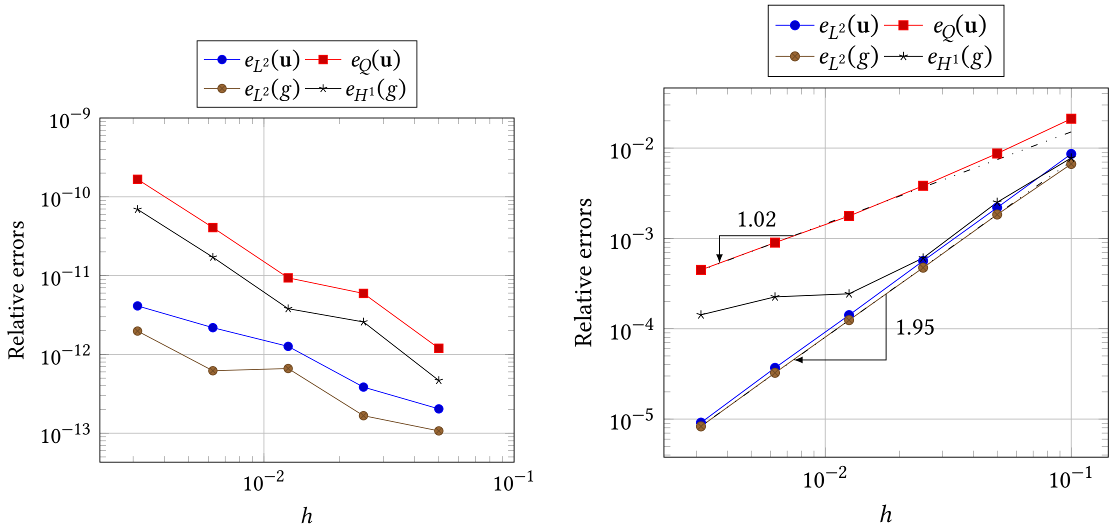


(b) $\mathcal{T}_h^{\Omega,unstr}$ with $h = 0.25$.



(c) $e_{L^2}(\mathbf{u})$ errors for $u = -K_0(\pi x)e^{i\pi y}$ with $\rho_1 = 10^{-5}$.

Figure 7.4: Influence of structuring $\mathcal{T}_h^{\Omega}$ on the stability of the method.



(a) For $u = 1$ with $\rho_1 = 10^{-3}$.

(b) For $u = -K_0(\pi x)e^{i\pi y}$ with $\rho_1 = 10^{-6}$.

Figure 7.5: Relative errors with $\mathscr{T}_h^{\Omega,str}$.

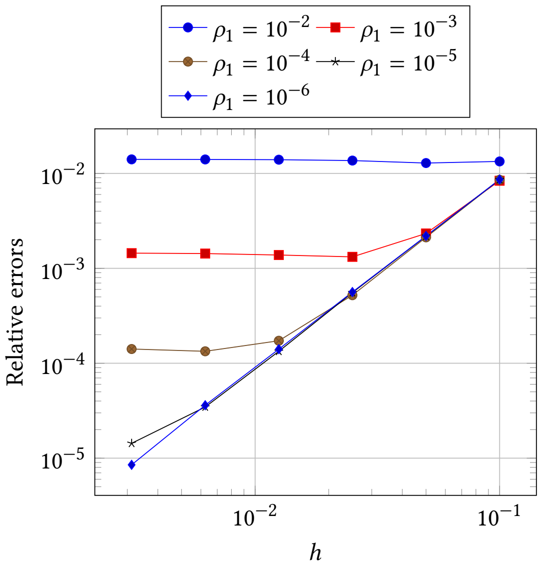
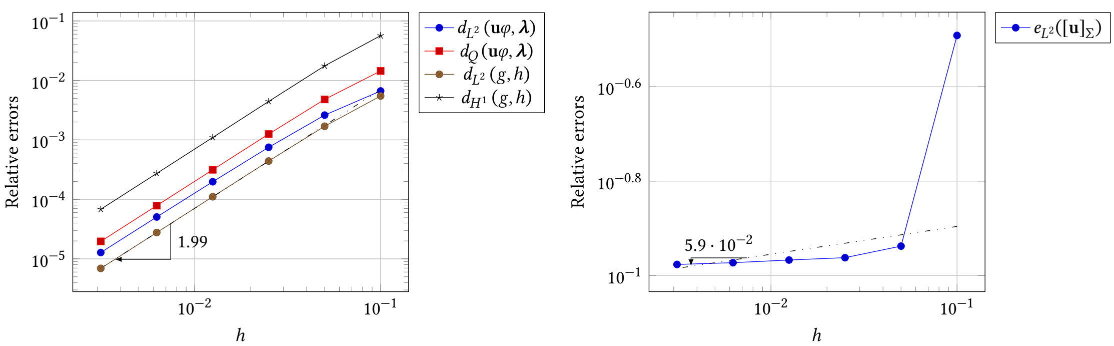


Figure 7.6: $e_{L^2}(\mathbf{u})$ errors with $\mathscr{T}_h^{\Omega,str}$ for $u = -e^{i\pi y}K_0(\pi x)$.



(a) Difference between $\mathbf{u}_h\varphi$ and $\boldsymbol{\lambda}_h$, and g and h.

(b) Convergence of the jump $[\mathbf{u}_h]$.

Figure 7.7: Experiment with $\alpha(x, y)$ that depends on y non-trivially.

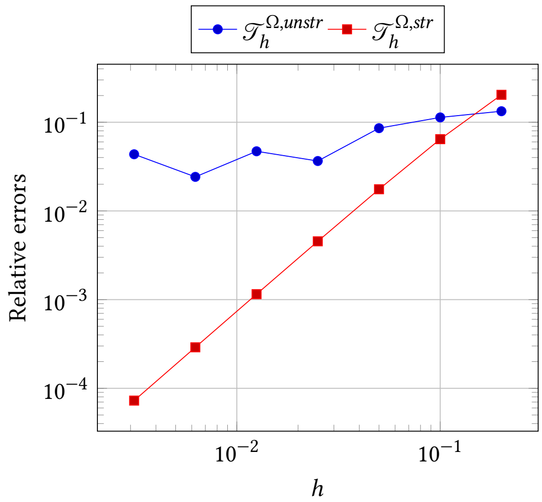


Figure 7.8: Relative error $e_{L^2}(\mathbf{u})$ for the first simplified formulation with structured meshes $\mathcal{T}_h^{\Omega,str}$ and unstructured meshes $\mathcal{T}_h^{\Omega,unstr}$, and $u = -K_0(\pi x)e^{i\pi y}$.

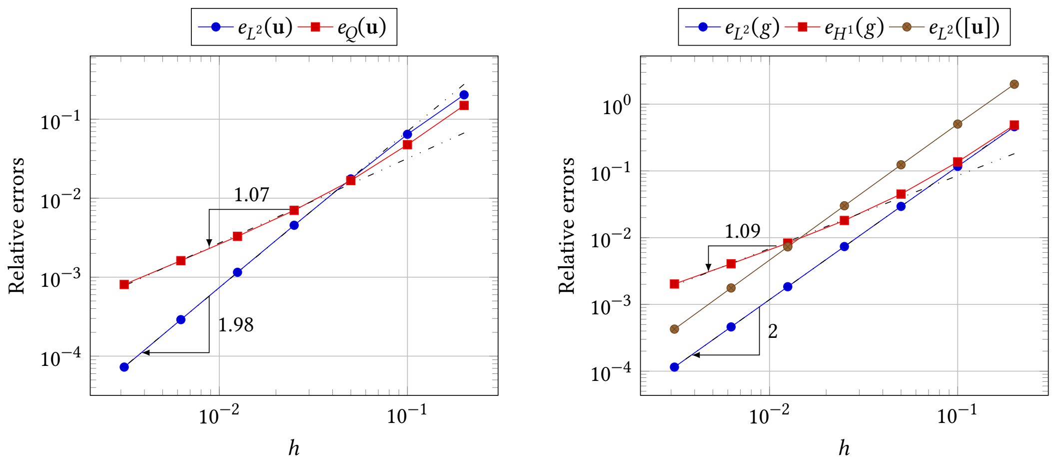
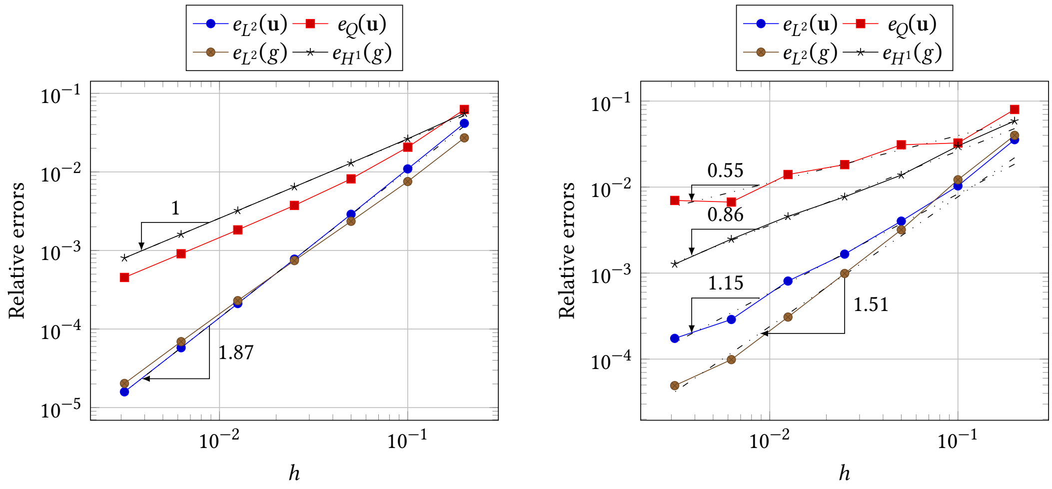
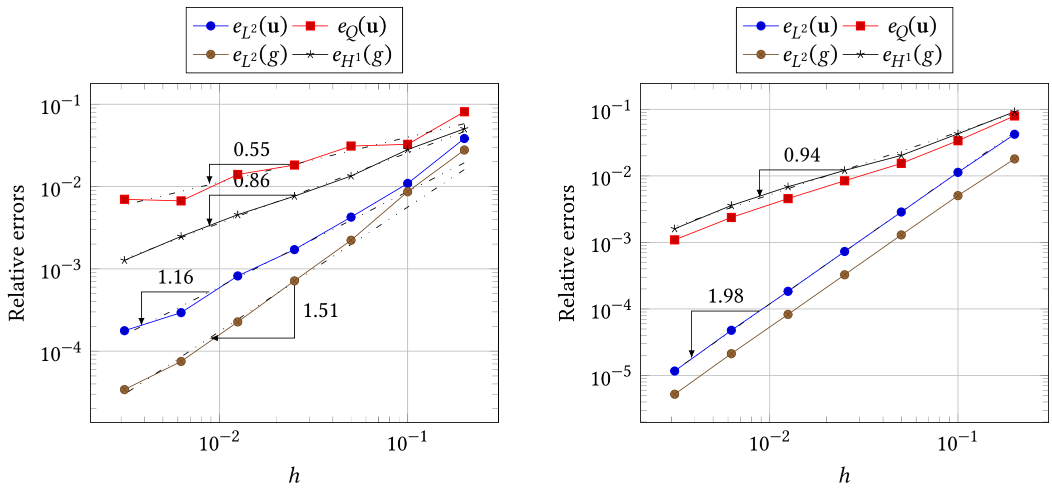


Figure 7.9: Relative errors for the simplified variational formulation with $\mathcal{T}_h^{\Omega,str}$, and $u = -K_0(\pi x)e^{i\pi y}$.



(a) With $\mathcal{T}_h^{\Omega,str}$. (b) With $\mathcal{T}_h^{\Omega,unstr}$.

Figure 7.10: Relative errors for the formulation with a discretized jump with $u = -e^{i\pi y}K_0(\pi x)$ and $\omega = 0$.



(a) For $u = -e^{i\pi y}K_0(\pi x)$. (b) For $u = (1 - x^2)e^{i\pi y}$.

Figure 7.11: Relative errors for the formulation with a discretized jump with $\mathcal{T}_h^{\Omega,unstr}$ and $\omega = 1$.

Conclusions and perspectives

Conclusions

In this work, we have conducted the analysis of two Maxwell's problems with sign-changing coefficients. Both these problems catch specific difficulties of the model of cold plasma from which they have been derived.

The first part I of this work proposes a mathematical analysis of electromagnetic wave propagation in a hyperbolic metamaterial. We proved the existence of smooth solution via the exhibition of a Newton potential. Moreover, radiation conditions in the spirit of Silver-Müller radiation conditions were established, which guarantee the uniqueness of the solution.

In the second part II, a degenerate PDE has been studied in detail. In particular, a limiting absorption principle theorem has been established in Chapter 5. More importantly, the ansatz of the singular part of the solution proposed in dimension 1 in [35], has been thoroughly refined in the case of a 2D interface and a source term away from this interface. This allowed to enhance in Chapter 6 the results obtained in [49]. In particular, we proved that the proposed method is consistent with the limiting absorption principle. Moreover, we proved in Chapter 5 and 6, using different techniques, that the regular part of the limiting absorption solution has a vanishing jump through the interface. This has naturally led in Chapter 7 to consider simple but efficient discrete problems and their continuous counterparts.

Perspectives

The first part of the thesis can obviously be considered as a preliminary work. The continuation of this work could consist in investigating the well-posedness of the problem expressed as boundary integral equation. In that case, the very first question to address is on which kind of domain this problem is well-posed. Can the domain be bounded or semi-bounded ? Are there constraints on the shape or the regularity of the boundary ? For the time being, these are open questions, but we refer to [29, 28] for interesting works in this direction.

The second part of the thesis is ended by two short-term prospects. On one hand, the Assumption 5.5.1 may lead to another interesting framework in which the variational formulation should be considered. On the other hand, the work done has shown the relevance of the simplified variational formulation 7.3, which justifies its further numerical analysis. Aside from this, only a scalar function α has been studied in detail. Therefore, it should be possible to replace it by a tensor of the form $\mathbb{\alpha} = \alpha\mathbb{H}$, with $\mathbb{H}$ an elliptic hermitian matrix. Finally, another prospective is the model described in Section 2.4.2 for which an even more general form of the tensor α is suggested. Of course, all these questions are open for the full 3D Maxwell system, whose mathematical and numerical analysis is the ultimate goal of this research.

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Simulation et analyse des équations de Maxwell avec changement de signe dans un plasma froid

Mots clés : Métamatériau hyperbolique, équations de Maxwell, condition de rayonnement, principe d'absorption limite, équations aux dérivées partielles dégénérées, solutions singulières, résonance « lower-hybrid » du plasma

Résumé : De nos jours, les plasmas sont principalement utilisés à des fins industrielles. L'un des exemples les plus fréquemment cités d'utilisation industrielle est la production d'énergie électrique via des réacteurs nucléaires à fusion. Pour contenir le plasma correctement à l'intérieur du réacteur, un champ magnétique est imposé en arrière-plan, et la densité et la température du plasma doivent être précisément contrôlées. Cela est effectué en envoyant des ondes électromagnétiques à des fréquences et dans des directions spécifiques en fonction des caractéristiques du plasma.
La première partie de cette thèse de doctorat est consacrée à l'étude du modèle du plasma avec un fort champ magnétique en arrière-plan, ce qui correspond à un métamatériau hyperbolique. L'objectif est d'étendre les résultats existant en 2D au cas 3D et de dériver une condition de radiation. Nous introduisons une séparation des champs électriques et magnétiques ressemblant à la décomposition TE et TM habituelle, puis nous présentons quelques résultats sur les deux problèmes résultants. Les résultats sont dans un état très partiel et constituent un brouillon approximatif sur le sujet.
La deuxième partie étudie l'EDP dégénérée associée aux ondes résonantes « lower-hybrid » dans le plasma. Le problème aux limites associé est bien posé dans un cadre variationnel « naturel ». Cependant, ce cadre n'inclut pas le comportement singulier présenté par les solutions physiques obtenues via le principe d'absorption limite. Ce comportement singulier est important du point de vue physique car il induit le chauffage du plasma mentionné précédemment. Un des résultats clés de cette deuxième partie est la définition d'une notion de saut à travers l'interface à l'intérieur du domaine, ce qui permet de caractériser la décomposition de la solution d'absorption limite en parties régulières et singulières.

Title : Simulation and analysis of sign-changing Maxwell's equations in cold plasma

Keywords : Hyperbolic metamaterial, Maxwell's equations, radiation condition, limiting absorption principle, degenerate partial differential equations, singular solutions, lower-hybrid plasma resonance

Abstract : Nowadays, plasmas are mainly used for industrial purpose. One of the most frequently cited example of industrial use is electric energy production via fusion nuclear reactors. Then, in order to contain plasma properly inside the reactor, a background magnetic field is imposed, and the density and temperature of the plasma must be precisely controlled. This is done by sending electromagnetic waves at specific frequencies and directions depending on the characteristics of the plasma.
The first part of this PhD thesis consists in the study of the model of plasma in a strong background magnetic field, which corresponds to a hyperbolic metamaterial. The objective is to extend the existing results in 2D to the 3D-case and to derive a radiation condition. We introduces a splitting of the electric and magnetic fields resembling the usual TE and TM decomposition, then, it gives some results on the two resulting problems. The results are in a very partial state, and constitute a rough draft on the subject.
The second part consists in the study of the degenerate PDE associated to the lower-hybrid resonant waves in plasma. The associated boundary-value problem is well-posed within a "natural" variational framework. However, this framework does not include the singular behavior presented by the physical solutions obtained via the limiting absorption principle. Notice that this singular behavior is important from the physical point of view since it induces the plasma heating mentioned before. One of the key results of this second part is the definition of a notion of weak jump through the interface inside the domain, which allows to characterize the decomposition of the limiting absorption solution into a regular and a singular parts.

Institut Polytechnique de Paris
91120 Palaiseau, France