Mathematical Models and Methods in Applied Sciences

Direct computation of stresses in planar linearized elasticity

PHILIPPE G. CIARLET

Department of Mathematics, City University of Hong Kong,
83 Tat Chee Avenue, Kowloon, Hong Kong
mapgc@cityu.edu.hk

PATRICK CIARLET, JR.

Laboratoire POEMS, UMR 2706 CNRS/ENSTA/INRIA, Ecole Nationale Supérieure de Techniques Avancées, 32, boulevard Victor, 75015 Paris, France
patrick.ciarlet@ensta.fr

Received
Revised
Communicated by (xxxxxxxxxx)

Dedicated with admiration to Professor Robert Dautray on the occasion of his 80th birthday.

Given a simply-connected domain Ω in $\mathbb{R}^2$, consider a linearly elastic body with $\overline{\Omega}$ as its reference configuration, and define the Hilbert space

$$\mathbb{E}(\Omega) = \{\mathbf{e} = (e_{\alpha\beta}) \in \mathbb{L}^2_s(\Omega); \partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11} = 0 \text{ in } H^{-2}(\Omega)\}.$$

Then we recently showed that the associated pure traction problem is equivalent to finding a 2×2 matrix field $\boldsymbol{\epsilon} = (\epsilon_{\alpha\beta}) \in \mathbb{E}(\Omega)$ that satisfies

$$j(\boldsymbol{\epsilon}) = \inf_{\mathbf{e} \in \mathbb{E}(\Omega)} j(\mathbf{e}), \quad \text{where } j(\mathbf{e}) = \frac{1}{2}\int_\Omega A_{\alpha\beta\sigma\tau} e_{\sigma\tau} e_{\alpha\beta} dx - \ell(\mathbf{e}),$$

where $(A_{\alpha\beta\sigma\tau})$ is the elasticity tensor, and ℓ is a continuous linear form over $\mathbb{E}(\Omega)$ that takes into account the applied forces. Since the unknown stresses $(\sigma_{\alpha\beta})$ inside the elastic body are then given by $\sigma_{\alpha\beta} = A_{\alpha\beta\sigma\tau} e_{\sigma\tau}$, this minimization problem thus directly provides the stresses.

We show here how the above Saint Venant compatibility condition $\partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11} = 0$ in $H^{-2}(\Omega)$ can be exactly implemented in a finite element space $\mathbb{E}^h$, which uses "edge" finite elements in the sense of J. C. Nédélec. We then establish that the unique solution $\boldsymbol{\epsilon}^h$ of the associated discrete problem, viz., find $\boldsymbol{\epsilon}^h \in \mathbb{E}^h$ such that

$$j(\boldsymbol{\epsilon}^h) = \inf_{\mathbf{e}^h \in \mathbb{E}^h} j(\mathbf{e}^h),$$

converges to $\boldsymbol{\epsilon}$ in the space $\mathbb{L}^2_s(\Omega)$.

We emphasize that, by contrast with a mixed method, only the approximate stresses are computed in this approach.

Keywords: Linearized elasticity, finite element methods, edge finite element, computation of stresses

AMS Subject Classification: 49N10, 65N30, 74B05

1. Introduction

The accurate computation of stresses inside an elastic body is a challenging problem. The main objective of this two-part article is to describe and analyze a new approach to this problem, based on a new formulation of linearized elasticity problems, recently proposed by the same authors; see Ref. 16.

Let $\mathbb{S}^3$ denote denote the space of all symmetric matrices of order three. Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. To fix ideas, consider a homogeneous, isotropic, linearly elastic body with Lamé constants $\lambda > 0$ and $\mu > 0$, with $\overline{\Omega}$ as its reference configuration, and subjected to applied body forces, of density $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ in its interior and of density $\mathbf{g} \in \mathbf{L}^{4/3}(\Gamma)$ on its boundary Γ. Given any matrix $\mathbf{e} = (e_{ij}) \in \mathbb{S}^3$, let the matrix $\mathbf{Ae} \in \mathbb{S}^3$ be defined by

$$\mathbf{Ae} = \lambda(\mathrm{tr}\,\mathbf{e})\mathbf{I} + 2\mu\mathbf{e}.$$

Finally, let $\boldsymbol{\epsilon} : \mathbf{e}$ denote the inner product of two matrices $\boldsymbol{\epsilon}$ and $\mathbf{e}$.

Then the associated *pure traction problem of three-dimensional linearized elasticity* classically consists in finding a *displacement vector field* $\mathbf{u} \in \mathbf{H}^1(\Omega)$ that satisfies

$$J(\mathbf{u}) = \inf_{\mathbf{v}\in\mathbf{H}^1(\Omega)} J(\mathbf{v}), \text{where } J(\mathbf{v}) = \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{\nabla}_s\mathbf{v} : \boldsymbol{\nabla}_s\mathbf{v}dx - L(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathbf{H}^1(\Omega),$$

where

$$L(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx + \int_\Gamma \mathbf{g}\cdot\mathbf{v}d\Gamma,$$

and

$$\boldsymbol{\nabla}_s\mathbf{v} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v}^T + \boldsymbol{\nabla}\mathbf{v}) \in L^2(\Omega; \mathbb{S}^3)$$

denotes the *linearized strain tensor field* associated with any vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$.

Clearly, $\inf_{\mathbf{v}\in\mathbf{H}^1(\Omega)} J(\mathbf{v}) > -\infty$ only if the applied body forces satisfy the *compatibility condition* $L(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{R}(\Omega)$, where

$$\mathbf{R}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \boldsymbol{\nabla}_s\mathbf{v} = \mathbf{0} \text{ in } \Omega\} = \{\mathbf{v} = \mathbf{a} + \mathbf{b}\wedge\mathbf{id}_\Omega;\ \mathbf{a}\in\mathbb{R}^3, \mathbf{b}\in\mathbb{R}^3\}$$

denotes the space of *infinitesimal rigid displacements* of the set Ω.

It is well known (see, e.g., Chapter 3 in Duvaut & Lions[22]) that this compatibility condition is also sufficient for the existence of solutions to the above minimization problem, as a consequence of *Korn's inequality*. Besides, such solutions are unique up to the addition of any vector field $\mathbf{v} \in \mathbf{R}(\Omega)$.

In Ciarlet & Ciarlet, Jr.[16], a new approach to the above pure traction problem has been proposed that consists in *considering the linearized strain tensor as the "primary" unknown instead of the displacement itself* (note that, back in 1976, Antman[4] already proposed that the "full" strain tensor be analogously considered as the new unknown in minimization problems arising in three-dimensional nonlinear elasticity). What follows is a brief outline of this approach (see also Section 2, where this approach is described in details in the two-dimensional case).

First, one characterizes those symmetric 3×3 matrix fields $\mathbf{e} = (e_{ij}) \in L^2(\Omega; \mathbb{S}^3)$ that can be written as $\mathbf{e} = \boldsymbol{\nabla}_s \mathbf{v}$ for some vector fields $\mathbf{v} \in \mathbf{H}^1(\Omega)$, again uniquely defined up to infinitesimal rigid displacements. As shown in Ciarlet & Ciarlet, Jr.[16], this is possible if the set Ω is simply-connected and the components e_{ij} of the field $\mathbf{e}$ satisfy the following *weak form of the classical Saint Venant compatibility conditions*:

$$R_{ijkl}(\mathbf{e}) = \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega)$$

for all $i, j, k, l \in \{1, 2, 3\}$. The proof crucially hinges on the following *H^{-2}-version of a classical lemma of Poincaré*: *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Let $h_k \in H^{-1}(\Omega)$ be distributions that satisfy $\partial_l h_k = \partial_k h_l$* in $H^{-2}(\Omega)$. *Then there exists a function $p \in L^2(\Omega)$, unique up to an additive constant, such that $h_k = \partial_k p$* in $H^{-1}(\Omega)$. See also Amrouche, Ciarlet and Ciarlet, Jr[2], Ciarlet, Ciarlet, Jr., Geymonat and Krasucki[19] and Geymonat and Krasucki[24,25] for various extensions. Note that another, although of a different nature, characterization of such matrix fields $\mathbf{e} = (e_{ij})$ has also been given by Ting[35]. See also Amrouche, Ciarlet, Gratie and Kesavan[3].

Let $\mathbb{E}(\Omega)$ denote the closed subspace of $L^2(\Omega; \mathbb{S}^3)$ formed by the 3×3 symmetric matrix fields that satisfy the above weak Saint Venant compatibility conditions. Then *the mapping*

$$\mathcal{F} : \mathbf{e} \in \mathbb{E}(\Omega) \to \dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega) = \mathbf{H}^1(\Omega)/\mathbf{R}(\Omega),$$

where $\dot{\mathbf{v}}$ is such that $\boldsymbol{\nabla}_s \dot{\mathbf{v}} = \mathbf{e}$, is an isomorphism between the Hilbert spaces $\mathbb{E}(\Omega)$ *and* $\dot{\mathbf{H}}^1(\Omega)$ (a property that incidentally yields a *new proof of the classical three-dimensional Korn's inequality*; see Section 4 in Ref. 16.).

Thanks to the isomorphism $\mathcal{F}$, the pure traction problem of linearized elasticity may thus be equivalently recast in terms of the new unknown $\mathbf{e} \in L^2(\Omega; \mathbb{S}^3)$ as the following *constrained minimization problem*: One now seeks a 3×3 *matrix field* $\boldsymbol{\epsilon} \in \mathbb{E}(\Omega)$ *that satisfies*

$$j(\boldsymbol{\epsilon}) = \inf_{\mathbf{e} \in \mathbb{E}(\Omega)} j(\mathbf{e}), \text{where } j(\mathbf{e}) = \frac{1}{2} \int_\Omega \mathbf{A}\mathbf{e} : \mathbf{e} dx - \ell(\mathbf{e}),$$

and the continuous linear form $\Lambda : \mathbb{E}(\Omega) \to \mathbb{R}$ is defined by $\ell = L \circ \boldsymbol{\mathcal{F}}$. As expected, one can further show that $\boldsymbol{\epsilon} = \boldsymbol{\nabla}_s \mathbf{v}$.

Since the unknown *stress tensor field* $\boldsymbol{\sigma} = (\sigma_{ij})$ inside the linearly elastic body under consideration is given by the *constitutive equation*

$$\boldsymbol{\sigma} = \mathbf{A}\mathbf{e},$$

the above quadratic minimization problem thus provides a *direct way of computing the stresses* σ_{ij} inside the body.

Classically, such stresses are approximated by a *mixed method*, i.e., one that approximates the *saddle point* $(\boldsymbol{\sigma}, \mathbf{u})$ of a *Lagrangian* defined over the product space

$\mathbb{H}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega)$, where

$$\mathbb{H}(\mathbf{div};\Omega) = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{S}^3); \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\},$$

and $\mathbf{div}\,\boldsymbol{\tau} = (\partial_j \tau_{ij})_{i=1}^3$. To this end, finite element subspaces of $\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{L}^2(\Omega)$ must be constructed that satisfy the classical *inf-sup condition* of Babuška[12] and Brezzi[13].

In this direction, note that *mixed finite element methods for linearized elasticity* have been studied in particular by Arnold, Brezzi and Douglas, Jr.[6], Arnold, Douglas, Jr. and Gupta[7], Stenberg[33,34], Arnold and Falk[8], Morley[26], Arnold and Winther[10,11], and Arnold and Awanou[5].

While there exist efficient finite element methods that satisfy the inf-sup condition in the "vector-scalar" case (see Raviart and Thomas[31] and Brezzi, Douglas, Jr. and Marini[14]), the "matrix-vector" case considered here is more challenging, basically because of the required *symmetry* of the stress tensor. In this direction, see the illuminating discussions about such questions and about the related "differential complexes" in Arnold, Falk and Winther[9] and Eastwood[23].

The objective of this two-part article is to describe and analyze a *direct finite element approximation of the minimization problem* $\inf_{\mathbf{e}\in\mathbb{E}(\Omega)} j(\mathbf{e})$, first for planar elasticity in the present paper, then for three-dimensional elasticity in Ref. 18. This means in particular that *only the approximate strains*, or equivalently *only the approximate stresses* (by means of the constitutive equation), are computed, by contrast with mixed methods, where the approximate displacements are simultaneously computed.

To this end, we show how the Saint Venant compability conditions

$$R_{ijkl}(\mathbf{e}) = 0 \text{ in } H^{-2}(\Omega) \text{ for all } i,j,k,l \in \{1,2,3\},$$

or their two-dimensional counterpart, viz.,

$$\mathrm{curl}\,\mathbf{curl}\,\mathbf{e} = \partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11} = 0 \text{ in } H^{-2}(\Omega),$$

can be *exactly satisfied in a finite element subspace* $\mathbb{E}^h$ of $\mathbb{E}(\Omega)$. This means that *the elements* $\mathbf{e}^h \in \mathbb{E}^h$ *satisfy exactly*

$$R_{ijkl}(\mathbf{e}^h) = 0 \text{ in } H^{-2}(\Omega)$$

for all $i,j,k,l \in \{1,2,3\}$ *in dimension three*, or

$$\mathrm{curl}\,\mathbf{curl}\,\mathbf{e}^h = 0 \text{ in } H^{-2}(\Omega)$$

in dimension two.

More specifically, given a triangulation of a polygonal or polyhedral domain Ω, we consider here discrete matrix-valued functions $\mathbf{e}^h \in \mathbb{E}^h$ whose components are *piecewise constant.* It then turns out that, in order to satisfy the inclusion $\mathbb{E}^h \subset \mathbb{E}(\Omega)$, i.e., in order to exactly satisfy either of the above compatibility conditions, *the degrees of freedom that define the elements* $\mathbf{e}^h \in \mathbb{E}^h$ *must be supported by the edges* (see Theorem 3.1) *and they must satisfy specific compatibility conditions* (see

Theorems 3.2 and 3.3). The associated finite elements thus provide examples of *edge finite elements* in the sense of Nédélec[27,28] (in this respect, see also the recent paper of Rapetti[30]).

Finally, we show that the *discrete problem*, which naturally consists in seeking a discrete matrix field $\boldsymbol{\epsilon}^h \in \mathbb{E}^h$ that satisfies

$$j(\boldsymbol{\epsilon}^h) = \inf_{\mathbf{e}^h \in \mathbb{E}^h} j(\mathbf{e}^h),$$

has one and only solution (Theorem 4.1), and we establish the *convergence* of the method (see Theorem 4.2).

Note that this approach is in a sense the "matrix-analog" of the approximation of the Stokes problem by means of the divergence-free finite elements of Crouzeix & Raviart[21] (although theirs are non-conforming, whereas ours are not).

One potential interest of this approach is thus the possibility of providing efficient numerical schemes that directly compute the *stresses inside an elastic body*. It could also pave the way for devising analogous schemes that could be likewise applied to *elastic plates* and *elastic shells*. Incidentally, note that the soundness of this kind of approach is also corroborated in the literature about shells, where it is known under the name of "*intrinsic equations of shell theory*"; in this direction, see notably Opoka & Pietraszkiewicz[29].

The results of this article have been announced in Ref. 17.

2. Intrinsic planar linearized elasticity

Greek indices range over the set $\{1, 2\}$, and the summation convention with respect to repeated Greek indices is used. The Euclidean inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and norm of $\mathbf{a} \in \mathbb{R}^2$ are denoted $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{a}|$. The matrix inner product $\epsilon_{\alpha\beta} e_{\alpha\beta}$ of two matrices $\boldsymbol{\epsilon} = (\epsilon_{\alpha\beta})$ and $\mathbf{e} = (e_{\alpha\beta})$ of order two is denoted $\boldsymbol{\varepsilon} : \boldsymbol{e}$, and $\mathbb{S}^2$ denotes the set of all symmetric matrices of order two. The restriction of a mapping f to a set X is denoted $f|_X$. If V is a vector space and R is a subspace of V, the quotient space of V modulo R is denoted V/R and the equivalence class of $v \in V$ modulo R is denoted $\dot{v}$. The duality pairing between a topological vector space X and its dual X' is denoted ${}_{X'}\langle \cdot, \cdot \rangle_X$.

Let x_α denote the coordinates of a point $x \in \mathbb{R}^2$, let $\partial_\alpha := \partial/\partial x_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial x_\alpha \partial x_\beta$. Given a smooth enough vector field $\mathbf{v} = (v_\alpha)$, we define the 2×2 matrix field $\boldsymbol{\nabla}\mathbf{v} := (\partial_\alpha v_\beta)$. Spaces of matrix-valued functions are denoted by capital roman letters, such as $\mathbb{L}^2_s(\Omega), \mathbb{E}(\Omega), \mathbb{E}^h$, etc.

A *domain* in $\mathbb{R}^2$, is an open, bounded, and connected subset of $\mathbb{R}^2$ whose boundary is Lipschitz-continuous. Let Ω be a domain in $\mathbb{R}^2$. Given any vector field $\mathbf{v} \in \mathbf{H}^1(\Omega) := H^1(\Omega; \mathbb{R}^2)$, viewed here as a *displacement field* of the set Ω, let

$$\boldsymbol{\nabla}_s \mathbf{v} := \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v}^T + \boldsymbol{\nabla}\mathbf{v}) \in \mathbb{L}^2_s(\Omega) := L^2(\Omega; \mathbb{S}^2) \qquad (2.1)$$

denote its associated *symmetrized gradient matrix field, also called linearized strain tensor field*. Let

$$\mathbf{R}(\Omega) := \{\mathbf{r} \in \mathbf{H}^1(\Omega);\ \boldsymbol{\nabla}_s \mathbf{r} = 0 \text{ in } \Omega\} \tag{2.2}$$

denote the space of *infinitesimal rigid displacement fields* of the set Ω. It is well known that $\mathbf{r} = (r_\alpha) \in \mathbf{R}(\Omega)$ if and only if there exist constants c_1, c_2, and δ such that

$$r_1(x) = c_1 - \delta x_2 \text{ and } r_2(x) = c_2 + \delta x_1 \text{ for all } x = (x_1, x_2) \in \Omega.$$

Consider a *planar linearly elastic body* with $\overline{\Omega}$ as its reference configuration in the absence of applied forces. The elastic material constituting the body is characterized by its *elasticity tensor* $\mathbf{A} = (A_{\alpha\beta\sigma\tau})$, whose elements

$$A_{\alpha\beta\sigma\tau} = A_{\beta\alpha\sigma\tau} = A_{\sigma\tau\alpha\beta} \in L^\infty(\Omega)$$

are such that there exists a constant $\alpha > 0$ such that

$$\alpha \mathbf{t} : \mathbf{t} \leq \mathbf{A}(x)\mathbf{t} : \mathbf{t} \quad \text{for almost all } x \in \Omega \text{ and all } \mathbf{t} \in \mathbb{S}^3, \tag{2.3}$$

where $(\mathbf{A}(x)\mathbf{t})_{\alpha\beta} := A_{\alpha\beta\sigma\tau}(x) t_{\sigma\tau}$. Then the associated *pure traction problem* classically consists in finding a *displacement field* $\dot{\boldsymbol{u}} \in \dot{\mathbf{H}}^1(\Omega) := \mathbf{H}^1(\Omega)/\mathbf{R}(\Omega)$ that satisfies

$$J(\dot{\mathbf{u}}) = \inf_{\dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega)} J(\dot{\mathbf{v}}) \quad \text{with} \qquad J(\dot{\mathbf{v}}) := \frac{1}{2} \int_\Omega \mathbf{A} \boldsymbol{\nabla}_s \dot{\mathbf{v}} : \boldsymbol{\nabla}_s \dot{\mathbf{v}} dx - L(\dot{\mathbf{v}}), \tag{2.4}$$

where $L : \mathbf{H}^1(\Omega) \to \mathbb{R}$ is a continuous linear form that takes into account the *applied forces* and satisfies the *compatibility condition* $L(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathbf{R}(\Omega)$. It is well known that the minimization problem (2.4) has one and only one solution, thanks to *Korn's inequality*; see Ref. 22.

By contrast, the *intrinsic approach* to the same problem consists in directly seeking the *linearized strain tensor field* $\boldsymbol{\varepsilon} := \boldsymbol{\nabla}_s \dot{\mathbf{u}} \in \mathbb{L}^2_s(\Omega)$, which thus becomes the *primary unknown*, instead of the displacement field in the classical approach. The mathematical justification of such an approach crucially hinges on the following results (cf. Theorems 2.1, 2.2, and 2.3), which were proved in Ref. 16 in the three-dimensional case, and which can be proved by the same methods as in *ibid* in the two-dimensional case considered here.

The curl of any vector field $\mathbf{v} = (v_\alpha) \in \mathcal{D}'(\Omega; \mathbb{R}^2)$ is given by

$$\text{curl } \mathbf{v} = \partial_1 v_2 - \partial_2 v_1 \in \mathcal{D}'(\Omega),$$

and the **curl** of any matrix field $\mathbf{e} = (e_{\alpha\beta}) \in \mathcal{D}'(\Omega; \mathbb{S}^2)$ is given by

$$\mathbf{curl}\ \mathbf{e} = \begin{pmatrix} \partial_1 e_{12} - \partial_2 e_{11} \\ \partial_1 e_{22} - \partial_2 e_{21} \end{pmatrix} \in \mathcal{D}'(\Omega; \mathbb{S}^2).$$

Hence

$$\text{curl } \mathbf{curl}\ \mathbf{e} = \partial_{11} e_{22} - 2\partial_{12} e_{12} + \partial_{22} e_{11} \in \mathcal{D}'(\Omega) \text{ if } \mathbf{e} = (e_{\alpha\beta}) \in \mathcal{D}'(\Omega; \mathbb{S}^2).$$

Theorem 2.1. *Let Ω be a simply-connected domain in $\mathbb{R}^2$ and let $\mathbf{e} = (e_{\alpha\beta}) \in \mathbb{L}^2_s(\Omega)$ be a tensor field that satisfies*

$$\mathrm{curl}\,\mathbf{curl}\,\mathbf{e} = \partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11} = 0 \text{ in } H^{-2}(\Omega).$$

Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\boldsymbol{\nabla}_s\mathbf{v} = \mathbf{e}$ in $\mathbb{L}^2_s(\Omega)$, and all the other solutions $\widetilde{\mathbf{v}}$ to the equation $\boldsymbol{\nabla}_s\widetilde{\mathbf{v}} = \mathbf{e}$ are of the form

$$\widetilde{\mathbf{v}} = \mathbf{v} + \mathbf{r} \text{ for some } \mathbf{r} \in \mathbf{R}(\Omega),$$

where $\mathbf{R}(\Omega)$ is the space defined in (2.2).

Theorem 2.2. *Let Ω be a simply-connected domain in $\mathbb{R}^3$. Define the space*

$$\mathbb{E}(\Omega) := \{\mathbf{e} \in \mathbb{L}^2_s(\Omega);\ \mathrm{curl}\,\mathbf{curl}\,\mathbf{e} = \mathbf{0} \text{ in } H^{-2}(\Omega)\}, \tag{2.5}$$

and, given any $\mathbf{e} \in \mathbb{E}_s(\Omega)$, let $\dot{\mathbf{v}} = \boldsymbol{\mathcal{F}}(\mathbf{e})$ denote the unique element in the space $\dot{\mathbf{H}}^1(\Omega)$ that satisfies $\mathbf{e} = \boldsymbol{\nabla}_s\dot{\mathbf{v}}$ (Theorem 2.1)*. Then the linear mapping*

$$\boldsymbol{\mathcal{F}} : \mathbb{E}(\Omega) \to \dot{\mathbf{H}}^1(\Omega)$$

defined in this fashion is an isomorphism between the Hilbert spaces $\mathbb{E}(\Omega)$ and $\dot{\mathbf{H}}^1(\Omega)$.

Thanks to the isomorphism $\boldsymbol{\mathcal{F}}$, the minimization problem (2.4) can be recast as another minimization problem (see (2.6) below), *where $\boldsymbol{\nabla}_s\dot{\mathbf{u}}$ is now the primary unknown.*

Theorem 2.3. *Let Ω be a simply-connected domain in $\mathbb{R}^2$. Then the minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbb{E}(\Omega)$ such that*

$$j(\boldsymbol{\varepsilon}) = \inf_{\mathbf{e}\in\mathbb{E}(\Omega)} j(\mathbf{e}), \text{ where } j(\mathbf{e}) := \frac{1}{2}\int_\Omega \mathbf{A}\mathbf{e} : \mathbf{e}dx - \ell(\mathbf{e}) \text{ with } \ell := L \circ \boldsymbol{\mathcal{F}}, \tag{2.6}$$

has one and only solution $\boldsymbol{\varepsilon}$. Besides,

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_s\dot{\mathbf{u}},$$

where $\dot{\mathbf{u}}$ is the unique solution to the minimization problem (2.4).

3. A curl **curl-free finite element space**

In what follows, we use standard definitions and notations for finite element methods (see, e.g., Ciarlet[15]). In particular, $P_k(T;\mathbb{X})$ denotes the space of all mappings from a subset T of $\mathbb{R}^2$ into a vector space $\mathbb{X}$, whose components are restrictions to T of polynomials of degree $\leq k$ in the variables x_α.

To begin with, we describe a *triangular finite element*, which provides a new type of *edge finite element*, in the sense of Nédélec[27,28]. The length element is denoted dl.

Theorem 3.1. *Let T be a non-degenerate triangle with edges $s_i, 1 \leq i \leq 3$. Given any edge s_i of T, let $\boldsymbol{\tau}^i$ denote a unit vector parallel to s_i, and let the degrees of freedom $d_i, 1 \leq i \leq 3$, be defined as*

$$d_i(\mathbf{e}) := \int_{s_i} \boldsymbol{\tau}^i \cdot \mathbf{e}\boldsymbol{\tau}^i dl \quad \text{for all } \mathbf{e} \in P_0(T; \mathbb{S}^2). \tag{3.1}$$

Then the set $\{d_i; 1 \leq i \leq 3\}$ is $P_0(T; \mathbb{S}^2)$-unisolvent, i.e., a tensor field $\mathbf{e} \in P_0(T; \mathbb{S}^2)$ is uniquely defined by the three numbers $d_i(\mathbf{e}), 1 \leq i \leq 3$.

Proof. The space $P_0(T; \mathbb{S}^2)$ is of dimension three and there are three degrees of freedom d_i. It thus suffices to show that, if $\mathbf{e} = (e_{\alpha\beta}) \in P_0(T; \mathbb{S}^2)$ satisfies $d_i(\mathbf{e}) = 0, 1 \leq i \leq 3$, then $\mathbf{e} = \mathbf{0}$ on T. Since the functions $e_{\alpha\beta}$ are constant on T, and the length of each edge s_i is > 0, the relations $d_i(\mathbf{e}) = 0$ are equivalent to the linear system

$$\tau^i_\alpha \tau^i_\beta e_{\alpha\beta} = 0, \qquad 1 \leq i \leq 3, \tag{3.2}$$

where τ^i_α designates the α-*th* coordinate of the vector $\boldsymbol{\tau}^i$.

For notational convenience, let $\alpha_i := \tau^i_1$ and $\beta_i := \tau^i_2, 1 \leq i \leq 3$. Assume first that one of the components β_i vanishes, say β_1; hence $\alpha_1 \neq 0$ since $\boldsymbol{\tau}^1 \neq \mathbf{0}$. The first equation of the linear system (3.2) then reduces to $e_{11}\alpha_1^2 = 0$, which shows that $e_{11} = 0$ since $\alpha_1 \neq 0$. The second and third equations of (3.2) then reduce to

$$\beta_i(2e_{12}\alpha_i + e_{22}\beta_i) = 0, \qquad i = 2, 3.$$

Since $\beta_2 \neq 0$ and $\beta_3 \neq 0$ (the vectors $\boldsymbol{\tau}^2$ and $\boldsymbol{\tau}^3$ are not parallel to $\boldsymbol{\tau}^1$), we are left with a linear system with two unknowns e_{12} and e_{22}, whose determinant $2(\alpha_2\beta_3 - \alpha_3\beta_2)$ is $\neq 0$ (the vectors $\boldsymbol{\tau}^2$ and $\boldsymbol{\tau}^3$ are not parallel). Hence $e_{12} = e_{22} = 0$.

Assume next that $\beta_i \neq 0, 1 \leq i \leq 3$, so that the system (3.2) can be rewritten as

$$\left(\frac{\alpha_i}{\beta_i}\right)^2 e_{11} + 2\left(\frac{\alpha_i}{\beta_i}\right) e_{12} + e_{22} = 0, \qquad 1 \leq i \leq 3.$$

This means that the polynomial $t \in \mathbb{R} \to e_{11}t^2 + 2e_{12}t + e_{22}$ has three distinct roots $\frac{\alpha_i}{\beta_i}, 1 \leq i \leq 3$ (if they were not all distinct, at least two of the three vectors $\boldsymbol{\tau}^i, 1 \leq i \leq 3$, would be parallel). Hence $e_{11} = e_{12} = e_{22} = 0$. $\square$

Notice in passing that an analogous result holds as well in dimension three: If T is a non-degenerate *tetrahedron* in $\mathbb{R}^3$, we will similarly show that the set $\{d_i; 1 \leq i \leq 6\}$, where d_i is again defined as in (3.1) along each edge $s_i, 1 \leq i \leq 6$, of T, is $P_0(T; \mathbb{S}^3)$ - unisolvent; cf. Ciarlet and Ciarlet, Jr.[18].

Remark 3.1. Consider the second degree polynomial $p := e_{\alpha\beta}x_\alpha x_\beta$ and remark that $\boldsymbol{\tau}^i \cdot \mathbf{e}\boldsymbol{\tau}^i = 0$ implies that the second tangential derivative of p on the i-th edge is zero. If the second degree polynomial p has zero second tangential derivatives on each edge of a triangle, then p is affine on each edge and then it is affine in the whole triangle (in effect this is precisely what is established in the above proof). Hence all its second derivatives are zero and $\mathbf{e}$ is identically zero.

From now on, Ω denotes a *polygonal domain* in $\mathbb{R}^2$, and we consider *triangulations* $\mathcal{T}^h$ of the set $\overline{\Omega}$ by triangles $T \in \mathcal{T}^h$ subjected to the usual conditions; in particular, all the triangles $T \in \mathcal{T}^h$ are non-degenerate.

Given such a triangulation $\mathcal{T}^h$ of $\overline{\Omega}$, let Σ^h denote the set of all *"interior" edges* found in $\mathcal{T}^h$ (i.e., that are not contained in the boundary $\partial\Omega$), let Σ^h_∂ denote the set of all "*boundary" edges* found in $\mathcal{T}^h$ (i.e., that are contained in $\partial\Omega$), and let A^h denote the set of all *"interior" vertices* found in $\mathcal{T}^h$ (i.e., that are contained in Ω).

We also assume that each interior or boundary edge $\sigma \in \Sigma^h \cup \Sigma^h_\partial$ is *oriented*. For instance, if $\{a^j; 1 \le j \le J\}$ denotes the set of all the vertices found in $\mathcal{T}^h$ and $\sigma = [a^i, a^j] \in \Sigma^h \cup \Sigma^h_\partial$ with $i < j$, one may let $\boldsymbol{\tau} := \dfrac{a^j - a^i}{|a^j - a^i|}$.

Theorem 3.2. *Given any triangulation $\mathcal{T}^h$ of $\overline{\Omega}$, define the finite element space*

$$\begin{aligned}\widetilde{\mathbb{E}}^h :=& \{\mathbf{e}^h \in \mathbb{L}^2_s(\Omega); \mathbf{e}^h|_T \in P_0(T;\mathbb{S}^2) \text{ for all } T \in \mathcal{T}^h \text{ and} \\ & \int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_{T_1})\boldsymbol{\tau}\,dl = \int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_{T_2})\boldsymbol{\tau}\,dl \text{ for all } \sigma = T_1\cap T_2 \in \Sigma^h \text{ with } T_1, T_2 \in \mathcal{T}^h\}.\end{aligned} \tag{3.3}$$

Then each tensor field $\mathbf{e}^h \in \widetilde{\mathbb{E}}^h$ is uniquely defined by the numbers $d_\sigma(\mathbf{e}^h), \sigma \in \Sigma^h \cup \Sigma^h_\partial$, where the degrees of freedom $d_\sigma : \widetilde{\mathbb{E}}^h \to \mathbb{R}$ are defined by

$$d_\sigma(\mathbf{e}^h) := \begin{cases} \int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_{T_1})\boldsymbol{\tau}\,dl = \int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_{T_2})\boldsymbol{\tau}\,dl \text{ if } \sigma = T_1\cap T_2 \in \Sigma^h, \\ \int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_\sigma)\boldsymbol{\tau}\,dl \text{ if } \sigma \in \Sigma^h_\partial. \end{cases} \tag{3.4}$$

Furthermore, given any interior vertex $a \in A^h$, let $\{T; T \in \mathcal{T}^h(a)\}$ denote the set formed by all the triangles of $\mathcal{T}^h$ that have the vertex a in common, and let

$$\widetilde{\Omega}(a) := \operatorname{int}\Big(\bigcup_{T\in\mathcal{T}^h(a)} T - a\Big).$$

Then

$$\operatorname{curl}\mathbf{curl}\,\mathbf{e}^h = 0 \text{ in } \mathcal{D}'(\widetilde{\Omega}(a)) \text{ for all } \mathbf{e}^h \in \widetilde{\mathbb{E}}^h.$$

Conversely, let $\mathbf{e}^h \in \mathbb{L}^2_s(\Omega)$ be such that $\mathbf{e}^h|_T \in P_0(T;\mathbb{S}^2)$ for all $T \in \mathcal{T}^h$ and

$$\operatorname{curl}\mathbf{curl}\,\mathbf{e}^h = 0 \text{ in } \mathcal{D}'(\widetilde{\Omega}(a)) \tag{3.5}$$

for some $a \in A^h$. Then

$$\int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_{T_1})\boldsymbol{\tau}\,dl = \int_\sigma \boldsymbol{\tau}\cdot(\mathbf{e}^h|_{T_2})\boldsymbol{\tau}\,dl$$

for all the edges $\sigma = T_1\cap T_2 \in \Sigma^h$, with $T_1, T_2 \in \mathcal{T}_h$, that have the vertex a as one of their end-points.

Proof. That each tensor field $\mathbf{e}^h \in \widetilde{\mathbb{E}}^h$ is uniquely defined by the numbers $d_\sigma(\mathbf{e}^h), \sigma \in \Sigma^h \cup \Sigma^h_\partial$, follows from the unisolvence established in Theorem 3.1.

Let T_1 and T_2 be two adjacent triangles with a common edge $\sigma = T_1 \cap T_2 \in \Sigma^h$, let $\boldsymbol{\nu} = (\nu_\alpha)$ denote the unit outer normal vector to T_1 along σ, and assume (to fix ideas) that σ is oriented with $\boldsymbol{\tau} = (\tau_\alpha)$ with $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$. For any function $\theta \in \mathcal{D}(\widehat{\Omega})$, where

$$\widehat{\Omega} := \text{int}\,(T_1 \cup T_2),$$

we have

$$\begin{aligned}
{}_{D'(\widehat{\Omega})}\langle \text{curl}\,\mathbf{curl}\,\mathbf{e}^h, \theta\rangle_{D(\widehat{\Omega})} &= {}_{D'(\widehat{\Omega})}\,\langle \partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11}, \theta\rangle_{D(\widehat{\Omega})} \\
&= \sum_{\alpha=1,2} \int_{T_\alpha} (e_{22}\partial_{11}\theta - e_{12}\partial_{12}\theta - e_{21}\partial_{21}\theta + e_{11}\partial_{22}\theta)dx \\
&= \sum_{\alpha=1,2} \int_{\partial T_\alpha} (e_{22}\nu_1\partial_1\theta - e_{12}\nu_2\partial_1\theta \\
&\qquad\qquad - e_{21}\nu_1\partial_2\theta + e_{11}\nu_2\partial_2\theta)dl,
\end{aligned}$$

since the functions $e_{\alpha\beta}$ are constant on each triangle. Noticing that the function θ vanishes on the boundary of $\widehat{\Omega}$, we are thus left with

$${}_{D'(\widehat{\Omega})}\langle \text{curl}\,\mathbf{curl}\,\mathbf{e}^h, \theta\rangle_{D(\widehat{\Omega})} = \int_\sigma \{([e_{22}]\tau_2 + [e_{12}]\tau_1)\partial_1\theta - ([e_{21}]\tau_2 + [e_{11}]\tau_1)\partial_2\theta)\}dl, \tag{3.6}$$

where

$$[e_{\alpha\beta}] := (e_{\alpha\beta}|_{T_1})|_\sigma - (e_{\alpha\beta}|_{T_2})|_\sigma.$$

By definition of the space $\widetilde{\mathbb{E}}^h$ (cf. (3.3)), the functions $[e_{\alpha\beta}]$, which are constant along σ, satisfy

$$[e_{\alpha\beta}]\tau_\alpha\tau_\beta = 0 \text{ along } \sigma. \tag{3.7}$$

Assume first that $\tau_1 \neq 0$, so that relation (3.7) can be rewritten in this case as

$$[e_{11}]\tau_1 + [e_{12}]\tau_2 + [e_{21}]\tau_2 + [e_{22}]\frac{\tau_2\tau_2}{\tau_1} = 0 \text{ along } \sigma,$$

Combining with relation (3.6), we thus find that

$${}_{D'(\widehat{\Omega})}\langle \text{curl}\,\mathbf{curl}\,\mathbf{e}^h, \theta\rangle_{D(\widehat{\Omega})} = \frac{1}{\tau_1}([e_{12}]\tau_1 + [e_{22}]\tau_2)\int_\sigma (\tau_1\partial_1\theta + \tau_2\partial_2\theta)dl \text{ if } \tau_1 \neq 0.$$

Assume next that $\tau_1 = 0$, so that relation (3.7) reduces in this case to

$$[e_{22}] = 0 \text{ along } \sigma.$$

Combining with relation (3.6), we thus find that

$${}_{D'(\widehat{\Omega})}\langle \text{curl}\,\mathbf{curl}\,\mathbf{e}^h, \theta\rangle_{D(\widehat{\Omega})} = -(\text{sign}\,\tau_2)[e_{21}]\int_\sigma \partial_2\theta dl \text{ if } \tau_1 = 0.$$

Since

$$\int_\sigma (\tau_1\partial_1\theta + \tau_2\partial_2\theta)dl = 0 \text{ if } \tau_1 \neq 0 \text{ and } \int_\sigma \partial_2\theta dl = 0 \text{ if } \tau_1 = 0$$

(the function θ vanishes at both end-points of the edge σ), it follows that

$$_{D'(\widehat{\Omega})}\langle \operatorname{curl} \mathbf{curl}\, \mathbf{e}^h, \theta\rangle_{D(\widehat{\Omega})} = 0.$$

Hence curl $\mathbf{curl}\, \mathbf{e}^h = 0$ in $\mathcal{D}'(\widehat{\Omega})$. The set $\widetilde{\Omega}(a)$ being defined as in the statement of Theorem 3.2, a similar argument shows that

$$_{\mathcal{D}'(\widetilde{\Omega}(a))}\langle \operatorname{curl} \mathbf{curl}\, \mathbf{e}^h, \theta\rangle_{\mathcal{D}(\widetilde{\Omega}(a))} = 0 \text{ for all } \theta \in \mathcal{D}(\widetilde{\Omega}(a)).$$

Therefore, curl $\mathbf{curl}\, \mathbf{e}^h = 0$ in $\mathcal{D}'(\widetilde{\Omega}(a))$.

Conversely, assume that relation (3.5) holds. Then, in particular,

$$\begin{aligned} 0 &= _{D'(\widehat{\Omega})} \langle \operatorname{curl} \mathbf{curl}\, \mathbf{e}^h, \theta\rangle_{D(\widehat{\Omega})} \\ &= \int_\sigma \{([e_{22}]\tau_2 + [e_{12}]\tau_1)\partial_1\theta - ([e_{21}]\tau_2 + [e_{11}]\tau_1)\partial_2\theta)\}dl \text{ for all } \theta \in \mathcal{D}(\widehat{\Omega}), \end{aligned} \tag{3.8}$$

where $\widehat{\Omega} := \operatorname{int}(T_1 \cup T_2)$ and T_1 and T_2 are any two adjacent triangles whose common edge σ has the vertex a as one of its end-points.

Assume first that $\tau_1 \neq 0$, so that relation (3.8) becomes in this case

$$0 = -\frac{1}{\tau_1}\int_\sigma [e_{\alpha\beta}]\tau_\alpha\tau_\beta\partial_2\theta dl \text{ for all } \theta \in \mathcal{D}(\widehat{\Omega}),$$

since $\int_\sigma \partial_1\theta dl = -\frac{\tau_2}{\tau_1}\int_\sigma \partial_2\theta dl$. Choosing any function $\theta \in \mathcal{D}(\widehat{\Omega})$ that satisfies $\partial_2\theta \geq 0$ along σ and $\partial_2\theta(c) > 0$ at one point c inside σ then shows that $[e_{\alpha\beta}]\tau_\alpha\tau_\beta = 0$ along σ.

Assume next that $\tau_1 = 0$, so that relation (3.8) becomes

$$0 = (\operatorname{sign} \tau_2)[e_{22}]\int_\sigma \partial_1\theta dl \text{ for all } \theta \in \mathcal{D}(\widehat{\Omega}),$$

since $\int_\sigma \partial_2\theta dl = 0$. Then we similarly conclude that, in this case, $[e_{22}] = [e_{\alpha\beta}]\tau_\alpha\tau_\beta = 0$ along σ. $\square$

The definition (3.3) of the finite element space $\widetilde{\mathbb{E}}^h$ and the definition (3.4) of its degrees of freedom d_σ together imply that the *dimension of* $\widetilde{\mathbb{E}}^h$ is equal to the number of edges in $\mathcal{T}^h$ and that the *support of the associated basis functions in* $\widetilde{\mathbb{E}}^h$ is either the union of the two adjacent triangles having the edge σ in common if $\sigma \in \Sigma^h$, or a single triangle if $\sigma \in \Sigma^h_\partial$.

As a first step towards the construction of a curl **curl**-free finite element space, we show that the symmetrized gradients of the simplest conforming finite element approximations of displacement fields belong to the space $\widetilde{\mathbb{E}}^h$ of (3.3).

Lemma 3.1. *Define the finite element space*

$$\widehat{\mathbb{E}}^h := \{\boldsymbol{\nabla}_s\dot{\mathbf{v}}^h \in \mathbb{L}^2_s(\Omega); \dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h\}, \tag{3.9}$$

where, the space $\mathbf{R}(\Omega)$ *being defined as in (2.2),*

$$\dot{\mathbf{V}}^h := \mathbf{V}^h/\mathbf{R}(\Omega) \text{ with } \mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{C}^0(\overline{\Omega}); \mathbf{v}^h|_T \in P_1(T;\mathbb{R}^2)\}. \tag{3.10}$$

Then,

$$\hat{\mathbb{E}}^h \subset \widetilde{\mathbb{E}}^h .$$

Besides,

$$\dim \widehat{\mathbb{E}}^h = \dim \mathbf{V}^h - 3.$$

Proof. Given any element $\mathbf{v}^h \in \mathbf{V}^h$, define $\mathbf{e}(\mathbf{v}^h) = \boldsymbol{\nabla}_s \mathbf{v}^h \in \mathbb{L}^2_s(\Omega)$. Then, by definition, one has $\mathbf{e}(\mathbf{v}^h)|_T \in P_0(T;\mathbb{S}^2)$ for all $T \in \mathcal{T}_h$.

Next, consider an interior edge $\sigma = T_1 \cap T_2 \in \Sigma^h$. We then remark that, since $\mathbf{v}^h \in \mathbf{C}^0(\overline{\Omega})$, one has $[\mathbf{v}^h] = 0$; in particular, one also has $[\tau \cdot \boldsymbol{\nabla} \mathbf{v}^h \tau] = 0$. Elementary computations yield

$$\tau \cdot \boldsymbol{\nabla} \mathbf{v}^h \tau = \partial_\alpha v^h_\beta \tau_\alpha \tau_\beta = \partial_1 v^h_1 \tau_1 \tau_1 + (\partial_1 v^h_2 + \partial_2 v^h_1)\tau_1\tau_2 + \partial_2 v^h_2 \tau_2 \tau_2 = e(\mathbf{v}^h)_{\alpha\beta}\tau_\alpha\tau_\beta \ ,$$

so that $[e(\mathbf{v}^h)_{\alpha\beta}]\tau_\alpha\tau_\beta = 0$ along σ, and thus $\mathbf{e}(\mathbf{v}^h) \in \widetilde{\mathbb{E}}^h$ according to (3.3). Hence the inclusion $\hat{\mathbb{E}}^h \subset \widetilde{\mathbb{E}}^h$ follows.

Noting that the components of a displacement field $\mathbf{v} = (v_\alpha) \in \mathcal{D}'(\Omega;\mathbb{R}^2)$ that satisfies $\boldsymbol{\nabla}_s \mathbf{v} = 0$ in $\mathcal{D}'(\Omega;\mathbb{S}^2)$ are of the form

$$v_1(x) = c_1 - \delta x_2 \quad \text{and} \quad v_2(x) = c_2 + \delta x_1 \text{ for all } (x_1,x_2) \in \Omega,$$

for arbitrary constants c_1, c_2, δ, we conclude that $\mathrm{Ker}\boldsymbol{\nabla}_s \subset \mathbf{V}^h$ and that $\dim \mathrm{Ker}\boldsymbol{\nabla}_s = 3$. Hence the second assertion follows. □

We next recall a few well-known results on a triangulation $\mathcal{T}^h$ of a simply-connected polygonal domain. We define the following integers:

$$\begin{aligned}
N &= [\text{number of vertices in } \mathcal{T}_h] = N_b + N_i, \text{where}\\
N_b &= [\text{number of boundary vertices] and } N_i = [\text{number of interior vertices}]\,;\\
A &= [\text{number of edges in } \mathcal{T}_h] = A_b + A_i, \text{where}\\
A_b &= [\text{number of boundary edges] and } A_i = [\text{number of interior edges}]\,;\\
F &= [\text{number of triangles in } \mathcal{T}_h].
\end{aligned}$$

It is thus clear that

$$\dim \widetilde{\mathbb{E}}^h = A, \quad \dim \mathbf{V}^h = 2N, \quad \dim \widehat{\mathbb{E}}^h = 2N - 3. \tag{3.11}$$

Then Euler relation, viz.,

$$N - A + F = 1,$$

combined with the relations

$$A_b + 2A_i = 3F \text{ and } N_b = A_b$$

(these last relations immediately follow from the definitions of A_b, A_i, F, and N_b), gives

$$\begin{aligned}2F &= 2A - 2N + 2 = 2(A_b + A_i) - 2(N_b + N_i) + 2\\ &= A_b + 2A_i - 2N_i - N_b + 2 = 3F - 2N_i - N_b + 2.\end{aligned}$$

Consequently,

$$F = 2N_i + N_b - 2.$$

This relation in turn implies that

$$2A = N_b + A_b + 2A_i = N_b + 3F = 6N_i + 4N_b - 6 = 2N_i + 4N - 6,$$

which shows that

$$A = N_i + 2N - 3. \tag{3.12}$$

The next theorem shows how to transform the finite element space $\widetilde{\mathbb{E}}^h$ into a curl **curl**-free one, denoted $\mathbb{E}^h$ (cf. (3.13)), by adding an appropriate *constraint* $\varphi_a(\mathbf{e}^h) = 0$ at each interior vertex $a \in A^h$. Note that, as shown in the next proof, the *explicit form* of the linear forms φ_a can be easily computed. Besides, this finite element space of curl **curl**-free elements can be identified with $\hat{\mathbb{E}}^h$, thanks to Lemma 3.1 and formulas (3.11) and (3.12).

Theorem 3.3. *Assume that the polygonal domain Ω is simply-connected. Given any interior vertex $a \in A^h$, there exists a linear form $\varphi_a : \widetilde{\mathbb{E}}^h \to \mathbb{R}$ such that*

$$\mathrm{curl}\,\mathbf{curl}\,\mathbf{e}^h = 0 \text{ in } \mathcal{D}'(\Omega) \text{ for all } \mathbf{e}^h \in \mathbb{E}^h\,,$$

where the space $\mathbb{E}^h$ is defined by

$$\mathbb{E}^h := \{\mathbf{e}^h \in \widetilde{\mathbb{E}}^h; \varphi_a(\mathbf{e}^h) = 0 \text{ for all } a \in A^h\}. \tag{3.13}$$

More specifically, the coefficients of each linear form φ_a are explicitly computable functions (cf. part (iii) of the proof) *of the coordinates of the vertex a and of the vertices of the triangles of $\mathcal{T}^h$ that have a as a vertex. In addition,*

$$\mathbb{E}^h = \hat{\mathbb{E}}^h.$$

Proof. The proof is carried out in several steps, corresponding to increasingly complex triangulations. We will use frequently the following relations:

$$\widehat{\mathbb{E}}^h \subset \widetilde{\mathbb{E}}^h, \tag{3.14}$$

$$\dim \widetilde{\mathbb{E}}^h = \dim \widehat{\mathbb{E}}^h + N_i \tag{3.15}$$

(the first one was established in Lemma 3.1, and the second one is a consequence of formulas (3.11) and (3.12)).

(i) Consider first the case of a triangulation that contains *no interior vertex* (see two examples in Figure 1). Then the space $\widetilde{\mathbb{E}}^h$ and its subspace $\mathbb{E}^h$ coincide since there is no interior vertex. Moreover, according to (3.14)-(3.15), $\widetilde{\mathbb{E}}^h = \widehat{\mathbb{E}}^h$.

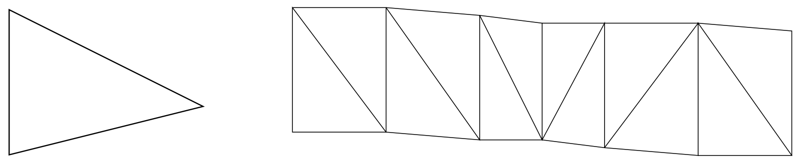

Fig. 1. Two triangulations *without interior vertex* of a simply-connected domain.

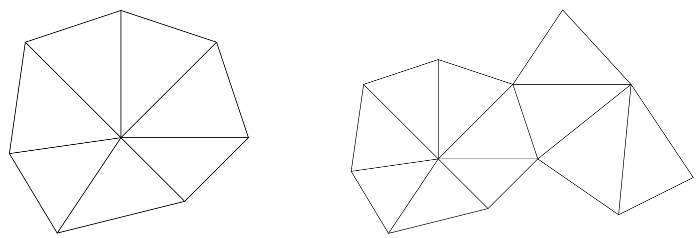

Fig. 2. Two triangulations *with a single interior vertex* of a simply-connected domain.

(ii) Assume now that there is *only one interior vertex* a in the triangulation (see two examples in Figure 2). In this configuration, formula (3.15) gives

$$\dim \widetilde{\mathbb{E}}^h = \dim \widehat{\mathbb{E}}^h + 1.$$

This relation, together with the inclusion $\widehat{\mathbb{E}}^h \subset \widetilde{\mathbb{E}}^h$ (see (3.14)), implies that there exists a non-zero linear form $\varphi_a : \widetilde{\mathbb{E}}^h \to \mathbb{R}$ such that $\widehat{\mathbb{E}}^h$ is equal to

$$\mathbb{E}^h := \{\mathbf{e}^h \in \widetilde{\mathbb{E}}^h ; \varphi_a(\mathbf{e}^h) = 0\}. \tag{3.16}$$

Let us compute the linear form φ_a explicitly in the case where there is *only one interior vertex* a in the triangulation, denoted $\mathcal{T}_h^a$, of a K-sided polygon Ω, *with* K *triangles* (see the first example in Figure 2). Let $a^k = (a^k_\alpha), 1 \le k \le K$, denote all the other vertices found in the triangulation $\mathcal{T}_h^a$, which are thus all "boundary" vertices in this case. For each $1 \le k \le K$, let T^k denote the triangle with vertices $a, a^k, a^{k+1}, 1 \le k \le K$, and define the vectors

$$\boldsymbol{\tau}^k = (\tau^k_\alpha) := a^k - a \text{ and } \widetilde{\boldsymbol{\tau}}^k = (\widetilde{\tau}^k_\alpha) := \boldsymbol{\tau}^{k+1} - \boldsymbol{\tau}^k,$$

the index k being understood as counted modulo K. A linear form φ_a which appears in (3.16) is necessarily a linear combination of the degrees of freedom $d_\sigma, \sigma \in \Sigma^h \cup \Sigma^h_\partial$, of the space $\widetilde{\mathbb{E}}^h$, since they form a basis of the dual space of $\widetilde{\mathbb{E}}^h$ (Theorem 3.2). In other words, there exist constants $\alpha_k, 1 \le k \le K$, and $\widetilde{\alpha}_k, 1 \le k \le K$, not all zero and uniquely defined only up to a single constant factor, such that

$$\varphi_a(\mathbf{e}^h) = \sum_{k=1}^K \alpha_k (\boldsymbol{\tau}^k \cdot \mathbf{e}^h|_{T^k} \boldsymbol{\tau}^k) + \sum_{k=1}^K \widetilde{\alpha}_k (\widetilde{\boldsymbol{\tau}}^k \cdot \mathbf{e}^h|_{T^k} \widetilde{\boldsymbol{\tau}}^k) \tag{3.17}$$

(recall that the components of each matrix field $(\mathbf{e}^h)$ are constant on each triangle; hence the integrals appearing in (3.4) may be replaced by their integrands).

We now show how to *explicitly compute the coefficients* α_k and $\widetilde{\alpha}_k, 1 \leq k \leq K$, that appear in the linear form φ_a used in the definition of the space $\mathbb{E}^h$ (cf. (3.16) and (3.17)).

To this end, we first note that, given any triangle T and any points $b = (b_\alpha)$ and $c = (c_\alpha)$ in T,

$$\mathbf{v}(b) - \mathbf{v}(c) = (b_\alpha - c_\alpha)\partial_\alpha \mathbf{v} \text{ for any } \mathbf{v} \in P_1(T; \mathbb{R}^2).$$

This relation then implies that, for any $\mathbf{v}^h = (\mathbf{v}_\alpha^h) \in P_1(T^k; \mathbb{R}^2)$,

$$\begin{aligned}
\boldsymbol{\tau}^k \cdot \mathbf{e}^h|_{T^k}\boldsymbol{\tau}^k &= e^h_{\alpha\beta}|_{T^k}\tau^k_\alpha\tau^k_\beta = \boldsymbol{\tau}^k \cdot \{\mathbf{v}^h(a^k) - \mathbf{v}^h(a)\},\ 1 \leq k \leq K,\\
\widetilde{\boldsymbol{\tau}}^k \cdot \mathbf{e}^h|_{T^k}\widetilde{\boldsymbol{\tau}}^k &= e^h_{\alpha\beta}|_{T^k}\widetilde{\tau}^k_\alpha\widetilde{\tau}^k_\beta = \widetilde{\boldsymbol{\tau}}^k \cdot \{\mathbf{v}^h(a^{k+1}) - \mathbf{v}^h(a^k)\},\ 1 \leq k \leq K,
\end{aligned}$$

where

$$\mathbf{e}^h|_{T^k} = (e^h_{\alpha\beta}|_{T^k}) \text{ with } (e^h_{\alpha\beta}|_{T^k}) = \frac{1}{2}\{\partial_\alpha(v^h_\beta|_{T^k}) + \partial_\beta(v^h_\alpha|_{T^k})\},\ 1 \leq k \leq K.$$

Hence relation (3.16) shows that, for any $\mathbf{e}^h = \boldsymbol{\nabla}_s\dot{\mathbf{v}}^h \in \mathbb{E}^h$ with $\mathbf{v}^h \in \mathbf{V}^h$,

$$\varphi_a(\mathbf{e}^h) = \sum_{k=1}^K \alpha_k\boldsymbol{\tau}^k \cdot \{\mathbf{v}^h(a^k) - \mathbf{v}^h(a)\} + \sum_{k=1}^K \widetilde{\alpha}_k\widetilde{\boldsymbol{\tau}}^k \cdot \{\mathbf{v}^h(a^{k+1}) - \mathbf{v}^h(a^k)\} = 0. \quad (3.18)$$

Since only the differences $\{\mathbf{v}^h(a^k) - \mathbf{v}^h(a)\},\ 1 \leq k \leq K$, appear in (3.18), we may assume without loss of generality that $\mathbf{v}^h(a) = \mathbf{0}$. In order to compute the unknown coefficients α_k and $\widetilde{\alpha}_k$, it then suffices to express that the $2K$ factors of $v^h_\alpha(a^k), 1 \leq k \leq K, \alpha = 1, 2$, appearing in (3.18) vanish, since these $2K$ numbers $v^h_\alpha(a^k)$ are arbitrary. This gives, after some elementary computations,

$$\left\{\begin{array}{l}
\widetilde{\alpha}_{k-1}(a_1^{k-1} - a_1^k) + \widetilde{\alpha}_k(a_1^{k+1} - a_1^k) = \alpha_k a_1^k, 1 \leq k \leq K,\\
\widetilde{\alpha}_{k-1}(a_2^{k-1} - a_2^k) + \widetilde{\alpha}_k(a_2^{k+1} - a_2^k) = \alpha_k a_2^k, 1 \leq k \leq K.
\end{array}\right. \quad (3.19)$$

Let

$$\Delta_k := (a_1^{k-1} - a_1^k)(a_2^{k+1} - a_2^k) - (a_1^{k+1} - a_1^k)(a_2^{k-1} - a_2^k), 1 \leq k \leq K,$$

and let us first consider the case where $\Delta_k = 0$ (note that $\Delta_k = 0$ if and only if the three vertices a^{k-1}, a^k, a^{k+1} are aligned). Since equations (3.19), viewed as a linear system with unknowns $\widetilde{\alpha}_{k-1}$ and $\widetilde{\alpha}_k$, must have a solution, the right-hand sides in (3.19) must satisfy the compatibility relation

$$\alpha_k(a_1^k a_2^{k+1} - a_1^{k+1}a_2^k) = 0.$$

Since

$$D_k := a_1^k a_2^{k+1} - a_1^{k+1}a_2^k$$

does not vanish (otherwise the three vertices a, a^k, a^{k+1} would be aligned), it follows that $\alpha_k = 0$ if $\Delta_k = 0$. We have thus shown that, if $\Delta_k = 0$ for some index k, then

$$\begin{cases} \widetilde{\alpha}_k = \dfrac{a_1^k - a_1^{k-1}}{a_1^{k+1} - a_1^k}\widetilde{\alpha}_{k-1} = \dfrac{a_2^k - a_2^{k-1}}{a_2^{k+1} - a_2^k}\widetilde{\alpha}_{k-1} \text{ if } a_1^{k+1} \neq a_1^k \text{ and } a_2^{k+1} \neq a_2^k, \\ \widetilde{\alpha}_k = \dfrac{a_2^k - a_2^{k-1}}{a_2^{k+1} - a_2^k}\widetilde{\alpha}_{k-1} \text{ if } a_1^{k+1} = a_1^k, \\ \widetilde{\alpha}_k = \dfrac{a_1^k - a_1^{k-1}}{a_1^{k+1} - a_1^k}\widetilde{\alpha}_{k-1} \text{ if } a_2^{k+1} = a_2^k. \end{cases} \tag{3.20}$$

Let us next consider the case where $\Delta_k \neq 0$. Then solving the linear system (3.19) gives

$$\widetilde{\alpha}_{k-1} = \frac{1}{\Delta_k}(a_1^k a_2^{k+1} - a_1^{k+1} a_2^k)\alpha_k = \frac{D_k}{\Delta_k}\alpha_k,$$
$$\widetilde{\alpha}_k = \frac{1}{\Delta_k}(a_1^{k-1} a_2^k - a_1^k a_2^{k-1})\alpha_k = \frac{D_{k-1}}{\Delta_k}\alpha_k,$$

and thus

$$\widetilde{\alpha}_k = \frac{D_{k-1}}{D_k}\widetilde{\alpha}_{k-1} \tag{3.21}$$

in this case.

To sum up, it follows from (3.19), (3.20), and (3.21) that *there exist coefficients $B_k, C_k, D_k, 1 \leq k \leq K$, that are explicitly computable in terms of the coordinates of the points a^{k-1}, a^k, a^{k+1}, such that*

$$\widetilde{\alpha}_k = B_k \widetilde{\alpha}_{k-1}, 1 \leq k \leq K, \tag{3.22}$$

$$\alpha_k = C_k \widetilde{\alpha}_{k-1} + D_k \widetilde{\alpha}_k, 1 \leq k \leq K \tag{3.23}$$

(since a_1^k and a_2^k cannot vanish simultaneously for a given k), at least one of the two equations (3.19) is of the form (3.23)). Hence the coefficients $\widetilde{\alpha}_k$ and $\alpha_k, 1 \leq k \leq K$, can be computed recursively, once a non-zero value has been assigned to, e.g., α_1.

Note in passing that $\widetilde{\alpha}_k \neq 0, 1 \leq k \leq K$. Otherwise all the coefficients $\widetilde{\alpha}_k$ and $\alpha_k, 1 \leq k \leq K$, would vanish.

(iii) We now return to the *general case*, i.e., where there are more than one interior vertex in the triangulation $\mathcal{T}_h$. Let then the space $\mathbb{E}^h$ be defined as

$$\mathbb{E}^h := \{\mathbf{e}^h \in \widetilde{\mathbb{E}}^h; \varphi_a(\mathbf{e}^h) = 0 \text{ for all } a \in A^h\},$$

where the space $\widetilde{\mathbb{E}}^h$ is defined as in (3.3), and, for each interior vertex $a \in A^h$, the linear form $\varphi_a : \widetilde{\mathbb{E}}^h \to \mathbb{R}$ is defined as in part (ii), where the triangulation, denoted $\mathcal{T}_h^a$ there, consists of all the triangles that have a as one of their vertices. This means that $\varphi_a(\mathbf{e}^h)$ is of the form (3.17), where the coefficients α_k and $\widetilde{\alpha}_k, 1 \leq k \leq K$, are computed as in (3.22) and (3.23) by means of the coordinates of those vertices a^k that are joined to a by an interior edge $\sigma \in \Sigma^h$.

Then the conditions $\varphi_a(\mathbf{e}^h) = 0$ satisfied by each $\mathbf{e}^h \in \mathbb{E}^h$ at all $a \in A^h$ imply by (ii) that, given any point $x \in \Omega$, the distribution curl $\mathbf{curl}\ \mathbf{e}^h$ vanishes in an open set containing x. Hence curl $\mathbf{curl}\ \mathbf{e}^h = 0$ in $\mathcal{D}'(\Omega)$ by the *principle of localization of distributions*; cf. Chapter 1 in Schwartz[32].

There remains to prove that $\mathbb{E}^h = \widehat{\mathbb{E}}^h$. Let $\mathbf{e}^h = (e^h_{\alpha\beta}) \in \mathbb{E}^h$ be given. Since accordingly $\mathbf{e}^h \in \mathbb{L}^2_s(\Omega)$ satisfies curl $\mathbf{curl}\ \mathbf{e}^h = 0$ in $H^{-2}(\Omega)$, Theorem 2.1 shows that there exists a vector field $\dot{\mathbf{v}}^h = (\dot{v}^h_\alpha) \in \mathbf{H}^1(\Omega)/\mathbf{R}(\Omega)$ such that $\mathbf{e}^h = \boldsymbol{\nabla}_s \dot{\mathbf{v}}^h$ in $\mathbb{L}^2_s(\Omega)$. Since, for each $T \in \mathcal{T}^h$,

$$\partial_{\alpha\beta} v^h_\tau = \partial_\alpha e_{\tau\beta}(\mathbf{v}^h) + \partial_\beta e_{\tau\alpha}(\mathbf{v}^h) - \partial_\tau e_{\alpha\beta}(\mathbf{v}^h) \text{ in } H^{-1}(\text{int } T),$$

and $\mathbf{e}^h|_T \in P_0(T; \mathbb{S}^2)$, it follows that $\mathbf{v}^h|_T \in P_1(T; \mathbb{R}^2)$. Consequently, $\mathbf{v}^h \in \mathbf{C}^0(\overline{\Omega})$ since $\mathbf{v}^h \in \mathbf{H}^1(\Omega)$. We have thus shown that

$$\mathbb{E}^h \subset \widehat{\mathbb{E}}^h.$$

The definition of the space $\mathbb{E}^h$ shows that

$$\dim \mathbb{E}^h = \dim \widetilde{\mathbb{E}}^h - N_i,$$

since the linear forms $\varphi_a, a \in A^h$, are linearly independent (this can be seen by applying them to the symmetrized gradients of the canonical “hat” basis functions of the space $\mathbf{V}^h$). According to (3.15), we finally conclude that $\mathbb{E}^h = \widehat{\mathbb{E}}^h$. $\square$

4. The discrete problem; convergence.

In what follows, Ω is again assumed to be a simply-connected polygonal domain in $\mathbb{R}^2$. The *discrete problem* is now defined, as the minimization problem (4.1) below.

Theorem 4.1. *Given any triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, let $\mathbb{E}^h$ be the finite element space defined in* (3.13). *Then there exists one and only one $\boldsymbol{\varepsilon}^h \in \mathbb{E}^h$ such that*

$$j(\boldsymbol{\varepsilon}^h) = \inf_{\mathbf{e}^h \in \mathbb{E}^h} j(\mathbf{e}^h), \tag{4.1}$$

where j is the functional defined in (2.6). *Let $\dot{\mathbf{V}}^h$ be the finite element space defined in* (3.10), *and let J be the functional defined in* (2.4). *Then $\boldsymbol{\varepsilon}^h = \boldsymbol{\nabla}_s \dot{\mathbf{u}}^h$, where $\dot{\mathbf{u}}^h \in \dot{\mathbf{V}}^h$ is the unique solution to the minimization problem*

$$J(\dot{\mathbf{u}}^h) = \inf_{\dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h} J(\dot{\mathbf{v}}^h).$$

Proof. We have (cf. (2.6))

$$j(\mathbf{e}) = \frac{1}{2}\int_\Omega \mathbf{A}\mathbf{e} : \mathbf{e}\, dx - (L \circ \boldsymbol{\mathcal{F}})(\mathbf{e}) = \frac{1}{2} b(\mathbf{e}, \mathbf{e}) - \ell(\mathbf{e}) \text{ for all } \mathbf{e} \in \mathbb{L}^2_s(\Omega),$$

where the bilinear form b and the linear form ℓ satisfy all the assumptions of the Lax-Milgram lemma over the space $\mathbb{E}(\Omega)$ of (2.5) (thanks in particular to the inequality (2.3)), hence over its subspace $\mathbb{E}^h$, which is closed ($\mathbb{E}^h$ is finite-dimensional). Consequently, there exists one, and only one, minimizer $\boldsymbol{\varepsilon}^h$ of the functional j over $\mathbb{E}^h$.

That $\dot{\mathbf{u}}^h$ minimizes the functional J over $\dot{\mathbf{V}}^h$ implies that $\boldsymbol{\nabla}_s \dot{\mathbf{u}}^h$ minimizes the functional j over $\mathbb{E}^h$ since $\boldsymbol{\nabla}_s \dot{\mathbf{u}}^h \in \mathbb{E}^h$ by Theorem 3.3. Hence $\boldsymbol{\varepsilon}^h = \boldsymbol{\nabla}_s \dot{\mathbf{u}}^h$ since the minimizer is unique. □

Finally, we examine the *convergence* of the method.

Theorem 4.2. *Consider a regular family of triangulations $\mathcal{T}^h$ of $\overline{\Omega}$. Then*

$$||\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h||_{\mathbb{L}^2_s(\Omega)} \to 0 \text{ as } h \to 0. \tag{4.2}$$

If $\mathbf{u} \in \mathbf{H}^2(\Omega)$, there exists a constant C independent of h such that

$$||\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h||_{\mathbb{L}^2_s(\Omega)} \leq C||\mathbf{u}||_{\mathbf{H}^2(\Omega)} h. \tag{4.3}$$

Proof. Let b denote as above the bilinear form that appears in the functional j. Observing that $\boldsymbol{\varepsilon}^h$ is the projection of $\boldsymbol{\varepsilon}$ onto $\mathbb{E}^h$ with respect to the inner product b and taking into account the assumptions made in Section 2 on the elasticity tensor $\mathbf{A}$, we infer that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1||\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h||^2_{\mathbb{L}^2_s(\Omega)} \leq b(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h) = \inf_{\mathbf{e}^h \in \mathbb{E}^h} b(\boldsymbol{\varepsilon} - \mathbf{e}^h, \boldsymbol{\varepsilon} - \mathbf{e}^h)$$
$$\leq C_2 \inf_{\mathbf{e}^h \in \mathbb{E}^h} ||\boldsymbol{\varepsilon} - \mathbf{e}^h||^2_{\mathbb{L}^2_s(\Omega)} = C_2 \inf_{\mathbf{v}^h \in \mathbf{V}^h} ||\boldsymbol{\nabla}_s \mathbf{u} - \boldsymbol{\nabla}_s \mathbf{v}^h||^2_{\mathbb{L}^2_s(\Omega)}.$$

The error estimates (4.2) and (4.3) then follow by standard error estimates. □

5. The not simply-connected case

The present analysis can be extended to a domain that is *not simply-connected*. In this case, Ω can be "reduced" to a simply-connected domain $\dot{\Omega}$ by means of a finite number N_Σ of piecewise affine cuts $\mathcal{C}_\sigma$, $\sigma = 1, \dots, N_\Sigma$, such that the boundary of each cut $\mathcal{C}_\sigma$ is contained in $\partial\Omega$. The simply-connected domain $\dot{\Omega} = \Omega \setminus \cup_{\sigma=1}^{N_\Sigma} \mathcal{C}_\sigma$ is pseudo-Lipschitz, in the sense of Amrouche, Bernardi, Dauge and Girault[1]. Analogs of Theorems 2.1, 2.2 and 2.3 hold (one needs here a characterization of the orthogonal complement of the space $\boldsymbol{\nabla}_s \mathbf{H}^1(\Omega)$ in the space $\mathbb{E}(\Omega)$ of (2.5), found in Ref. 19).

We then consider *triangulations* $\mathcal{T}^h$ of the set $\overline{\Omega}$ by triangles $T \in \mathcal{T}^h$ that are subjected to the usual conditions, and also such that $\text{int}(T) \cap \mathcal{C} = \emptyset$ for all $T \in \mathcal{T}^h$. The statement of Theorem 3.1 being localized to a triangle, it is unmodified in this case. The corresponding finite element space, now denoted $\widetilde{\mathbb{E}}^h_{nsc}$, is defined as before (see (3.3)), and the statement of Theorem 3.2 is also unmodified. But the finite element space of displacements is now equal to

$$\mathbf{V}^h_{nsc} := \{\mathbf{v}^h \in \mathbf{C}^0(\overline{\Omega} \setminus \mathcal{C}); \left[\left[\mathbf{v}^h\right]\right]_{\mathcal{C}_\sigma} \in \mathbf{R}(\Omega),\ \sigma = 1, \dots, N_\Sigma,\ \mathbf{v}^h|_T \in P_1(T; \mathbb{R}^2)\},$$

where $\left[\left[\mathbf{v}^h\right]\right]_{\mathcal{C}_\sigma}$ denotes the jump of $\mathbf{v}^h$ across the cut $\mathcal{C}_\sigma$. As a consequence, the definition of the space $\widehat{\mathbb{E}}^h$ is now replaced by

$$\widehat{\mathbb{E}}^h_{nsc} := \{\boldsymbol{\nabla}_s \dot{\mathbf{v}}^h \in \mathbb{L}^2_s(\Omega); \dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h_{nsc}\},$$

where $\dot{\mathbf{V}}^h_{nsc} := \mathbf{V}^h_{nsc}/\mathbf{R}(\Omega)$. Then, $\widehat{\mathbb{E}}^h_{nsc} \subset \widetilde{\mathbb{E}}^h_{nsc}$ like in Lemma 3.1.

In view of establishing the analog of Theorem 3.3, let us recall some relations on the numbers of vertices, edges and triangles N, A and F, of $\mathcal{T}_h$. On a triangulation $\mathcal{T}^h$ of a polygonal domain with h holes ($h = 0$ corresponds to the case of a simply-connected domain), Euler relation becomes

$$N - A + F = 1 - h.$$

Combined with the relations $A_b + 2A_i = 3F$ and $N_b = A_b$, this relation gives

$$F = 2N_i + N_b - 2 + 2h.$$

Next, one easily establishes the relations

$$A = N_i + 2N - 3 + 3h\,, \tag{5.1}$$

and

$$\dim \widetilde{\mathbb{E}}^h = A$$

(since there is one degree of freedom per edge). However, the dimension of $\widehat{\mathbb{E}}^h_{nsc}$ is now given by

$$\dim \widehat{\mathbb{E}}^h_{nsc} = 2N - 3 + 3h. \tag{5.2}$$

Let us briefly mention how the relation (5.2) is obtained. By definition, the dimension of $\mathbf{V}^h_{nsc}$ is equal to $2N + 3N_\Sigma$, since there are exactly three additional degrees of freedom per cut when the domain $\dot{\Omega}$ is obtained by removing N_Σ cuts from Ω. It follows that $\dim \widehat{\mathbb{E}}^h_{nsc} = 2N - 3 + 3N_\Sigma$. Now, for a domain Ω in $\mathbb{R}^2$, it is well-known that the number N_Σ of cuts is equal to the number of holes h. Hence formula (5.2) holds.

Thanks to these observations, we conclude that Theorem 3.3 can be generalized as follows.

Theorem 5.1. *Given any interior vertex $a \in A^h$, there exists a linear form φ_a : $\widetilde{\mathbb{E}}^h_{nsc} \to \mathbb{R}$ such that*

$$\text{curl}\,\mathbf{curl}\,\mathbf{e}^h = 0 \text{ in } \mathcal{D}'(\Omega) \text{ for all } \mathbf{e}^h \in \mathbb{E}^h_{nsc}\,,$$

where the space $\mathbb{E}^h_{nsc}$ is defined by

$$\mathbb{E}^h_{nsc} := \{\mathbf{e}^h \in \widetilde{\mathbb{E}}^h_{nsc}; \varphi_a(\mathbf{e}^h) = 0 \text{ for all } a \in A^h\}\,. \tag{5.3}$$

The coefficients of each linear form φ_a are again explicitly computable. Besides,

$$\mathbb{E}^h_{nsc} = \hat{\mathbb{E}}^h_{nsc}.$$

Interestingly, the definition (5.3) of the finite element space $\mathbb{E}^h_{nsc}$ is *independent* of the cuts and holes, an observation which could lead to simpler (with respect to a discretization of the displacements) numerical algorithms.

Finally, we note that the convergence analysis of the method can be carried out as in Section 4.

6. Commentary

What precedes is admittedly only a modest "first stone", since there remains a variety of yet to be solved questions, which we briefly mention below. These questions will be studied in future works, referred to here under a single Ref. 20 for brevity.

(a) In order to *numerically solve the discrete minimization problem* (4.1), two approaches are possible. Either one tries to explicitly construct basis functions in the space $\widehat{\mathbb{E}}^h$ of (3.9), e.g., as the images of the canonical basis functions or the space $\mathbf{V}^h$ of (3.10) through the symmetrized gradient operator $\boldsymbol{\nabla}_s$ defined in (2.1). Or one views this problem as a quadratic minimization problem, constrained by the linear equations $\varphi_a(\mathbf{e}^h) = 0$ for all $a \in A^h$.

(b) Naturally, finite element spaces based on piecewise polynomials of higher degree or on quadrilaterals instead of triangles, can be likewise studied.

(c) A *non-conforming method* would consist in minimizing the quadratic functional j over the space $\widetilde{\mathbb{E}}^h$ of (3.3) instead of the space $\mathbb{E}^h$ of (3.13). Since the elements $\mathbf{e}^h$ of the space $\mathbb{E}^h$ are in a sense "not far from being curl **curl**-free" (they satisfy curl $\mathbf{curl}\ \mathbf{e}^h = 0$ in $\mathcal{D}'(\Omega - \cup_{a\in A^h}\{a\})$, thanks to Theorem 3.2 and the principle of localization of distributions), convergence may still occur. This would yield an appealing method, since the minimization problem is no longer constrained as in (a), and since the basis functions of the space $\widetilde{\mathbb{E}}^h$ have very small supports (either two adjacent triangles or a single triangle).

(d) Another *non-conforming method* would consist in replacing the constraint curl $\mathbf{curl}\ \mathbf{e}^h = 0$ in $H^{-2}(\Omega)$ by relations of the form $\langle \text{curl}\ \mathbf{curl} \mathbf{e}^h, \theta^h\rangle = 0$ for all $\theta^h \in M^h$ where M^h is an appropriate finite element subspace of the space $H^2_0(\Omega)$.

(e) One virtue of the present approach is that it overcomes the *three-dimensional indeterminacy* (since dim Ker $\boldsymbol{\nabla}_s = 3$ in two dimensions), or the *six-dimensional indeterminacy* in dimension three (since dim Ker $\boldsymbol{\nabla}_s = 6$ in three-dimensions), traditionally encountered in solving *pure traction problems* by means of the displacement approach.

(f) Another key issue will be to see how *boundary conditions of Dirichlet type* and the *linear form L* can be conveniently re-expressed in terms of the new discrete unknowns in the formulation of the discrete problems, since they are naturally expressed in terms of displacements, not of stresses (it is likely, however, that this issue is more challenging for the "continuous" problem than for its discretisation).

More specifically, the definition of the linear form $\ell : \mathbb{E}(\Omega) \to \mathbb{R}$ that appears in the definition of the constrained quadratic minimization problem that we wish to approximate (cf. (2.6)), depends on the isomorphism $\boldsymbol{\mathcal{F}} : \mathbb{E}(\Omega) \to \dot{\mathbf{H}}^1(\Omega)$ found in Theorem 2.2. Hence another important goal will be to construct efficient approximations $\boldsymbol{\mathcal{F}}_h$ of this isomorphism. Once this is achieved, an approximate displacement field $\boldsymbol{u}^h$ (up to infinitesimal rigid displacements) can be recovered from the approximate strain tensor $\boldsymbol{\epsilon}^h$, simply by computing $\boldsymbol{u}^h = \boldsymbol{\mathcal{F}}_h(\boldsymbol{\epsilon}^h)$.

With such an approximation $\boldsymbol{\mathcal{F}}_h$ at hand, *Dirichlet boundary conditions* of the form $\boldsymbol{u} = \mathbf{0}$ on $\Gamma_0 \subset \partial\Omega$ could be incorporated in the discrete problems in the form of the constraint $\mathcal{F}_h(\boldsymbol{\epsilon}^h) = \mathbf{0}$ on Γ_0.

Acknowledgement. This work was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No.9041101, City U 100706].

References

1. C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, *Math. Meth. Appl. Sci.*, **21** (1998), 823-864.
2. C. Amrouche, P. G. Ciarlet, P. Ciarlet, Jr., Vector and scalar potentials, Poincaré's theorem and Korn's inequality, *C. R. Acad. Sci. Paris*, *Ser. I*, **345** (2007), 603-608.
3. C. Amrouche, P. G. Ciarlet, L. Gratie, S. Kesavan, On the characterization of matrix fields as linearized strain tensor fields, *J. Math. Pures Appl.* **86** (2006), 116-132.
4. S. S. Antman, Ordinary differential equations of nonlinear elasticity I: Foundations of the theories of non-linearly elastic rods and shells, *Arch. Rational Mech. Anal.* **61** (1976), 307-351.
5. D. N. Arnold, G. Awanou, Rectangular mixed finite elements for elasticity, *Math. Models Methods Appl. Sci.* **15** (2005), 1417-1429.
6. D. N. Arnold, F. Brezzi, J. Douglas, Jr., PEERS, A new mixed finite element for plane elasticity, *Japan J. Appl. Math.* **1** (1984), 347-367.
7. D. N. Arnold, J. Douglas, Jr., C. P. Gupta, A family of higher order mixed finite element methods for plane elasticity, *Numer. Math.* **45** (1984), 1-22.
8. D. N. Arnold, R. S. Falk, A new mixed formulation for elasticity, *Numer. Math.* **53** (1988), 13-30.
9. D. N. Arnold, R. S. Falk, R. Winther, Finite element exterior calculus, homological techniques, and applications, *Acta Numerica* **15** (2006), 1-155.
10. D. N. Arnold, R. Winther, Mixed finite element methods for elasticity, *Numer. Math.* **92** (2002), 401-419.
11. D. N. Arnold, R. Winther, Mixed finite elements for elasticity in the stress-displacement formulation, in *Current Trends in Scientific Computing* (Z. Chen, R. Glowinski, K. Li, Editors), Contemporary Mathematics, Vol. 329, American Mathematical Society, 2003, pp. 33-42.
12. I. Babuška, The finite element method with Lagrangian multipliers, *Numer. Math.* **20** (1973), 179-192.
13. F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *Rev. Française Automat. Informat. Recherche Opérationelle Sér. Rouge* **8** (1974), 129-151.
14. F. Brezzi, J. Douglas, Jr., D. Marini, Two families of mixed finite elements for second order elliptic problems, *Numer. Math.* **47** (1985), 217-235.
15. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978; reprinted as SIAM Classics in Applied Mathematics, Vol. 40, SIAM, Philadelphia, 2002.
16. P. G. Ciarlet, P. Ciarlet, Jr., Another approach to linearized elasticity and a new proof of Korn's inequality, *Math. Models Methods Appl. Sci.* **15** (2005), 259-271.
17. P. G. Ciarlet, P. Ciarlet, Jr., A new approach for approximating linear elasticity problems, *C. R. Acad. Sci. Paris, Ser. I*, **346** (2008), 351-356.
18. P. G. Ciarlet, P. Ciarlet, Jr., Direct computation of stresses in three-dimensional linearized elasticity, in preparation.

19. P. G. Ciarlet, P. Ciarlet, Jr., G. Geymonat, F. Krasucki, Characterization of the kernel of the operator CURL CURL, *C. R. Acad. Sci. Paris*, *Ser. I*, **344** (2007), 305-308.
20. P. G. Ciarlet, P. Ciarlet, Jr., S. Sauter, A new finite element approximation of linear elasticity problems, based on the Saint Venant compatibility conditions, several works in preparation.
21. M. Crouzeix, P. A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, I, *Rev. Française Automat. Informat. Recherche Opérationnelle, Sér. Rouge Anal. Numér.* **R-3** (1973), 77-104.
22. G. Duvaut, J. L. Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, 1972. English translation: *Inequalities in Mechanics and Physics*, Springer-Verlag, 1976.
23. M. Eastwood, Notes on projective differential geometry, in *Symmetries and Overdetermined Systems of Partial Differential Equations*, IMA Volume 144, Springer, 2007, 41-60.
24. G. Geymonat, F. Krasucki, Some remarks on the compatibility conditions in elasticity, *Rend. Accad. Naz. Sci.*, **XL 123** (2005), 175-182.
25. G. Geymonat, F. Krasucki, Beltrami's solutions of general equilibrium equations in continuum mechanics, *C. R. Acad. Sci. Paris*, *Ser. I*, **342** (2006), 359-363.
26. M. E. Morley, A family of mixed finite elements for linear elasticity, *Numer. Math.* **55** (1989), 633-666.
27. J. C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* **35** (1980), 315-341.
28. J. C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* **50** (1986), 57-81.
29. S. Opoka, W. Pietraszkiewicz, Intrinsic equations for nonlinear deformation and stability of thin elastic shells, *Internat. J. Solids Structures* **41** (2004), 3275-3292.
30. F. Rapetti, High order edge elements on simplicial meshes, *Math. Modelling Numer. Anal.* **41** (2007), 1001-1020.
31. P. A. Raviart, J. M. Thomas, A mixed finite element method for second order elliptic problems, in *Mathematical Aspects of the Finite Element Method* (I. Galligani, E. Magenes, Editors), Lecture Notes in Mathematics, Vol. 606, Springer-Verlag, New York, 1977, pp. 292-315.
32. L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
33. R. Stenberg, On the construction of optimal mixed finite element methods for the linear elasticity problem, *Numer. Math.* **48** (1986), 447-462.
34. R. Stenberg, A family of mixed finite elements for the elasticity problem, *Numer. Math.* **53** (1988), 513-538.
35. T. W. Ting, St. Venant's compatibility conditions, *Tensor, N.S.* **28** (1974), 5-12.