Theoretical tools to solve the axisymmetric Maxwell equations*

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Contents

Abstract

In this paper, the mathematical tools, which are required to solve the axisymmetric Maxwell equations, are presented. An in-depth study of the problems posed in the meridian half-plane, numerical algorithms, as well as numerical experiments, based on the implementation of the theory described hereafter, shall be presented in forthcoming papers. In the present paper, the attention is focused on the (orthogonal) splitting of the electromagnetic field in a regular part and a singular part, the former being in the Sobolev space H^1 component-wise. It is proven that the singular fields are related to singularities of Laplace-like operators, and, as a consequence, that the space of singular fields is finite dimensional. This paper can be viewed as the continuation of [5, 4].

Introduction

A number of real-life electromagnetic devices present an axial symmetry, at least close to the axis. It is therefore important, in the first place, to be able to model correctly the axisymmetric problem, that is, the resolution of Maxwell equations in an axisymmetric domain with symmetric data. Then, in the second place, some directions can be easily inferred to study the more general problems, that is with any data and/or in a perturbed axisymmetric geometry. In the third place, it is a common practice to approximate problems by their axisymmetric counterpart. Note that a number of axisymmetric problems have been studied, such as the Laplace equation, the Stokes and Navier-Stokes systems: in particular, we refer the reader to [6], in which, moreover, extensions are given for any data, and for a domain presenting a perturbed geometry.

The systems that we study hereafter are the static and time-dependent Maxwell equations. As for the boundary conditions, the domain is assumed to be enclosed in a perfectly conducting material. On a mathematical point of view, notice that the theory which is presented in this paper can not be considered as a plain application of the existing theory for curvilinear polyhedra [3, 12]. Indeed, an axisymmetric domain is never a curvilinear polyhedron, due to the presence of the conical vertices, except in the very special case when both conical angles (see Figure 1) are equal to $\pi/2$.

The paper is organized as follows. In the first Section, we briefly recall some well-known results about the existence and uniqueness of solution of the *static* Maxwell equations in three-dimensional domains, which we call the div-curl problems. In the second Section, we introduce the geometry in which we intend to solve Maxwell equations and after that, we describe in detail some useful properties of first order and second order differential operators related to the *symmetry of revolution*. In the third Section, we recall some properties of the distributions, smooth fields and elements of Sobolev spaces, which are *invariant by rotation*. We also focus on the trace mappings, which allow one to define alternatively those concepts *via* their trace on any given meridian half-plane. In the fourth Section, we prove several closedness results, which are related to the lack of density of regular (i.e. H^1 component-wise) fields in the natural spaces of electromagnetic fields. This further leads to the splitting of the electromagnetic field in regular and singular parts. Then, in Section 5, we relate the singular fields to the singular solutions of some Laplace-like problems. Finally, we consider the case of the *time-dependent* Maxwell equations, for which we prove existence and uniqueness of the solution, and provide a continuous decomposition (in time) into a regular and a singular parts.

Note that, throughout the paper, we also address the case of the div-curl problem with mixed boundary conditions, i.e. existence and uniqueness of the solution, its regularity, and closedness of the subspace of regular fields satisfying those boundary conditions.

In the remainder of this paper, we write vector fields or spaces with boldface or calligraphic characters (We mainly use calligraphic characters for the three-dimensional electromagnetic fields and spaces.)

1 The div-curl problems

Let Ω be a bounded and simply connected domain of $\mathbb{R}^3$, Γ its Lipschitz boundary, and $\mathbf{n}$ the unit outward normal to Γ. Note that the case of a domain, which is not simply connected, is treated very carefully in [14, 1].

1.1 The static Maxwell equations

There are two div-curl problems, depending on the boundary condition. The first one is, for $\mathbf{f}$ in $\mathbf{L}^2_0(\Omega)$ such that $\operatorname{div}\mathbf{f} = 0$ and $\mathbf{f}\cdot\mathbf{n}_{|\Gamma} = 0$, and g in $L^2(\Omega)$:
Find $\mathcal{E} \in \mathbf{L}^2(\Omega)$ *such that*

$$\mathbf{curl}\,\mathcal{E} = \mathbf{f} \text{ in } \Omega, \tag{1}$$
$$\operatorname{div}\mathcal{E} = g \text{ in } \Omega, \tag{2}$$
$$\mathcal{E}\times\mathbf{n}_{|\Gamma} = 0. \tag{3}$$

The boundary condition on $\mathbf{f}$ is imposed by the condition (3) (cf. [15]). In order to prove the existence and uniqueness of the solution $\mathcal{E}$ to (1)-(3), a possible way is to reformulate these equations as a saddle-point formulation, and to check that the Lagrange multiplier is equal to 0 (see [10] for details).

The second div-curl problem is, given $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$ such that $\operatorname{div}\mathbf{f} = 0$, and g in $L^2_0(\Omega)$:
Find $\mathcal{B} \in \mathbf{L}^2(\Omega)$ *such that*

$$\mathbf{curl}\,\mathcal{B} = \mathbf{f} \text{ in } \Omega, \tag{4}$$
$$\operatorname{div}\mathcal{B} = g \text{ in } \Omega, \tag{5}$$
$$\mathcal{B}\cdot\mathbf{n}_{|\Gamma} = 0. \tag{6}$$

The fact that g has a mean zero value stems from (6). The existence and uniqueness of $\mathcal{B}$ can also be inferred by using a saddle-point approach.

In both cases, the existence and uniqueness result can be achieved thanks to the *Weber inequality*, which stems from the compactness result of [16]:

Proposition 1.1 *In* $\mathbf{H}_0(\mathbf{curl}\,;\Omega)\cap\mathbf{H}(\operatorname{div}\,;\Omega)$ *and* $\mathbf{H}(\mathbf{curl}\,;\Omega)\cap\mathbf{H}_0(\operatorname{div}\,;\Omega)$, *the seminorm* $\mathbf{u}\mapsto\left(||\mathbf{curl}\,\mathbf{u}||_0^2+||\operatorname{div}\mathbf{u}||_0^2\right)^{1/2}$ *is a norm, which is equivalent to the canonical norm* $||\cdot||_{0,\mathbf{curl},\operatorname{div}}$. *In other words, there exists a constant* $C>0$ *such that*

$$||\mathbf{u}||_0 \le C\,\left(||\mathbf{curl}\,\mathbf{u}||_0^2+||\operatorname{div}\mathbf{u}||_0^2\right)^{1/2},$$

for all $\mathbf{u}$ *in* $\mathbf{H}_0(\mathbf{curl}\,;\Omega)\cap\mathbf{H}(\operatorname{div}\,;\Omega)$ *and* $\mathbf{H}(\mathbf{curl}\,;\Omega)\cap\mathbf{H}_0(\operatorname{div}\,;\Omega)$.

1.2 Mixed boundary conditions

As for the mixed boundary conditions, we shall follow Fernandes and Gilardi [14]. One has first to define a splitting of the boundary : $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$, where $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are compact (Lipschitz) submanifolds of Γ; $(\Gamma_1^k)_{0\le k\le K_m-1}$ are the connected components of Γ_1; the interface $\partial\Gamma = \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ is a one-dimensional, (Lipschitz) submanifold of Γ.

The third div-curl problem is, given $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$ and g in $L^2(\Omega)$:
Find $\mathbf{M} \in \mathbf{L}^2(\Omega)$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{M} &= \mathbf{f} \text{ in } \Omega, &(7)\\ \operatorname{div}\mathbf{M} &= g \text{ in } \Omega, &(8)\\ \mathbf{M}\times\mathbf{n}_{|\Gamma_1} &= 0, &(9)\\ \mathbf{M}\cdot\mathbf{n}_{|\Gamma_2} &= 0. &(10)\end{aligned}$$

The assumptions on $\mathbf{f}$ and g will be specified later on.

Uniqueness of the solution The space

$$\mathbb{H} = \{\mathbf{z} \in \mathbf{L}^2(\Omega) : \ \mathbf{curl}\,\mathbf{z} = 0,\ \operatorname{div}\mathbf{z} = 0,\ \mathbf{z}\times\mathbf{n}_{|\Gamma_1} = 0,\ \mathbf{z}\cdot\mathbf{n}_{|\Gamma_2} = 0\}, \tag{11}$$

is of dimension $K_m - 1$. Let $(\mathbf{h}_k)_{1\le k\le K_m-1}$ be a basis of $\mathbb{H}$. In order to have a unique solution to (7)-(10), it is necessary to add

$$(\mathbf{M}, \mathbf{h}_k)_0 = \alpha_k,\ 1 \le k \le K_m - 1, \tag{12}$$

where $(\alpha_k)_{1\le k\le K_m-1}$ is an element of $\mathbb{R}^{K_m-1}$.

Existence and uniqueness of the solution As $\mathbb{H}$ is a finite dimensional vector space, it is possible to split the space of solutions

$$\{\mathbf{z} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\operatorname{div}\,;\Omega) : \ \mathbf{z}\times\mathbf{n}_{|\Gamma_1} = 0,\ \mathbf{z}\cdot\mathbf{n}_{|\Gamma_2} = 0\}$$

into $\mathbb{H} \overset{\perp}{\oplus} \mathbb{H}^\perp$, and $\mathbf{M}$ as $\mathbf{M} = \mathbf{h} + \mathbf{M}_\perp$. Note that the orthogonality can be understood with respect to either the $\mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\operatorname{div}\,;\Omega)$ scalar product, or the $\mathbf{L}^2(\Omega)$ scalar product, thanks to the definition of $\mathbb{H}$.
It is clear that (12) characterizes the part of the solution which belongs to $\mathbb{H}$, i.e. $\mathbf{h}$.
Then, one can carry out a saddle-point reformulation of the problem in $\mathbf{M}_\perp$, that is (7)-(10) and (12) with $(\alpha_k)_k = 0$, with two Lagrange multipliers, one for (8) and the other for the vanishing (12). It is an interesting, but somewhat lengthy exercise, to prove that the solution is indeed unique, with zero Lagrange multipliers and then to recover the original equations.

Let us mention that, in order to achieve existence and uniqueness of the saddle point problem with zero Lagrange multipliers, one has to use the next three ingredients:

- A generalization of Weber's inequality

  Proposition 1.2 *In* $\mathbb{H}^\perp$, $\mathbf{u} \mapsto \left(||\mathbf{curl}\,\mathbf{u}||_0^2 + ||\operatorname{div}\mathbf{u}||_0^2\right)^{1/2}$ *is a norm, which is equivalent to the canonical norm* $||\cdot||_{0,\mathbf{curl},\operatorname{div}}$.

- An orthogonal decomposition of $\mathbf{L}^2(\Omega)$.
- An integration by parts formula, for elements of $H^1(\Omega)$, which vanish on Γ_1.

To conclude, let us also mention that to recover the original set of equations (7)-(10) and the vanishing (12), one has to choose $\mathbf{f}$ such that $\operatorname{div} \mathbf{f} = 0$, $\mathbf{f} \cdot \mathbf{n}_{|\Gamma_1} = 0$, and $\mathbf{f}$ is orthogonal to

$$\{\mathbf{z} \in \mathbf{L}^2(\Omega) : \ \mathbf{curl}\, \mathbf{z} = 0,\ \operatorname{div} \mathbf{z} = 0,\ \mathbf{z} \cdot \mathbf{n}_{|\Gamma_1} = 0,\ \mathbf{z} \times \mathbf{n}_{|\Gamma_2} = 0\}.$$

2 The axisymmetric geometry and operators

2.1 Notations

Let us consider the surface of revolution Γ generated by the rotation around the (Oz) axis of a polygonal line γ_b, the extremities of which stand on (Oz). Let Ω be the volume limited by Γ, ω the intersection of Ω and a meridian half-plane, and $\gamma = \gamma_a \cup \gamma_b$ its boundary, where γ_a corresponds to the segment of (Oz) lying between the extremities of γ_b. By definition, Γ is piecewise smooth, and the domain Ω is Lipschitz continuous; the same holds for γ and ω respectively.

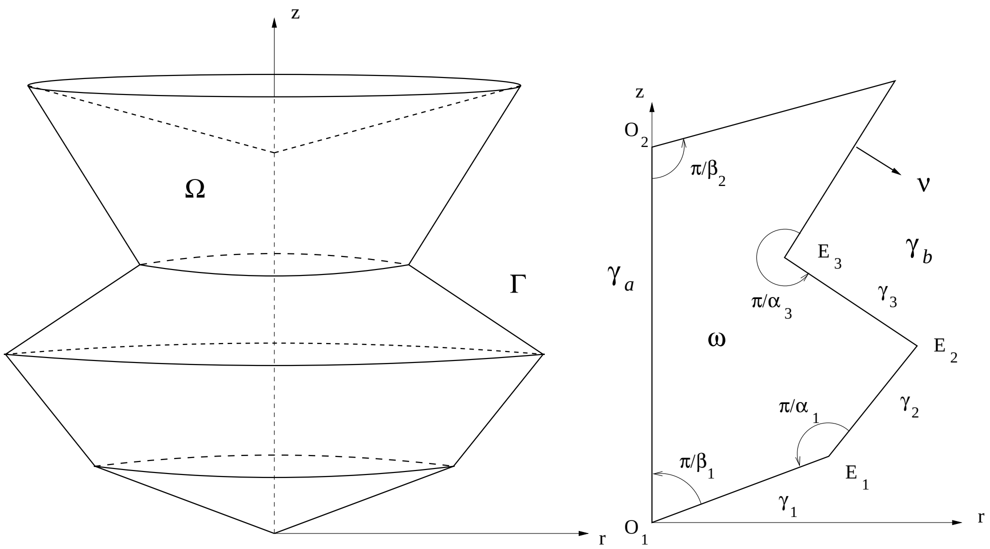

Figure 1: The domains Ω and ω.

Let us denote by $\gamma_1, \ldots, \gamma_{n+1}$ the sides of γ_b labeled counterclockwise, and $\Gamma_1, \ldots, \Gamma_{n+1}$ the conical sectors, or faces, they generate. $E_1, \ldots, E_n$ stand for the corners of γ_b, which are not on the axis (Oz) (i.e. E_j is the intersection of γ_j and γ_{j+1}); the corresponding angles are called π/α_j. In the same way, O_1 and O_2 are the extremities of γ_b, and π/β_1 and π/β_2 are the corresponding conical angles (see Figure 1).

The natural coordinates for this type of domain are the cylindrical coordinates (r, θ, z), with the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. In these coordinates, a meridian half-plane is defined by the equation $\theta = cst$: (r, z) thus correspond to cartesian coordinates in this half-plane. Note that the components of $\mathbf{n}$ read $(n_r, 0, n_z)$ – thanks to the symmetry, there is no θ-component.

2.2 Axisymmetric operators in cylindrical coordinates

In cylindrical coordinates, the formulas of the gradient, divergence and curl operators are given by (i)-(iii) (see the Annex).

According to its definition, the fact that there is a *symmetry of revolution* means that all (partial) derivatives with respect to θ of either scalar fields or the cylindrical coordinates of vector fields vanish. Those fields will be called *invariant by rotation* (resp. *axisymmetric*) if they are scalar (resp. vector). In the paper, it is therefore usually assumed that $\partial_\theta \cdot = 0$, except at paragraph 4.2 and in the last Section. For axisymmetric vector fields, this yields a decoupling between, on the one hand, the divergence and the θ-component of the curl, which are functions of (u_r, u_z), and, on the other hand, the r- and z-components of the curl, which are functions of u_θ. Now, one has the

Definition 2.1 *For any vector field* $\mathbf{u}$, *let* $\mathbf{u}_m = \varpi_m(\mathbf{u}) = u_r\,\mathbf{e}_r + u_z\,\mathbf{e}_z$ *and* $\mathbf{u}_\theta = \varpi_\theta(\mathbf{u}) = u_\theta\,\mathbf{e}_\theta$. *They are respectively called meridian and azimuthal components of* $\mathbf{u}$.

After that, one can easily check the

Proposition 2.2 *For any axisymmetric vector field* $\mathbf{u}$, *in the sense of distributions:*

- *if* $\mathbf{u}$ *is meridian* ($\varpi_\theta(\mathbf{u}) = 0$), $\mathbf{curl}\,\mathbf{u}$ *is azimuthal and* $\Delta\mathbf{u}$ *is meridian,*
- *if* $\mathbf{u}$ *is azimuthal* ($\varpi_m(\mathbf{u}) = 0$), $\mathbf{curl}\,\mathbf{u}$ *is meridian,* $\Delta\mathbf{u}$ *is azimuthal and* $\operatorname{div}\mathbf{u} = 0$,
- *the following identities hold:* $\mathbf{curl}\,\mathbf{u}_m = \varpi_\theta(\mathbf{curl}\,\mathbf{u})$, $\mathbf{curl}\,\mathbf{u}_\theta = \varpi_m(\mathbf{curl}\,\mathbf{u})$, $\operatorname{div}\mathbf{u}_m = \operatorname{div}\mathbf{u}$, $\Delta\mathbf{u}_m = \varpi_m(\Delta\mathbf{u})$, $\Delta\mathbf{u}_\theta = \varpi_\theta(\Delta\mathbf{u})$.

As a consequence, when one solves the static or time-dependent Maxwell equations with axisymmetric data and initial conditions in an axisymmetric domain, this allows to decouple the set of equations in two unrelated parts, one in $(\mathbf{E}_m, \mathbf{B}_\theta)$, and the other in $(\mathbf{E}_\theta, \mathbf{B}_m)$.

Proposition 2.3 *In the meridian half-plane, let the* modified Laplace *operator* Δ^- *be defined as follows*

$$\Delta^-\varphi = \frac{\partial^2\varphi}{\partial r^2} - \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2}. \tag{13}$$

Then, for an axisymmetric, azimuthal vector field $\mathbf{u}$, *there holds :*

$$\mathbf{u} = \frac{\varphi}{r}\,\mathbf{e}_\theta \implies \Delta\mathbf{u} = \frac{1}{r}\,\Delta^-\varphi\,\mathbf{e}_\theta. \tag{14}$$

In cylindrical coordinates, the expression of the Jacobian of any vector field $\mathbf{u}$ is given by (iv) in the Annex. Once again, there is a decoupling of the meridian and azimuthal components for an axisymmetric vector field.

Last, let us emphasize an obvious orthogonality property about the meridian and azimuthal components of vector fields. As there are mutually orthogonal pointwise, the same is true in the sense of the $\mathbf{L}^2(\Omega)$ scalar product.

Proposition 2.4 *For any* $(\mathbf{u}, \mathbf{v}) \in [\mathbf{L}^2(\Omega)]^2$, *there holds* $\displaystyle\int_\Omega \varpi_\theta(\mathbf{u}) \cdot \varpi_m(\mathbf{v})\, d\Omega = 0.$

This property also holds for the curl and the vector Laplace ($\Delta = \mathbf{grad}\operatorname{div} - \mathbf{curl}\,\mathbf{curl}$) operators, or the Jacobian of a field, provided that they belong to $\mathbf{L}^2(\Omega)$.

2.3 Axisymmetric operators in spherical coordinates

Close to the conical vertices O_1 and O_2, it is more favourable to use the spherical coordinates centered at $O \in \{O_1, O_2\}$, that is (ρ, θ, ϕ), where θ is the *cylindrical azimuth* and ϕ is the *angle with* (Oz) (see Figure 2). Notice that, in order to keep the same basis vector $\mathbf{e}_\theta$, we use a *non-standard* representation of the two angular variables θ and ϕ. In these coordinates, the expressions for the first order differential operators are given in

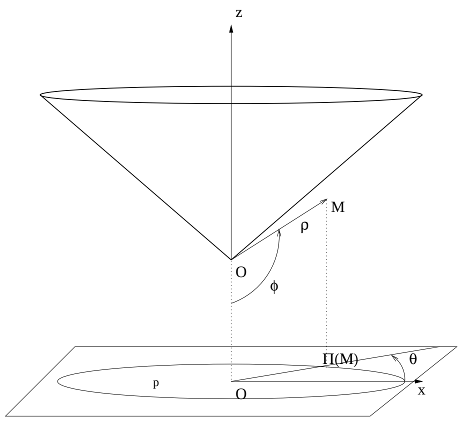

Figure 2: Spherical coordinates.

the Annex (v)-(viii). In these coordinates, the *symmetry of revolution* still amounts to the property that all (partial) derivatives with respect to θ of either scalar fields or the spherical coordinates of vector fields vanish.

3 Sobolev spaces and the symmetry of revolution

In this Section, Ω stands for any axisymmetric domain, possibly unbounded. The property of symmetry of revolution extends to distributions and thus, to Sobolev spaces. We shall investigate in some detail the induced properties, as they are of upmost importance in the solution to the axisymmetric Maxwell equations. Note that a number of results, necessary to the overall comprehension of the theory, come from [6], and are stated without proof.

3.1 Distributions

Let $\mathcal{R}_\eta$ be the rotation of axis (Oz) and angle η. We first consider scalar distributions and, in the second place, vector distributions.

Definition 3.1 *The right-action of $\mathcal{R}_\eta$ on a distribution $T \in \mathcal{D}'(\Omega)$ is defined by:*

$$\forall f \in \mathcal{D}(\Omega), \quad \langle T \circ \mathcal{R}_\eta, f \rangle = \left\langle T, f \circ \mathcal{R}_\eta^{-1} \right\rangle .$$

The distribution $\breve{T}$ is labeled invariant by rotation *if and only if:*

$$\forall \eta \in [-\pi, \pi], \quad \breve{T} \circ \mathcal{R}_\eta = \breve{T}.$$

Let $\breve{\mathcal{D}}'(\Omega)$ be the space of distributions invariant by rotation, and let $\breve{\mathcal{D}}(\Omega) = \mathcal{D}(\Omega) \cap \breve{\mathcal{D}}'(\Omega)$.

Evidently, this definition means that an element $\breve{T}$ of $\breve{\mathcal{D}}'(\Omega)$ is "independent" of the variable θ. Actually, if $\breve{T}$ is sufficiently smooth, so that one can consider its values almost everywhere, it is characterized by the datum of its trace in a meridian half-plane $T(r, z) = \breve{T}(x, y, z)$. In particular,

Definition 3.2 *Let $\mathcal{D}_+(\omega)$ be the subspace of $\mathcal{C}^\infty(\overline{\varpi})$ defined by*

$$\mathcal{D}_+(\omega) = \left\{ \varphi \in \mathcal{C}_c^\infty(\overline{\varpi}) : \ \operatorname{supp}\varphi \cap \gamma_b = \emptyset \ \textit{and} \ \forall j \in \mathbb{N}, \ \left.\frac{\partial^{2j+1}\varphi}{\partial r^{2j+1}}\right|_{\gamma_a} = 0 \right\},$$

and let $\mathcal{D}'_+(\omega)$ be its dual.

One has the following

Proposition 3.3 *The trace operator is an isomorphism from $\breve{\mathcal{D}}(\Omega)$ to $\mathcal{D}_+(\omega)$.*

Definition 3.4 *Let the "trace" T of a distribution $\breve{T}$ invariant by rotation be defined as*

$$_{\mathcal{D}'_+(\omega)}\langle T, \varphi\rangle_{\mathcal{D}_+(\omega)} = {}_{\breve{\mathcal{D}}'(\Omega)}\left\langle \breve{T}, \breve{\varphi}\right\rangle_{\breve{\mathcal{D}}(\Omega)}, \quad \forall \breve{\varphi} \in \breve{\mathcal{D}}(\Omega).$$

Proposition 3.5 *The trace operator is an isomorphism from $\breve{\mathcal{D}}'(\Omega)$ to $\mathcal{D}'_+(\omega)$.*

We turn now to vector distributions. For that we need the

Definition 3.6 *Let $\mathbf{T} \in \mathbf{D}'(\Omega)$; the left-action of $\mathcal{R}_\eta$ is defined by:*

$$\forall \mathbf{f} \in \mathbf{D}(\Omega), \langle \mathcal{R}_\eta \circ \mathbf{T}, \mathbf{f}\rangle = \left\langle \mathbf{T}, \mathcal{R}_\eta^{-1} \circ \mathbf{f}\right\rangle.$$

The vector distribution $\breve{\mathbf{T}}$ is labeled axisymmetric *if and only if:*

$$\forall \eta \in [-\pi, \pi], \ \breve{\mathbf{T}} \circ \mathcal{R}_\eta = \mathcal{R}_\eta \circ \breve{\mathbf{T}}.$$

Let $\breve{\mathbf{D}}'(\Omega)$ be the space of axisymmetric vector distributions, and let $\breve{\mathbf{D}}(\Omega) = \mathbf{D}(\Omega) \cap \breve{\mathbf{D}}'(\Omega)$.

Naturally, a vector distribution $\breve{\mathbf{T}}$ is axisymmetric if and only if its cylindrical components $(\breve{T}_r, \breve{T}_\theta, \breve{T}_z)$ are invariant by rotation. If the vector distribution $\breve{\mathbf{T}}$ is sufficiently smooth, its trace is defined as the triple $\mathbf{T} = (T_r, T_\theta, T_z) \in \mathcal{D}'_+(\omega)^3$ of traces. In order to characterize the traces, we introduce yet another subspace of $\mathcal{C}^\infty(\overline{\varpi})$.

Definition 3.7 *Let $\mathcal{D}_-(\omega)$ be defined by*

$$\mathcal{D}_-(\omega) = \left\{ \varphi \in \mathcal{C}_c^\infty(\overline{\varpi}) : \ \operatorname{supp}\varphi \cap \gamma_b = \emptyset \ \textit{and} \ \forall j \in \mathbb{N}, \ \left.\frac{\partial^{2j}\varphi}{\partial r^{2j}}\right|_{\gamma_a} = 0 \right\},$$

and let $\mathcal{D}'_-(\omega)$ be its dual.

Proposition 3.8 *The trace operator is an isomorphism*

from $\breve{\mathbf{D}}(\Omega)$ to $\mathcal{D}_-(\omega) \times \mathcal{D}_-(\omega) \times \mathcal{D}_+(\omega)$.

Thanks to the above Proposition, one finally gets the

Definition 3.9 *The trace of an axisymmetric vector distribution $\breve{\mathbf{T}}$ is defined by*

$$\langle \mathbf{T}, \mathbf{f}\rangle = {}_{\breve{\mathbf{D}}'(\Omega)}\left\langle \breve{\mathbf{T}}, \breve{\mathbf{f}}\right\rangle_{\breve{\mathbf{D}}(\Omega)}, \quad \forall \breve{\mathbf{f}} \in \breve{\mathbf{D}}(\Omega),$$

where the brackets on the left-hand side are between $\mathcal{D}_-(\omega) \times \mathcal{D}_-(\omega) \times \mathcal{D}_+(\omega)$ and its dual.

3.2 Sobolev spaces

The results, which have been obtained for distributions and smooth fields, are now extended to Sobolev spaces. Let us introduce the spaces $\breve{L}^2(\Omega) = L^2(\Omega) \cap \breve{\mathcal{D}}'(\Omega)$, $\breve{H}^s(\Omega) = H^s(\Omega) \cap \breve{\mathcal{D}}'(\Omega)$ (for $s \in \mathbb{R}$), $\breve{\mathbf{L}}^2(\Omega) = \mathbf{L}^2(\Omega) \cap \breve{\mathbf{D}}'(\Omega)$, $\breve{\mathbf{H}}(\mathbf{curl}\,;\Omega) = \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \breve{\mathbf{D}}'(\Omega)$, and so on. We have to study the range of those spaces by the trace operator. For that, we introduce the weighted Lebesgue spaces on ω

$$L^2_\alpha(\omega) = \left\{ f : \ f \text{ is measurable on } \omega, \ \int_\omega |f|^2\, r^\alpha\, dr\, dz < +\infty \right\}, \quad \alpha \in \mathbb{R},$$

with its canonical norm $||\cdot||_{0,\alpha}$, and the related scale of Sobolev spaces $H^s_\alpha(\omega)$, with the canonical norms $||\cdot||_{s,\alpha}$. In the remainder of the paper, we shall only use the scale up to $s = 2$, so we give only those results. The more general ones can be found in [6].

Proposition 3.10 *The mapping $L^2_\alpha(\omega) \to L^2_{\alpha-2}(\omega)$, $f \mapsto r\, f$, is an isometry.*

Proposition 3.11 *The trace mapping $\breve{f} \mapsto f$ is an isometry (up to a factor $\sqrt{2\pi}$) from $\breve{L}^2(\Omega)$ to $L^2_1(\omega)$. The same holds for the reciprocal lifting, $L^2_1(\omega) \to \breve{L}^2(\Omega)$, $f \mapsto \breve{f}$.*

Definition 3.12 *Let $s \in]0, 2]$ and set*

- *if $s \neq 2$, $H^s_+(\omega) = H^s_1(\omega)$;*
- *if $s = 2$, $H^2_+(\omega) = \left\{ w \in H^2_1(\omega) : \ \dfrac{\partial^2 w}{\partial r^2} \in L^2_{-1}(\omega) \right\}$, which is a Hilbert space endowed with the norm $||w||_{2,+} = \left(||w||^2_{2,1} + ||\dfrac{\partial^2 w}{\partial r^2}||^2_{0,-1} \right)^{1/2}$.*

Then, for s in $]0, 2]$, one has the

Proposition 3.13 *The trace operator is an isomorphism from $\breve{H}^s(\Omega)$ to $H^s_+(\omega)$.*

It is also possible, in the same manner, to characterize traces of axisymmetric elements of Sobolev spaces.

Proposition 3.14 *The trace mapping $\breve{\mathbf{f}} \mapsto \mathbf{f}$ is an isometry (up to a factor $\sqrt{2\pi}$) from $\breve{\mathbf{L}}^2(\Omega)$ to $L^2_1(\omega)^3$.*

Definition 3.15 *Let $s \in]0, 2]$ and set*

- *if $s \neq 1$, $H^s_-(\omega) = H^s_1(\omega)$;*
- *if $s = 1$, $H^1_-(\omega) = H^1_1(\omega) \cap L^2_{-1}(\omega)$, which is a Hilbert space endowed with the norm $||w||_{1,-} = \left(||w||^2_{1,1} + ||w||^2_{0,-1} \right)^{1/2}$.*

Proposition 3.16 *The trace operator is an isomorphism from $\breve{\mathbf{H}}^s(\Omega)$ to $H^s_-(\omega) \times H^s_-(\omega) \times H^s_+(\omega)$, for s in $]0, 2]$.*

When s is fixed to 1, which is of practical importance in the following, one can further improve this result, by the

Proposition 3.17 *The trace mapping $\breve{\mathbf{f}} \mapsto \mathbf{f}$ is an isometry (up to a factor $\sqrt{2\pi}$) from $\breve{\mathbf{H}}^1(\Omega)$ to $H^1_-(\omega) \times H^1_-(\omega) \times H^1_+(\omega)$.*

Proof : Thanks to the general expression (iv) of the Jacobian of an element of $\breve{\mathbf{H}}^1(\Omega)$, and Proposition 3.14, one gets the identity

$$\frac{1}{2\pi}\,||\breve{\mathbf{f}}||^2_{1,\Omega} = ||f_r||^2_{1,1,\omega} + ||f_\theta||^2_{1,1,\omega} + ||f_z||^2_{1,1,\omega} + ||f_r||^2_{0,-1,\omega} + ||f_\theta||^2_{0,-1,\omega},$$

which is indeed equal to the square of the canonical norm on $H^1_-(\omega)\times H^1_-(\omega)\times H^1_+(\omega)$. ■

There is an additional property, worth mentioning, about the traces of elements of $H^1_-(\omega)$ on γ_a.

Proposition 3.18 *Let v belong to $H^1_-(\omega)$: $v_{|\gamma_a}\in L^2(\gamma_a)$ and $v_{|\gamma_a}=0$.*

Proof : Assume for now that we are interested in $v\in H^1_-(]0,1[)$. One can easily prove that v vanishes at $r=0$. Indeed, for any given sequence $(r_k)_k$, which tends to 0,

$$v^2(r_l)-v^2(r_m) = 2\int_{r_l}^{r_m} v\; v'\; dr \le \left\{\int_{r_l}^{r_m} v^2\;\frac{1}{r}\;dr\right\}^{1/2}\left\{\int_{r_l}^{r_m}(v')^2\;r\;dr\right\}^{1/2}.$$

This yields that $(v^2(r_k))_k$ is a Cauchy sequence. Now, has v belongs to $L^2_{-1}(]0,1[)$, its limit can only be 0. In addition, if $\omega=\gamma_a\times]0,1[$, $H^1_-(\omega)$ is embedded in $L^2(\gamma_a,H^1_-(]0,1[))$. Thus, elements of $H^1_-(\omega)$ have a trace on γ_a, which is moreover equal to zero in the L^2-sense. This can then be generalized to any polygon ω by localization. ■

Proposition 3.19 *The range of the trace operator from $\breve{\mathbf{H}}(\mathbf{curl}\,;\Omega)$ is:*
$R(\omega)=\left\{\mathbf{w}=(w_r,w_\theta,w_z):\ \mathbf{w}_m=(w_r,w_z)\in L^2_1(\omega)^2,\ \operatorname{curl}\mathbf{w}_m\in L^2_1(\omega),\ rw_\theta\in H^1_{-1}(\omega)\right\}$.

Proof : Let $\breve{\mathbf{w}}\in\breve{\mathbf{H}}(\mathbf{curl}\,;\Omega)$. According to Proposition 3.14, $\mathbf{w}$ belongs to $L^2_1(\omega)^3$. In addition, for an axisymmetric field, (iii) leads to

$$\begin{aligned}
(\mathbf{curl}\,\breve{\mathbf{w}})_r &= -\frac{\partial\breve{w}_\theta}{\partial z}\quad\left(=-\frac{1}{r}\frac{\partial(r\breve{w}_\theta)}{\partial z}\right),\\
(\mathbf{curl}\,\breve{\mathbf{w}})_\theta &= \frac{\partial\breve{w}_r}{\partial z}-\frac{\partial\breve{w}_z}{\partial r}\quad\left(=-\operatorname{curl}\mathbf{w}_m\right),\\
(\mathbf{curl}\,\breve{\mathbf{w}})_z &= \frac{1}{r}\frac{\partial(r\breve{w}_\theta)}{\partial r}.
\end{aligned}$$

Thus $\operatorname{curl}\mathbf{w}_m\in L^2_1(\omega)$. Now, as far as $r\breve{w}_\theta$ is concerned, its gradient is equal to the product of r and the vector field with cylindrical coordinates $((\mathbf{curl}\,\breve{\mathbf{w}})_z,0,-(\mathbf{curl}\,\breve{\mathbf{w}})_r)$: thanks to Proposition 3.10, $rw_\theta\in H^1_{-1}(\omega)$.

Conversely, given $(w_r,w_\theta,w_z)\in R(\omega)$, it is possible to build a lifting $\breve{\mathbf{w}}\in\breve{\mathbf{L}}^2(\Omega)$. By the definition of $R(\omega)$, the curl of the lifting belongs to $\breve{\mathbf{L}}^2(\Omega)$. ■

From now on, an axisymmetric field $\breve{\mathbf{u}}$ and its trace on a meridian half-plane $\mathbf{u}$ shall be merged.

4 Closedness results

In this Section and the next, our aim is to demonstrate that when the domain Ω is not convex, the regular axisymmetric fields are not dense in the "natural" spaces of axisymmetric electromagnetic fields. In this Section, we carry out the first step: we prove that the subspaces of regular axisymmetric fields are closed in their "natural" couterpart (except in one special case). In Section 5, we further prove that when the domain Ω is not convex, the orthogonal of the subspaces is not reduced to $\{0\}$, thus leading to the lack of density.

First recall that the "natural" spaces of electromagnetic fields are, cf. Section 1,

$$\mathcal{X} = \mathbf{H}_0(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,;\Omega) \text{ and } \mathcal{Y} = \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}_0(\mathrm{div}\,;\Omega).$$

What is called in the following *regular* is assumed to belong to the "regularized" subspaces

$$\mathcal{X}_R = \mathcal{X} \cap \mathbf{H}^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) :\ \mathbf{u} \times \mathbf{n}_{|\Gamma} = 0\},\ \mathcal{Y}_R = \mathcal{Y} \cap \mathbf{H}^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) :\ \mathbf{u} \cdot \mathbf{n}_{|\Gamma} = 0\}.$$

Their respective axisymmetric subspaces are denoted by $\breve{\mathcal{X}}$, $\breve{\mathcal{Y}}$, $\breve{\mathcal{X}}_R$ and $\breve{\mathcal{Y}}_R$. This Section is devoted to proving the

Theorem 4.1 *In the spaces $\breve{\mathcal{X}}_R$ and $\breve{\mathcal{Y}}_R$, the canonical norm of* $\mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,;\Omega)$ *is equivalent to the $\mathbf{H}^1$-norm. In other words, there exists a constant α, which depends only of Ω, such that*

$$\alpha\,||\mathbf{u}||_1^2 \leq ||\mathbf{u}||^2_{0,\mathbf{curl},\mathrm{div}}\,. \tag{15}$$

As an immediate consequence, $\breve{\mathcal{X}}_R$ and $\breve{\mathcal{Y}}_R$ are closed in $\breve{\mathcal{X}}$ and $\breve{\mathcal{Y}}$ respectively.

In the magnetic case, a constructive proof is built as follows. First, an integration by parts formula produced by Costabel [11], which holds for $\mathbf{H}^2$-regular fields, is recalled. After that, one has to check a technical point, i.e. that fields of $\mathcal{C}^\infty(\overline{\Omega})$, which satisfy the magnetic boundary condition, are dense in $\mathcal{Y}_R$, in order to generalize the integration by parts formula to fields of $\mathcal{Y}_R$. Finally, it is proved that

$$\exists K, \quad \forall \mathbf{u} \in \breve{\mathcal{Y}}_R, \quad ||\nabla \mathbf{u}||_0^2 \leq K\,\left(||\mathbf{curl}\,\mathbf{u}||_0^2 + ||\mathrm{div}\,\mathbf{u}||_0^2\right), \tag{16}$$

which amounts to (15).

Remark 4.1 The density of smooth fields is also true in $\mathcal{X}_R$. Thus, the integration by parts formula can be extended to elements of $\mathcal{X}_R$. ■

In the electric case, an inductive proof stems from the continuous splitting of fields of $\mathcal{X}$ in a regular part, which belongs to $\mathcal{X}_R$, and the gradient of a potential, which belongs to $H^1_0(\Omega)$. This continuous splitting has been obtained by Birman and Solomyak [7]. Note that the reason why an inductive proof can not be applied to the magnetic case is addressed in the course of the reasoning.

Corollary 4.2 *There holds*

$$\breve{\mathcal{X}} = \breve{\mathcal{X}}_R \overset{\perp}{\oplus} \breve{\mathcal{X}}_S \quad \text{and} \quad \breve{\mathcal{Y}} = \breve{\mathcal{Y}}_R \overset{\perp}{\oplus} \breve{\mathcal{Y}}_S, \tag{17}$$

which allows to split (orthogonally) the fields into a regular part and a singular part.

4.1 The Costabel integration by parts formula

Costabel [11, 12] proved that, as soon as Γ is piecewise smooth (at least piecewise-$\mathcal{C}^2$), one has, for all $(\mathbf{u}, \mathbf{v}) \in [\mathbf{H}^2(\Omega)]^2$,

$$(\nabla \mathbf{u} \mid \nabla \mathbf{v})_0 = (\mathbf{curl}\,\mathbf{u} \mid \mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u} \mid \mathrm{div}\,\mathbf{v})_0 - b(\mathbf{u}, \mathbf{v}) + d(\mathbf{u}, \mathbf{v}). \tag{18}$$

The boundary terms b and d are derived *via* the tangential gradient and divergence operators $\mathbf{grad}_\top$ and $\mathrm{div}_\top$. If we let, for $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $u_n = \mathbf{u}\cdot\mathbf{n}_{|\Gamma}$ and $\mathbf{u}_\top = \mathbf{n}\times(\mathbf{u}\times\mathbf{n})_{|\Gamma}$, there holds

$$\forall(\mathbf{u},\mathbf{v}) \in [\mathbf{H}^2(\Omega)]^2, \quad d(\mathbf{u},\mathbf{v}) = \int_\Gamma \{\mathbf{grad}_\top u_n \cdot \mathbf{v}_\top - (\mathrm{div}_\top \mathbf{u}_\top)\, v_n\}\, d\Gamma.$$

Provided that $\{\mathbf{u} \in \mathbf{H}^2(\Omega) :\ \mathbf{u}\cdot\mathbf{n}_{|\Gamma} = 0\}$ is dense in $\mathcal{Y}_R$, it is possible to extend $d(\mathbf{u},\mathbf{v})$ to $[\mathcal{Y}_R]^2$ by 0.

The term $b(\mathbf{u},\mathbf{v})$ is defined through the second fundamental form $\mathsf{B} = \nabla\mathbf{n}$ of Γ, by

$$\forall(\mathbf{u},\mathbf{v}) \in [\mathbf{H}^1(\Omega)]^2, \quad b(\mathbf{u},\mathbf{v}) = \int_\Gamma \{\mathbf{u}_\top \cdot \mathsf{B} \cdot \mathbf{v}_\top + (\mathrm{tr}\,\mathsf{B})\, u_n\, v_n\}\, d\Gamma.$$

Ω being axisymmetric, the normal $\mathbf{n}$ is such that $n_\theta = 0$. Moreover, on each face, $n_r = cst$ and $n_z = cst$. Thus, there remains a single non zero component in the expression of $\nabla\mathbf{n}$ (cf. (iv)) restricted to each face, that of indices (θ,θ), equal to n_r/r. Then

$$b(\mathbf{u},\mathbf{v}) = \int_\Gamma \frac{n_r}{r}\,(u_\theta\, v_\theta + u_n\, v_n)\; d\Gamma. \tag{19}$$

With the proposed boundary condition, there is a single non vanishing term, in $u_\theta\, v_\theta$.

Note that, when the test-fields have a vanishing normal or tangential trace, the inequality (16) is now equivalent, thanks to (18), to

$$\exists k < 1, \quad -b(\mathbf{u},\mathbf{u}) \le k\, ||\nabla\mathbf{u}||_0^2. \tag{20}$$

Remark 4.2 k in (20) and K in (16) depend only on the domain Ω. It is worth mentioning that, when Ω is convex, one has $n_r \ge 0$, so the bilinear form b is positive. Hence the result with $K = 1$ or $k = 0$. ■

4.2 Density of smooth fields

In order to obtain the density result, we provide a proof which follows step by step that of Costabel *et al.* [12]. Along the way, technical results are added to handle the case of the conical vertices. Let us begin by some Hardy inequalities, stated without proof.

Lemma 4.3 1. *Let $I =]0,R[$, with $R \in \mathbb{R}_*^+ \cup \{+\infty\}$.*
The mapping $\mathcal{L} : f \mapsto \mathcal{L}f(x) = \dfrac{1}{x}\displaystyle\int_0^x f(y)\,dy$, is continuous from $L^2_\alpha(I)$ to itself for $\alpha < 1$, and its norm is bounded by $2/(1-\alpha)$.

2. *Let $I =]R,+\infty[$, with $R \in \mathbb{R}^+$.*
The mapping $\mathcal{L} : f \mapsto \mathcal{L}f(x) = \dfrac{1}{x}\displaystyle\int_x^{+\infty} f(y)\,dy$, is continuous from $L^2_\alpha(I)$ to itself for $\alpha > 1$, and its norm is bounded by $2/(\alpha-1)$.

Then, one has the

Lemma 4.4 *Let E be a point of $\mathbb{R}^2$, and (ρ,ϕ) be some local polar coordinates with E as the origin. Let ω_e be the angular sector defined by $\{(\rho,\phi) :\ 0 < \rho < \rho_0,\ 0 < \phi < \phi_0\}$. Finally, let $\chi \in \mathcal{D}(\mathbb{R})$ be a cut-off function, which is equal to 1 in a neighborhood of 0. Consequently, for all h in $H^1(\omega_e)$, h belongs to the closure (in $H^1(\omega_e)$) of the set*

$$S(h) = \{\rho^\alpha\,[1-\chi(n\,\rho)]\,h,\ \alpha \in]0,1[,\ n \in \mathbb{N}\}.$$

Proof : This corresponds to Lemmas 2.3 and 2.4 of [12]. ■

The proof of the following Lemma is omitted.

Lemma 4.5 *Let ω_e be defined as above. It is further assumed that, drawn in a plane (r,z), one has $\overline{\omega}_e \cap (Oz) = \emptyset$. Let Ω_e be the domain generated by the rotation of ω_e around (Oz). Then, for all $h \in H^1(\Omega_e)$, h belongs to the closure (in $H^1(\Omega_e)$) of the set $S(h)$ defined above, where ρ now corresponds to the distance to the edge generated by the rotation of E.*

For the conical vertices, the following result holds true.

Lemma 4.6 *Let (ρ,ϕ) be the local polar coordinates of origin O and axis (Oz) in a meridian half-plane. Let ω_c be defined as $\{(\rho,\phi)\, :\ 0<\rho<\rho_0,\ 0\le\phi<\phi_0\}$, and let Ω_c be the domain generated by the rotation of ω_c around (Oz). With* our *spherical coordinates, it reads $\Omega_c = \{(\rho,\theta,\phi)\, :\ (\rho,\phi)\in\omega_c,\ \theta\in[0,2\pi[\}$. Finally, let $h\in H^1(\Omega_c)$. Then the sequence $([1-\chi(n\,\rho)]\,h)_n$, where χ is the same cut-off function as before, goes to h in $H^1(\Omega_c)$.*

Proof : Let us prove an equivalent statement: namely, that $(\chi(n\,\rho)\,h)_n$ goes to 0 in $H^1(\Omega_c)$. There exists a, such that the support of $\chi(n\,\rho)$ is a subset of $B(O,a/n)$. Consequently, as $|\chi|\le 1$, one can show by the bounded convergence Theorem that the four sequences with terms $\chi(n\,\rho)\,h$, $\chi(n\,\rho)\,\partial_\rho\,h$, $1/\rho\,\chi(n\,\rho)\,\partial_\phi\,h$, $1/(\rho\sin\phi)\,\chi(n\,\rho)\,\partial_\theta\,h$ all go to zero in $L^2(\Omega_c)$.

Then, as $\partial_\rho[\chi(n\,\rho)\,h] = \chi(n\,\rho)\,\partial_\rho\,h + n\,\chi'(n\,\rho)\,h$, there remains only to prove that the limit of the last term is indeed 0.
The function $x\mapsto x\,\chi'(x)$ is continuous with a compact support ; therefore, it is bounded by a constant C, and one gets

$$||n\,\chi'(n\,\rho)\,h||_0^2 \le C^2 \int_{\Omega_c\cap B(O,a/n)} \frac{h^2}{\rho^2}\,d\Omega. \tag{21}$$

In addition, as the function $\chi(n\,\rho)\,h$ vanishes for ρ large enough, for almost all (ρ,θ,ϕ), there holds

$$\frac{1}{\rho}\,\chi(n\,\rho)\,h(\rho,\theta,\phi) = -\frac{1}{\rho}\int_\rho^{+\infty} \frac{\partial}{\partial s}(\chi(n\,s)\,h(s,\theta,\phi))\,ds.$$

Now, as $\chi(n\,\rho)\,h$ belongs to $H^1(\Omega_c)$, its partial derivative with respect to ρ is in $L^2(\Omega_c)$: thanks to the Fubini Theorem, $\rho\mapsto\partial_\rho[\chi(n\,\rho)\,h]$ is, for almost all (θ,ϕ), in $L^2_2(]0,\rho_0[)$. As it vanishes near infinity, it is also an element of $L^2_2(]0,+\infty[)$.
It is therefore possible to use the Hardy inequality (in ρ) with $\alpha=2$ and $R=0$. Thanks again to the Fubini Theorem, one gets that $1/\rho\,\chi(n\,\rho)\,h$ belongs to $L^2(\Omega_c)$, and therefore h/ρ too.
As a consequence, the right-hand side of (21) goes to 0 when n goes to infinity, which allows to conclude the proof. ■

Proposition 4.7 *Let Ω be the domain of Figure 1. Then, the space*

$$\{\mathbf{v}\in\mathcal{C}^\infty(\overline{\Omega})^3\cap\mathcal{Y}_R :\ \mathbf{v}\text{ vanishes in a neighborhood of vertices and edges}\} \tag{22}$$

is dense in $\mathcal{Y}_R$.

Proof : Let $\mathbf{u}\in\mathcal{Y}_R$. For any $\varepsilon>0$, let us build an element $\widetilde{\mathbf{u}}$ of the space defined by (22), and such that $||\mathbf{u}-\widetilde{\mathbf{u}}||_1\le\varepsilon$. The construction proceeds in four steps

1ST STEP: LOCALIZATION. Let us apply Lemma 4.6 to each component of $\mathbf{u}$, and each conical vertex:
$\mathbf{u}_1 = (1 - \sum_{i=1,2} \chi(n\,\rho_{O_i}))\,\mathbf{u}$, for a sufficiently large n, is such that $||\mathbf{u} - \mathbf{u}_1||_1 \leq \varepsilon$.
After that, Lemma 4.5 is applied to each component of $\mathbf{u}_1$, and each edge: for a sufficiently large m and a sufficiently small α, $\mathbf{u}_2$, defined by

$$\mathbf{u}_2 = \sum_{1 \leq j \leq n} {\rho_{E_j}}^{\alpha} \left(1 - \chi(m\,\rho_{E_j})\right)\,\mathbf{u}_1$$

is such that $||\mathbf{u}_1 - \mathbf{u}_2||_1 \leq \varepsilon$.
Note that, by construction, $\mathbf{u}_2$ belongs to $\mathcal{Y}_R$, and that it vanishes in a neighborhood $\mathcal{V}_0$ of the vertices and edges.

2ND STEP: SMOOTHING. The field $\mathbf{u}_2$ is smoothed by convolution with a regularizing sequence $(\eta_k)_k$. The resulting field $\mathbf{u}_3$ belongs to $\mathcal{C}^\infty(\overline{\Omega})^3$ and, for a sufficiently large k:

- $\mathbf{u}_3$ vanishes in a neighborhood $\mathcal{V}_1 \subset\subset \mathcal{V}_0$ of the vertices and edges,
- $||\mathbf{u}_2 - \mathbf{u}_3||_1 \leq \varepsilon$,
- $||\mathbf{u}_3 \cdot \mathbf{n}||_{1/2,\Gamma} \leq \varepsilon$.

3RD STEP: ENFORCEMENT OF THE BOUNDARY CONDITION. On each face Γ_j, let us consider the (localized) left inverse R_j of the trace operator $\gamma_j h = h_{|\Gamma_j}$ with the following properties (cf. [3]). Let $\mathcal{V}$ be a neighborhood of $\partial\Gamma_j$ on $\mathbb{R}^3$.

- For any element g of $\{g \in H^{1/2}(\Gamma_j) :\ g$ vanishes in $\mathcal{V} \cap \Gamma_j\}$, its lifting $R_j g$ is equal to zero in $\mathcal{V}$, on (Oz) and has a vanishing trace on the other faces.
- R_j is continuous from $H^{1/2}(\Gamma_j)$ to $H^1(\Omega)$ (independently of $\mathcal{V}$).

Note that, as $\Gamma_j \setminus \mathcal{V}$ is a smooth surface, and moreover, $R_j\gamma_j h = h$ locally for any $h \in H^1(\Omega)$, R_j is regularizing.

From the operators $(R_j)_{1 \leq j \leq n+1}$, we build the lifting R_n of the trace $\gamma_n \mathbf{v} = \mathbf{v} \cdot \mathbf{n}_{|\Gamma}$, from $\{g \in \Pi_j H^{1/2}(\Gamma_j) :\ g$ vanishes in a neighborhood of vertices and edges$\}$ to $\mathbf{H}^1(\Omega)$. It is such that:

- R_n is continuous.
- For any g in $\Pi_j H^{1/2}(\Gamma_j)$ such that it vanishes in a neighborhood $\mathcal{V}$ of the vertices and edges, the support of $R_n g$ does not intersect $\mathcal{V}$, nor (Oz). If, morevoer, the support of g is imbedded in $\Gamma_j \setminus \mathcal{V}$ for a given j, the support of $R_n g$ does not intersect any other face.
- R_n is regularizing.

Let C_n denote the norm of the operator R_n.

4TH STEP: CONCLUSION. Let us define $\widetilde{\mathbf{u}} = \mathbf{u}_3 - R_n \gamma_n \mathbf{u}_3$. Thanks to the regularizing property, this fields belongs to $\mathcal{C}^\infty(\overline{\Omega})^3$. In addition, it satisfies $\widetilde{\mathbf{u}} \cdot \mathbf{n}_{|\Gamma} = 0$ and it vanishes in $\mathcal{V}_1$: thus it is an element of the space defined by (22). Last, $||\widetilde{\mathbf{u}} - \mathbf{u}_3||_1 \leq C_n\, \varepsilon$, and therefore $||\widetilde{\mathbf{u}} - \mathbf{u}||_1 \leq (3 + C_n)\, \varepsilon$. ■

Remark 4.3 The case of the density in $\mathcal{X}_R$ follows in the same manner, one has simply to note that, in Step 2, there holds $||\mathbf{u}_3 \times \mathbf{n}||_{1/2,\Gamma} \leq \varepsilon$, and then to replace Step 3 by the construction of a lifting operator $R_\top$ of the tangential trace with the same properties as the ones of R_n. ■

As a conclusion of the two previous paragraphs, one gets the

Corollary 4.8 *There holds*

$$\forall (\mathbf{u}, \mathbf{v}) \in [\mathcal{Y}_R]^2,\ \ (\nabla \mathbf{u} \mid \nabla \mathbf{v})_0 = (\mathbf{curl}\, \mathbf{u} \mid \mathbf{curl}\, \mathbf{v})_0 + (\operatorname{div} \mathbf{u} \mid \operatorname{div} \mathbf{v})_0 - \int_\Gamma \frac{n_r}{r}\, u_\theta\, v_\theta\, d\Gamma; \quad (23)$$

$$\forall (\mathbf{u}, \mathbf{v}) \in [\mathcal{X}_R]^2,\ \ (\nabla \mathbf{u} \mid \nabla \mathbf{v})_0 = (\mathbf{curl}\, \mathbf{u} \mid \mathbf{curl}\, \mathbf{v})_0 + (\operatorname{div} \mathbf{u} \mid \operatorname{div} \mathbf{v})_0 - \int_\Gamma \frac{n_r}{r}\, u_n\, v_n\, d\Gamma. \quad (24)$$

4.3 Closedness of $\breve{\mathcal{Y}}_R$ in $\breve{\mathcal{Y}}$

Let us begin with an extension of the Hardy inequalities.

Lemma 4.9 *The space $H^1_{-1}(\omega)$ is continuously imbedded into $L^2_{-3}(\omega)$, i.e. there exists a constant K_1 such that*

$$\forall u \in H^1_{-1}(\omega), \quad ||u|||^2_{0,-3,\omega} \leq K_1\, ||\mathbf{grad}\, u||^2_{0,-1,\omega}. \quad (25)$$

Proof : Assume first that ω is locally convex at the conical vertices (the conical angle is equal to or less than $\pi/2$): it can be described by

$$\omega = \{(r, z) :\ z_{\min} < z < z_{\max},\ 0 < r < R(z)\}.$$

Then, as $\partial_r u \in L^2_{-1}(\omega)$, the Fubini Theorem yields — in the remainder of the proof, the symbol $\forall$ will stand for *almost everywhere*:

$$\forall z \in]z_{\min}, z_{\max}[, \quad \int_0^{R(z)} \frac{1}{r} \left(\frac{\partial u}{\partial r}\right)^2 dr < +\infty \text{ and } \int_{z_{\min}}^{z_{\max}} dz \int_0^{R(z)} \frac{1}{r} \left(\frac{\partial u}{\partial r}\right)^2 dr = \left\|\frac{\partial u}{\partial r}\right\|^2_{0,-1}.$$

On the other hand, $u_{|\gamma_a} = 0$ in $L^2(\gamma_a)$ (Proposition 3.18), so that $\forall (r, z) \in \omega,\ u(r, z) = \displaystyle\int_0^r \partial_r u(s, z)\, ds$. For a given z, one can then apply Lemma 4.3 to $f = \partial_r u$, with $R = R(z)$ and $\alpha = -1$:

$$\int_0^{R(z)} \frac{1}{r^3}\, u(r, z)^2\, dr \leq \int_0^{R(z)} \frac{1}{r} \left(\frac{\partial u}{\partial r}(r, z)\right)^2 dr.$$

Consequently, the integrands being positive, the Fubini Theorem allows to prove that (25) is fulfilled with $K_1 = 1$.

As for the general case, this proves in particular that the only remaining problem is due to the conical vertices. For a general ω, let us consider a three-way splitting. Near the first conical vertex, there exists oblique coordinates (ξ, z), such that, locally, ω coincides with $\omega_1 = \{(\xi, z) : \ z_{\min} < z < z_{\max}, \ 0 < \xi < \Xi(z)\}$. Il we let ψ_1 denote the angle between $(O\xi)$ and (Oz), it is a simple matter to check that there holds

$$||u||^2_{0,-3,\omega_1} \leq (\sin\psi_1)^{-2} \, ||\mathbf{grad}\, u||^2_{0,-1,\omega_1}.$$

A similar result can also be derived in a domain ω_2, which coincides locally with ω near the other conical vertex.

Now, the bounds on z defining ω_1 and ω_2 can be chosen in such a way that $\omega_3 = \omega \setminus (\omega_1 \cup \omega_2)$ is at a distance $R_{\min}$ of (Oz), with $R_{\min} > 0$. Thus, for $u \in H^1_{-1}(\omega)$

$$||u||^2_{0,-3,\omega_3} \leq \frac{||u||^2_{0,\omega_3}}{{R_{\min}}^3} \leq \frac{||u||^2_{0,\omega}}{{R_{\min}}^3} \leq \frac{C_P^2}{{R_{\min}}^3} \, ||\mathbf{grad}\, u||_{0,\omega} \leq \frac{C_P^2 \, R_{\max}}{{R_{\min}}^3} \, ||\mathbf{grad}\, u||_{0,-1,\omega},$$

where C_P is the Poincaré constant (recall that $u_{|\gamma_a} = 0$). The conclusion follows, with $K_1 = (\sin\psi_1)^{-2} + (\sin\psi_2)^{-2} + C_P^2 \, R_{\max}/{R_{\min}}^3$ in (25). ■

Proposition 4.10 *The inequality (16) is satisfied for all* $\mathbf{u} \in \breve{\mathcal{Y}}_R$.

Proof : It is sufficient to check that it is fulfilled, for all $\mathbf{u}$ in

$$E^1_\theta = \left\{\mathbf{u} \in \breve{\mathbf{H}}^1(\Omega) : \ \mathbf{u}\cdot\mathbf{e}_r = \mathbf{u}\cdot\mathbf{e}_z = 0\right\} = \left\{\mathbf{u} = u\,\mathbf{e}_\theta, \ u \in H^1_-(\Omega)\right\}.$$

As a matter of fact, (16) is equivalent to (20). If the latter is fulfilled for $\mathbf{u} \in E^1_\theta$, then one splits $\mathbf{u} \in \breve{\mathbf{H}}^1(\Omega)$ in $\mathbf{u}_m + \mathbf{u}_\theta$: $\nabla\mathbf{u}_m$ and $\nabla\mathbf{u}_\theta$ are $\mathbf{L}^2$-orthogonal. In addition, the form $b(\mathbf{u}, \mathbf{u})$ depends only on u_θ. Thanks to

$$-b(\mathbf{u}, \mathbf{u}) = -b(\mathbf{u}_\theta, \mathbf{u}_\theta) \leq k \, ||\nabla\mathbf{u}_\theta||^2_0 \leq k \, ||\nabla\mathbf{u}||^2_0,$$

(16) also holds for $\mathbf{u}$.

Now, for a given $\mathbf{u} \in E^1_\theta$, let $v = r\,u_\theta$. On the one hand, as $\operatorname{div}\mathbf{u} = 0$, thanks to Proposition 3.19, one has $2\pi \, ||\mathbf{grad}\, v||^2_{1,-1,\omega} = ||\mathbf{curl}\,\mathbf{u}||^2_{0,\Omega} + ||\operatorname{div}\mathbf{u}||^2_{0,\Omega}$. On the other hand, the condition $\mathbf{u} \in \breve{\mathbf{H}}^1(\Omega)$ translates to $u_\theta \in H^1_-(\omega)$, that is $v \in H^1_{-1}(\omega) \cap L^2_{-3}(\omega) = H^1_{-1}(\omega)$ (see the previous Lemma). Using Proposition 3.17 and (25), one finds

$$\begin{aligned}
\frac{|\mathbf{u}|^2_{1,\Omega}}{2\,\pi} &= ||u_\theta||^2_{0,-1,\omega} + \left\|\frac{\partial u_\theta}{\partial r}\right\|^2_{0,1,\omega} + \left\|\frac{\partial u_\theta}{\partial z}\right\|^2_{0,1,\omega} \\
&= ||v||^2_{0,-3,\omega} + \left\|\frac{\partial v}{\partial r} - \frac{v}{r}\right\|^2_{0,-1,\omega} + \left\|\frac{\partial v}{\partial z}\right\|^2_{0,-1,\omega} \\
&\leq ||v||^2_{0,-3,\omega} + 2\left\|\frac{\partial v}{\partial r}\right\|^2_{0,-1,\omega} + 2\,||v||^2_{0,-3,\omega} + \left\|\frac{\partial v}{\partial z}\right\|^2_{0,-1,\omega} \\
&\leq (3\,K_1 + 2)\,||\mathbf{grad}\, v||^2_{1,-1,\omega}.
\end{aligned}$$

The bound in (16) is finally obtained with $K = (3\,K_1 + 2)$. ■

4.4 Closedness of $\breve{\mathcal{X}}_R$ in $\breve{\mathcal{X}}$

If one tries to use the same constructive techniques as in the previous paragraphs, it is only possible to obtain a positive result for a domain Ω, which is "slightly non convex" at the conical vertices.

Fortunately, there is an inductive way, which yields the closedness result (cf. [9]). It is based on a powerful result, obtained by Birman and Solomyak. For that, let

$$\Phi = \{\phi \in H^1_0(\Omega) : \ \Delta\phi \in L^2(\Omega)\}.$$

Theorem 4.11 *Let Ω be a bounded Lipschitz domain. Then, for all $\mathbf{u}$ in $\mathcal{X}$, there exist $\mathbf{u}_0$ in $\mathcal{X}_R$ and $\phi \in \Phi$ such that*

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + \mathbf{grad}\,\phi, & (26)\\ C\,||\mathbf{u}||^2_{0,\mathbf{curl},\mathrm{div}} &\geq ||\mathbf{u}_0||^2_1 + ||\Delta\phi||^2_0. & (27)\end{aligned}$$

Here, C denotes a nonnegative constant, which is independent of $\mathbf{u}$.

Proof : See [7]. ■

As we are considering an axisymmetric domain and axisymmetric data, let us focus on the

Lemma 4.12 *In the case when the domain Ω and the field $\mathbf{u}$ are axisymmetric, one can find suitable $\mathbf{u}_0$ and ϕ such that (26) and (27) are valid.*

Proof : In order to prove this result, a possibility is to check that the process of Birman and Solomyak, applied to an axisymmetric field, yields an axisymmetric decomposition. As a matter of fact, it is based upon mappings and continuation operators, which preserve axisymmetry.

Another approach consists in introducing an averaging mapping: $\mathcal{M}_\theta(f) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\,d\theta$.

The mapping $\mathcal{M}_\theta$ is continuous from $L^2(\Omega)$ to $\breve{L}^2(\Omega)$, and it fulfills, for any function f of $\mathcal{C}^\infty(\overline{\Omega})$,

$$\mathcal{M}_\theta(\partial_r f) = \partial_r\mathcal{M}_\theta(f), \quad \mathcal{M}_\theta(\partial_z f) = \partial_z\mathcal{M}_\theta(f) \text{ and } \mathcal{M}_\theta(\partial_\theta f) = 0 = \partial_\theta\mathcal{M}_\theta(f).$$

In other words, it commutes with differential operators, with sufficiently smooth data. By density, it defines a continuous mapping from $\mathbf{H}(\mathbf{curl}\,,\mathrm{div}\,;\Omega)$, resp. Φ, onto their respective axisymmetric subspaces. Thus, for an axisymmetric $\mathbf{u}$, one simply uses (26) to get $\mathbf{u}_0$ and ϕ, and consequently

$$\mathbf{u} = \mathcal{M}_\theta(\mathbf{u}) = \mathcal{M}_\theta(\mathbf{u}_0) + \mathbf{grad}\,\mathcal{M}_\theta(\phi);$$

the inequality (27) remains valid, thanks to the continuity of $\mathcal{M}_\theta$. ■

This amounts to saying that the *singular part* of the electric field (if it exists) is linked to a *gradient of a (well chosen) singular potential*; this relationship will be developed in Section 5.

Now, let $\Phi_R = \Phi \cap H^2(\Omega)$. Then, one proceeds with the

Lemma 4.13 *The following inequalities are equivalent, with C_1 and C_2 two nonnegative constants:*

$$\begin{aligned} ||\mathbf{u}||_1 &\leq C_1\,||\mathbf{u}||_{0,\mathbf{curl},\mathrm{div}}\,, \quad \forall \mathbf{u} \in \mathcal{X}_R, & (28)\\ ||\phi||_2 &\leq C_2\,||\Delta\phi||_0, \quad \forall \phi \in \Phi_R. & (29)\end{aligned}$$

Proof : Assume first that (28) holds: given ϕ in Φ_R, one gets

$$\begin{aligned} ||\phi||_2^2 &= ||\phi||_0^2 + |\phi|_1^2 + |\phi|_2^2,\\ &\leq C\,\{|\phi|_1^2 + |\phi|_2^2\}, \text{ thanks to the Poincaré inequality,}\\ &= C\,||\mathbf{grad}\,\phi||_1^2,\\ &\leq C\,||\mathbf{grad}\,\phi||^2_{0,\mathbf{curl},\mathrm{div}}\,, \text{ as } \mathbf{grad}\,\phi \in \mathcal{X}_R,\\ &\leq C\,||\Delta\phi||_0^2, \text{ by applying Weber's inequality (see Proposition 1.1).}\end{aligned}$$

Conversely, if (29) is true, then, for $\mathbf{u}$ in $\mathcal{X}_R$, one uses (26)-(27) with $\phi \in \Phi_R$ ($\mathbf{grad}\,\phi = \mathbf{u} - \mathbf{u}_0 \in \mathcal{X}_R$), to obtain

$$\begin{aligned} ||\mathbf{u}||_1^2 &\leq 2\{||\mathbf{u}_0||_1^2 + ||\phi||_2^2\}, \\ &\leq C\,\{||\mathbf{u}_0||_1^2 + ||\Delta\phi||_0^2\}, \text{ thanks to (29)}, \\ &\leq C\,||\mathbf{u}||_{0,\mathbf{curl},\mathrm{div}}^2 . \end{aligned}$$

■

This leads to the

Theorem 4.14 *Let Ω be a Lipschitz domain, such that its geometrical singularities are either conical vertices, or edges. Then (28) is satisfied in Ω if and only if all the conical angles are different from a prescribed value π/β_-. In addition, this value corresponds exactly to the case when 3/4 is an eigenvalue of the Laplace operator, considered in the vicinity of the conical vertex.*
As a consequence, if Ω is axisymmetric, (15) is satisfied in Ω if and only if all the conical angles are different from the prescribed value π/β_-.

Proof : Thanks to the above Lemma, the inequality (28) is satisfied if and only if (29) is true. Dauge [13] has proved that this is the case if the conical angles are different from π/β_- (numerical value close to 130°). Conversely, she also proved that the statement (29) is false when at least one conical angle takes the value π/β_-, and that it corresponds to the existence of the eigenvalue 3/4 in the spectrum of the Laplace operator, defined near the same conical vertex.
The result in an axisymmetric domain follows immediately. ■

Remark 4.4 It is not possible to apply the same technique to the magnetic fields, as the equivalent of (29) *had not been established* for elements of $\{\psi \in H^2(\Omega)/\mathbb{R} :\ \partial_n\psi_{|\Gamma} = 0\}$. However, as the transpositions of Theorem 4.11 and Lemma 4.13 are both valid in $\mathcal{Y}$ (cf. [7, 8] for the former, use of the equivalence of norms in $H^1(\Omega)/\mathbb{R}$ instead of the Poincaré inequality for the later), it is clear that (29) holds for axisymmetric elements of $\{\psi \in H^2(\Omega)/\mathbb{R} :\ \partial_n\psi_{|\Gamma} = 0\}$. ■

4.5 Mixed boundary conditions

Let us consider now the case of mixed boundary conditions. More precisely, given a splitting of the boundary $\gamma_b = \overline{\gamma}_1 \cup \overline{\gamma}_2$, with $\gamma_1 \cap \gamma_2 = \emptyset$, which yields an axisymmetric splitting of Γ in Γ_1 and Γ_2, define

$$\mathbf{Z} = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,;\Omega) :\ \mathbf{v}\times\mathbf{n}_{|\Gamma_1} = 0,\ \mathbf{v}\cdot\mathbf{n}_{|\Gamma_2} = 0\},\ \text{and } \mathbf{Z}_R = \mathbf{Z}\cap\mathbf{H}^1(\Omega).$$

Remark 4.5 It is important to note that, due to the axisymmetry, a single boundary condition holds in the neighborhood of each conical vertex. ■

Next, one can prove, using the same technique as the one developed in the proof of Proposition 4.7

Proposition 4.15 *The space*

$$\left\{\mathbf{v} \in \mathcal{C}^\infty(\overline{\Omega})^3 \cap \mathbf{Z}_R :\ \mathbf{v} \text{ vanishes in a neighborhood of vertices and edges}\right\}$$

is dense in $\mathbf{Z}_R$.

By plugging this density result in the integration by parts formula (18), one gets

Corollary 4.16 *Let* $(\mathbf{u}, \mathbf{v})$ *belong to* $[\mathbf{Z}_R]^2$. *There holds*

$$
\begin{aligned}
(\nabla \mathbf{u} \mid \nabla \mathbf{v})_0 &= (\mathbf{curl}\, \mathbf{u} \mid \mathbf{curl}\, \mathbf{v})_0 + (\operatorname{div} \mathbf{u} \mid \operatorname{div} \mathbf{v})_0 \\
&\quad - \int_{\Gamma_1} \frac{n_r}{r} u_n\, v_n\, d\Gamma - \int_{\Gamma_2} \frac{n_r}{r} u_\theta\, v_\theta\, d\Gamma. \qquad (30)
\end{aligned}
$$

Recall that the finite-dimensional subspace $\mathbb{H}$ (11), composed of elements with both vanishing curl and divergence, is not reduced to $\{0\}$, contrarily to the similarly defined subspaces of $\mathcal{X}$ and $\mathcal{Y}$. As a matter of fact, thanks to [14], a basis of $\mathbb{H}$ is given by $(\mathbf{grad}\, p_k)_{1 \le k \le K_m - 1}$, with

$$
p_k \in H^1(\Omega),\ \Delta p_k = 0,\ p_{k|\Gamma_1^0} = 0,\ p_{k|\Gamma_1^l} = \delta_{kl},\ 1 \le l \le K_m - 1,\ \frac{\partial p_k}{\partial n}_{|\Gamma_2} = 0. \qquad (31)
$$

By construction, the $\mathbf{grad}\, p_k$ are axisymmetric.

From this point on, in order to establish the closedness property, that of $\breve{\mathbf{Z}}_R$ in $\breve{\mathbf{Z}}$, it is possible to reuse the results of the previous two paragraphs to construct a proof.

Let us begin with elements of $\mathbb{H}$.
If Γ_1 (resp. Γ_2) contains one or two conical vertices, one can apply the inductive (resp. constructive) method to get (29) locally. The same reasoning also applies for edges which are either included in Γ_1 or Γ_2.
Last, in the neighborhood of the interface $\partial\Gamma$, one can take advantage of the following result, obtained by Dauge [13].

Proposition 4.17 *In the neighborhood of* $\partial\Gamma$, *(29) holds for any element of*

$$
\{\phi \in H^2(\Omega) :\ \phi_{|\Gamma_1} = 0,\ \frac{\partial \phi}{\partial n}_{|\Gamma_2} = 0\},
$$

provided that the angles at the interface are not equal to $\pi/2$, *nor to* $3\pi/2$.

This takes care of elements of $\mathbb{H}$.

Now, let us consider an element $\mathbf{u}$ of $\mathbb{H}^\perp$.
Given the usual cut-off function χ, split

$$
\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3,\ \text{with}\ \mathbf{u}_1 = \chi_1\, \mathbf{u},\ \mathbf{u}_2 = \chi_2\, \mathbf{u};
$$

here, $\chi_i = \chi(\rho_i)$, where ρ_i denotes the distance to the conical vertex O_i, and χ is such that $\partial\Gamma \cap \operatorname{supp}(\chi) = \emptyset$.
On the one hand, $\mathbf{u}_1$ and $\mathbf{u}_2$ do belong to $\breve{\mathcal{X}}_R$ or $\breve{\mathcal{Y}}_R$. Thus (16) is valid for both fields, provided the conical angle differs from π/β_- in the $\breve{\mathcal{X}}_R$-case.
On the other hand, $\mathbf{u}_3$ is such that its trace vanishes in a neighborhood of the conical vertices: this allows to prove that (16) holds too for the third field.

Lemma 4.18 *Let* $\mathcal{V}_0$ *be a neighborhood of the conical vertices. Then, there exists a constant* K *such that (16) is valid, for all elements of* $\mathbf{Z}_R \cap \mathbb{H}^\perp$, *the trace of which vanishes on* $\Gamma \cap \mathcal{V}_0$.

Proof : The result is obtained by contradiction. Let $(\mathbf{v}^k)_k$ be a sequence of elements such that

$$||\nabla \mathbf{v}^k||_0^2 = 1\ \forall k,\ ||\mathbf{curl}\,\mathbf{v}^k||_0^2 + ||\mathrm{div}\,\mathbf{v}^k||_0^2 \to 0.$$

According to the Weber inequality, stated in Proposition 1.2, $(\mathbf{v}^k)_k$ converges to 0 in $\mathbf{H}(\mathbf{curl}\,;\Omega)\cap\mathbf{H}(\mathrm{div}\,;\Omega)$. Now, as the sequence is bounded in $\mathbf{H}^1(\Omega)$, it admits a weakly convergent subsequence. Thanks to the above, its limit is 0. What is more, as $\mathbf{H}^1(\Omega)$ is compactly imbedded in $\mathbf{H}^{1-\varepsilon}(\Omega)$ (for a non negative ε), a subsequence converges to 0 in the latter space: thus, its trace vanishes in $\mathbf{L}^2(\Gamma)$. In particular, this implies that $b(\mathbf{v}^k,\mathbf{v}^k)$ also vanishes, as

$$b(\mathbf{v}^k,\mathbf{v}^k) = \int_{\Gamma\setminus\mathcal{V}_0} \frac{n_r}{r}\,||\mathbf{v}^k||^2\,d\Gamma \le \frac{1}{R_{\min}}||\mathbf{v}^k||_{0,\Gamma}^2.$$

The contradiction then follows from (30). ∎

As a consequence, one obtains

$$||\nabla\mathbf{u}||_0^2 = ||\sum_{i=1}^3 \nabla\mathbf{u}_i||_0^2 \le 3\sum_{i=1}^3 ||\nabla\mathbf{u}_i||_0^2 \le K\sum_{i=1}^3 \left\{||\mathbf{curl}\,\mathbf{u}_i||_0^2 + ||\mathrm{div}\,\mathbf{u}_i||_0^2\right\},$$

with K a constant independent of $\mathbf{u}$.
Last, there exists a constant C such that

$$\sum_{i=1}^3 \left\{||\mathbf{curl}\,\mathbf{u}_i||_0^2 + ||\mathrm{div}\,\mathbf{u}_i||_0^2\right\} \le C\ \left\{||\mathbf{curl}\,\mathbf{u}||_0^2 + ||\mathrm{div}\,\mathbf{u}||_0^2\right\};$$

otherwise, given $(\mathbf{v}^k)_k$ a sequence of elements such that

$$\sum_{i=1}^3 \left\{||\mathbf{curl}\,\mathbf{v}_i^k||_0^2 + ||\mathrm{div}\,\mathbf{v}_i^k||_0^2\right\} = 1\ \forall k,\ ||\mathbf{curl}\,\mathbf{v}^k||_0^2 + ||\mathrm{div}\,\mathbf{v}^k||_0^2 \to 0,$$

we infer that $(\mathbf{v}^k)_k$ converges to 0 in $\mathbf{H}(\mathbf{curl}\,;\Omega)\cap\mathbf{H}(\mathrm{div}\,;\Omega)$. Now, due to the identities

$$\mathbf{curl}\,\mathbf{v}_1 = \chi_1\,\mathbf{curl}\,\mathbf{v} + (\mathbf{grad}\,\chi_1)\times\mathbf{v},\ \mathrm{div}\,\mathbf{v}_1 = \chi_1\,\mathrm{div}\,\mathbf{v} + (\mathbf{grad}\,\chi_1)\cdot\mathbf{v},\ \text{etc.}$$

the same is true for the three sequences $(\mathbf{v}_i^k)_k$. This contradicts the assumption.

Thus (16) is valid for elements of $\breve{\mathbf{Z}}_R\cap\mathbb{H}^\perp$.

To conclude, one has the

Theorem 4.19 *Let Ω be an axisymmetric domain. $\breve{\mathbf{Z}}_R$ is closed in $\breve{\mathbf{Z}}$, provided that*

- *for the conical vertices included in Γ_1, the corresponding conical angle is different from the prescribed value π/β_-,*
- *the angles at the interface $\partial\Gamma$ are not equal to $\pi/2$, nor to $3\pi/2$.*

Corollary 4.20 *There holds*

$$\breve{\mathbf{Z}} = \breve{\mathbf{Z}}_R \overset{\perp}{\oplus} \breve{\mathbf{Z}}_S. \tag{32}$$

5 A characterization of singular fields

The aim of this Section is, for both the electric and the magnetic fields, to relate the singular fields to scalar singularities of Laplace-like operators. Again, the building of the relationship depends crucially on the way the closedness result has been derived. As a conclusion of each Subsection, the result on the dimension of the subspace of singular fields is stated, without proof. Last, in a short Subsection, a result is given on the regularity of the solutions of the mixed problem.

Before we begin to characterize the singular part of the electromagnetic field, let us recall the following Proposition, stated in the axisymmetric domain Ω.

Proposition 5.1 *If the data is axisymmetric for problems (1)-(3), (4)-(6) and (7)-(10), then their respective solution is also axisymmetric.*

5.1 Singular electric fields

In this Subsection, we assume that the conical angle values differ from π/β_-.

We apply next the technique of the Singular Complement Method [2]: our starting point is inequality (29), together with the Poincaré's and Weber's inequalities.

Proposition 5.2 *Φ can be orthogonally decomposed in the following way:*

In Φ, $||\phi||_\Phi = ||\Delta\phi||_0$ is a norm, which is equivalent to the canonical graph norm.

$\Delta\Phi_R$ is a closed subspace of $L^2(\Omega)$. Let N denote its orthogonal.

Define Φ_S as the subspace of Φ such that $\Delta\Phi_S = N$. Then both Φ_R and Φ_S are closed in Φ and

$$\Phi = \Phi_R \overset{\perp}{\oplus} \Phi_S. \tag{33}$$

As a consequence, thanks to Theorem 4.11, it is possible to prove the

Theorem 5.3 *The following decomposition is direct and continuous*

$$\mathcal{X} = \mathcal{X}_R \overset{c}{\oplus} \mathbf{grad}\,\Phi_S. \tag{34}$$

Proof : It follows from (26) and the inclusion $\mathbf{grad}\,\Phi_R \subset \mathcal{X}_R$ that $\mathcal{X} = \mathcal{X}_R + \mathbf{grad}\,\Phi_S$.

After that, let $\mathbf{v} \in \mathcal{X}_R \cap \mathbf{grad}\,\Phi_S$. By construction, $\mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathbf{grad}\,\Phi$, i.e. $\mathbf{v} \in \mathbf{grad}\,\Phi_R$. On the other hand, it is clear from the previous Proposition that $\mathbf{grad}\,\Phi$ can be split orthogonally in $\mathcal{X}$ into $\mathbf{grad}\,\Phi_R \overset{\perp}{\oplus} \mathbf{grad}\,\Phi_S$; as a consequence, $\mathbf{v} = 0$, and the sum is direct.

Last, the application

$$\begin{aligned} &\mathcal{X}_R \times \mathbf{grad}\,\Phi_S \to \mathcal{X} \\ &(\mathbf{v}_R, \mathbf{grad}\,\phi_S) \mapsto \mathbf{v} = \mathbf{v}_R + \mathbf{grad}\,\phi_S \end{aligned}$$

is linear, continuous and bijective. Now, as $\mathcal{X}_R \times \mathbf{grad}\,\Phi_S$ and $\mathcal{X}$ are Banach spaces, it stems from the open mapping Theorem that the inverse of the application is also continuous. ∎

So, as $\breve{\mathcal{X}}$ is the natural space of electric fields, and $\breve{\mathcal{X}}_R$ the subspace of regular electric fields, we derive from the above Theorem the direct and continuous decomposition

$$\breve{\mathcal{X}} = \breve{\mathcal{X}}_R \overset{c}{\oplus} \breve{\mathcal{X}}_S, \text{with } \breve{\mathcal{X}}_S = \mathbf{grad}\, \breve{\Phi}_S. \tag{35}$$

In other words, the electric singular fields are one-to-one with the gradients of the axisymmetric singularities of the Laplace operator. In addition, the elements of $\breve{\Phi}_S$ are characterized by their Laplacian, i.e. there remains to study

$$\breve{N} = \Delta \breve{\Phi}_S. \tag{36}$$

As we are dealing with an orthogonality property ($\breve{N}$ is, by definition, orthogonal to $\Delta\breve{\Phi}_R$ in $\breve{L}^2(\Omega)$), we shall use an integration by parts formula to obtain the relevant information on the elements of $\breve{N}$. This method has been introduced in [5, 3] and the references therein, for problems posed in a polygon or a polyhedron, in order to characterize N.

Definition 5.4 *On any face Γ_i, $1 \le i \le n+1$, let ρ_i be the distance to its boundary, and define*

$$H^{1/2}_{00}(\Gamma_i) = \{f \in H^{1/2}(\Gamma_i) :\ \frac{f}{\sqrt{\rho_i}} \in L^2(\Gamma_i)\}, \text{ and } \breve{H}(\Gamma_i) = H^{1/2}_{00}(\Gamma_i) \cap \mathcal{D}'(\Gamma).$$

By adapting the strategy developed in [3] (with a treatment specifically designed to handle the difficulties related to the conical vertices), we can prove the following on any face Γ_i, about the trace of the normal derivative of elements of $\breve{\Phi}_R$. Let γ_1^i be the corresponding trace application.

Lemma 5.5 *The application γ_1^i is continuous from $\breve{\Phi}_R$ to $\breve{H}(\Gamma_i)$.*
Moreover, it is surjective from $G_i = \{u \in \breve{\Phi}_R :\ \gamma_1^j u = 0,\ \forall j \neq i\}$ onto $\breve{H}(\Gamma_i)$, and there exists a continuous lifting operator from $\breve{H}(\Gamma_i)$ into G_i.

As a consequence, γ_1^i is surjective from $\breve{\Phi}_R$ onto $\breve{H}(\Gamma_i)$. This result permits to prove an integration by parts formula, between elements of $\breve{\Phi}_R$ and elements of the space $D(\Delta,\Omega) = \{g \in L^2(\Omega) :\ \Delta g \in L^2(\Omega)\}$. As a matter of fact, it is clear that elements of N possess a vanishing Laplacian, and therefore that they belong to $D(\Delta,\Omega)$. One has the

Lemma 5.6 *Let $p \in D(\Delta,\Omega)$ and $u \in \breve{\Phi}_R$. There holds*

$$\int_\Omega (p\,\Delta u - u\,\Delta p)\, d\Omega = \sum_{i=1}^{n+1} {}_{\breve{H}(\Gamma_i)'} \langle p, \gamma_1^i u \rangle_{\breve{H}(\Gamma_i)}.$$

Again, the method of proof is an adaptation of [3]. This leads to the first characterization of elements of $\breve{N}$.

Theorem 5.7 *Let $p \in \breve{L}^2(\Omega)$: p belongs to $\breve{N}$ if and only if*

$$\begin{aligned} &\Delta p = 0 \text{ in } \Omega, \\ &p_{|\Gamma_i} = 0 \text{ in } \breve{H}(\Gamma_i)',\ 1 \le i \le n+1. \end{aligned}$$

Proof : It is a straightforward consequence of the above Lemmas. ■

On the meridian half-plane, the trace of the operator Δ is defined by

$$\Delta^+ = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The second characterization of elements of $\breve{N}$ is then

Corollary 5.8 *Let $p \in L^2_1(\omega)$: p belongs to $\breve{N}$ if and only if*

$$\begin{aligned}&\Delta^+ p = 0 \ \textit{in } \omega,\\ &p_{|\gamma_i} = 0,\ 1 \le i \le n+1,\\ &p \in \mathcal{C}^\infty(\overline{\omega} \setminus \mathcal{V}_b),\ \textit{for any neighborhood } \mathcal{V}_b \textit{ of } \gamma_b.\end{aligned}$$

(The trace on γ_i is understood in the suitable trace space of $\breve{H}(\Gamma_i)$.)

Remark 5.1 Since p is smooth in the neighborhood of any segment included in γ_a, one infers that $\partial_r p(M_a) = 0$, for all points M_a of γ_a. This additional boundary condition is used in the actual computation of the singularity p. ∎

Proof : Thanks to Proposition 3.14, $p \in L^2_1(\omega)$. Then, Δ^+ being the trace of the three-dimensional Laplacian, one has $\Delta^+ p = 0$ in ω; the boundary condition $p = 0$ on γ_b is clear. Finally, as p is harmonic in Ω, it is smooth and so is its trace.
The reciprocal assertion is straightforward. ∎

Finally, by studying the properties of the Laplace-like operator Δ^+ (cf. a forthcoming paper), one finds that the dimension of $\breve{N}$, and thus that of $\breve{\Phi}_S$ and of the subspace of electric singular fields $\breve{\mathcal{X}}_S$, is equal to

  the number of conical vertices with conical angle larger than π/β_-

$+$ the number of reentrant edges.

5.2 Singular magnetic fields

As for the magnetostatic equations, one has to solve
Find $\mathcal{B} \in \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned}\mathbf{curl}\,\mathcal{B} &= \mathbf{f} \text{ in } \Omega,\\ \operatorname{div}\mathcal{B} &= 0 \text{ in } \Omega,\\ \mathcal{B}\cdot\mathbf{n}_{|\Gamma} &= 0,\end{aligned}$$

where the datum $\mathbf{f}$ of $\mathbf{L}^2(\Omega)$ is divergence-free.
Note that we rewrote the equations above to stress the fact that the magnetic field is always divergence-free. So, the natural space of axisymmetric magnetic fields is

$$\breve{\mathcal{W}} = \{\mathbf{v} \in \breve{\mathcal{Y}} :\ \operatorname{div}\mathbf{v} = 0\}.$$

In this space, an equivalent norm is $\mathbf{v} \mapsto ||\mathbf{curl}\,\mathbf{v}||_0$.
Then, if we let $\breve{\mathcal{W}}_R = \breve{\mathcal{W}} \cap \breve{\mathbf{H}}^1(\Omega)$ be the space of regular fields, we infer from Subsection 4.3 the

Proposition 5.9 *$\breve{\mathcal{W}}_R$ is closed in $\breve{\mathcal{W}}$.*

Proof : Let $(\mathbf{v}_n)_n$ be a sequence of elements of $\breve{\mathcal{W}}_R$, which converges to $\mathbf{v}$ in $\breve{\mathcal{W}}$.
In $\breve{\mathcal{Y}}$, this amounts to saying that $(\mathbf{v}_n)_n$ is a sequence of divergence-free elements of $\breve{\mathcal{Y}}_R$, which converges to the divergence-free $\mathbf{v}$.
As $\breve{\mathcal{Y}}_R$ is closed in $\breve{\mathcal{Y}}$, $\mathbf{v}$ belongs to $\breve{\mathcal{Y}}_R$, and therefore to $\breve{\mathcal{W}}_R$, as it is divergence-free. ∎

Let $\breve{\mathcal{W}}_S$ be its orthogonal, i.e.

$$\breve{\mathcal{W}} = \breve{\mathcal{W}}_R \overset{\perp}{\oplus} \breve{\mathcal{W}}_S. \tag{37}$$

(The orthogonal alternative of the Singular Complement Method [2].)

In order to characterize the singular magnetic fields, we proceed in two steps. First, we prove that they are *meridian*, and second, we use their definition by orthogonality to relate them to singular solutions of a Laplace-like problem.

Theorem 5.10 *Let* $\mathcal{B} \in \breve{\mathcal{W}}_S$*:* $\varpi_\theta(\mathcal{B}) = 0$.

Proof : Given $\mathcal{B}$ of $\breve{\mathcal{W}}_S$, let $\mathcal{B} = \mathbf{B}_\theta + \mathbf{B}_m$ be its decomposition into azimuthal and meridian parts. One further has

$$\mathbf{B}_\theta = \mathbf{B}_\theta^R + \mathbf{B}_\theta^S, \quad (\mathbf{B}_\theta^R, \mathbf{B}_\theta^S) \in \breve{\mathcal{W}}_R \times \breve{\mathcal{W}}_S,$$
$$\mathbf{B}_m = \mathbf{B}_m^R + \mathbf{B}_m^S, \quad (\mathbf{B}_m^R, \mathbf{B}_m^S) \in \breve{\mathcal{W}}_R \times \breve{\mathcal{W}}_S.$$

As $\mathcal{B} \in \breve{\mathcal{W}}_S$, $\mathbf{B}_\theta^R + \mathbf{B}_m^R = 0$. These two vectors are pointwise orthogonal, so $\mathbf{B}_\theta^R = \mathbf{B}_m^R = 0$. This means that both $\mathbf{B}_\theta$ and $\mathbf{B}_m$ belong to $\breve{\mathcal{W}}_S$. Let us focuse now on the azimuthal part, $\mathbf{B}_\theta$, and define $\mathbf{f} = \mathbf{curl}\,\mathbf{B}_\theta$. Thanks to Proposition 2.2, $\mathbf{f}$ is meridian. In addition,

$$\int_\Omega \mathbf{f} \cdot \mathbf{curl}\,\mathbf{C}\, d\Omega = 0, \quad \forall \mathbf{C} \in \breve{\mathcal{W}}_R. \tag{38}$$

Let us prove that this orthogonality property is true, for any vector field $\mathbf{C}$ of $\breve{\mathbf{H}}^1(\Omega)$: it is split into $\mathbf{C} = \mathbf{C}_\theta + \mathbf{C}_m$. Thanks again to Proposition 2.2, $\mathbf{curl}\,\mathbf{C}_m$ is azimuthal, and as a consequence, it is pointwise orthogonal to $\mathbf{f}$.
Now, $\mathbf{C}_\theta$ is independent of θ, and therefore it is divergence-free. Also, it has a vanishing normal component on the boundary ($\mathbf{n}$ is orthogonal to $\mathbf{e}_\theta$). Last, $\nabla \mathbf{C}_\theta$ and $\nabla \mathbf{C}_m$ are pointwise orthogonal (cf. (iv)): in order for $\nabla \mathbf{C}$ to be in $L^2(\Omega)^9$, they are both required to be in $L^2(\Omega)^9$ too. This yields $\mathbf{C}_\theta \in \breve{\mathcal{W}}_R$. As (38) holds for both $\mathbf{C}_\theta$ and $\mathbf{C}_m$, it does for $\mathbf{C}$ also. Consequently

$$\mathbf{curl}\,\mathbf{f} = 0 \text{ in } \Omega, \text{ and } \mathbf{f} \times \mathbf{n}_{|\Gamma} = 0.$$

Added to the definition of $\mathbf{f}$, this can be complemented with $\operatorname{div} \mathbf{f} = 0$ in Ω, i.e. $\mathbf{f} = 0$. As elements of $\breve{\mathcal{W}}$ are characterized by their curl, one concludes that $\mathbf{B}_\theta = 0$. ■

As mentioned in the proof, elements of $\breve{\mathcal{W}}$ are determined *via* their curl. Then, given $\mathcal{B} \in \breve{\mathcal{W}}_S$, define $\mathbf{P} = \mathbf{curl}\,\mathcal{B}$: $\mathcal{B}$ is meridian, so its curl $\mathbf{P}$ is azimuthal; let us further define $P_\theta \mathbf{e}_\theta = \mathbf{P}$.
In the remainder of the current Subsection, we shall characterize $\mathbf{P}$, first in Ω, and after that in the meridian half-plane.
In order to handle the case of the singular electric field, recall that we introduced an integration by parts formula, related to the (scalar) Laplacian operator. Here, we shall consider an integration by parts formula, which involves the vector Laplacian operator. How? Simply, by introducing the space of regular potentials

$$\mathsf{A}_R = \{\mathbf{A} \in \breve{\mathbf{H}}_0(\mathbf{curl}\,, \Omega) \cap \breve{\mathbf{H}}(\operatorname{div}\,, \Omega) :\ \operatorname{div} \mathbf{A} = 0, \mathbf{curl}\,\mathbf{A} \in \mathbf{H}^1(\Omega)\}$$

of elements of $\breve{\mathcal{W}}_R$. The equivalent of formula (38) is

$$\int_\Omega \mathbf{P} \cdot \Delta \mathbf{A}\, d\Omega = 0, \quad \forall \mathbf{A} \in \mathsf{A}_R, \tag{39}$$

as $\Delta = -\mathbf{curl}\,\mathbf{curl} + \mathbf{grad}\operatorname{div}$. One additional remark is that $\mathbf{P}$ is azimuthal, so the relevant part of $\Delta \mathbf{A}$ is $\varpi_\theta(\Delta \mathbf{A})$: in other words, it is enough to consider only elements of

$$\mathsf{A}_{\theta R} = \{\mathbf{A} \in \mathsf{A}_R :\ \mathbf{A} \parallel \mathbf{e}_\theta\}$$

(For all $\mathbf{A} \in \mathsf{A}_R$, both $\varpi_\theta(\mathbf{A})$ and $\varpi_m(\mathbf{A})$ belong to A_R.)

Then, let us proceed similarly to the previous Subsection. Let Γ_i be a given face, and $\gamma^i_{1\theta}$ be defined as $\gamma^i_{1\theta}\mathbf{u} = \gamma^i_1 u_\theta$.

Lemma 5.11 *$\gamma^i_{1\theta}$ is continuous from $\mathsf{A}_{\theta R}$ to $\breve{H}(\Gamma_i)$. Moreover, it is surjective from $\mathbf{G}_i = \{\mathbf{u} \in \breve{\mathbf{H}}^2(\Omega) \cap \breve{\mathbf{H}}^1_0(\Omega) \;:\; \mathbf{u} \parallel \mathbf{e}_\theta,\ \gamma^j_{1\theta}\mathbf{u} = 0,\ \forall j \neq i\}$ onto $\breve{H}(\Gamma_i)$, and there exists a continuous lifting operator from $\breve{H}(\Gamma_i)$ into $\mathbf{G}_i$.*

As a consequence, $\gamma^i_{1\theta}$ is surjective from $\mathsf{A}_{\theta R}$ onto $\breve{H}(\Gamma_i)$ (Again, some specific treatment has to be designed to handle the conical vertices.)

By essence, $\mathbf{P}$ belongs to $D(\Delta, \Omega)^3$ (cf. (39) with $\mathbf{A} \in \breve{\mathbf{D}}(\Omega)$). There follows the

Lemma 5.12 *Let $\mathbf{P} \in D(\Delta, \Omega)^3$ and $\mathbf{A} \in \mathsf{A}_{\theta R}$. There holds*

$$\int_\Omega (\mathbf{P} \cdot \Delta \mathbf{A} - \mathbf{A} \cdot \Delta \mathbf{P})\, d\Omega = \sum_{i=1}^{n+1} {}_{\breve{H}(\Gamma_i)'} \left\langle P_\theta, \gamma^i_{1\theta}\mathbf{A} \right\rangle_{\breve{H}(\Gamma_i)}.$$

With the help of this formula, one can derive the first characterization of elements of $\mathbf{curl}\,\breve{\mathcal{W}}_S$.

Theorem 5.13 *Let $\mathcal{B} \in \breve{\mathcal{W}}_S$: then $\mathbf{P} = P_\theta \mathbf{e}_\theta = \mathbf{curl}\,\mathcal{B}$ satisfies*

$$\begin{aligned} &\mathbf{P} \in \breve{\mathbf{L}}^2(\Omega), \\ &\Delta \mathbf{P} = 0 \ \text{in } \Omega, \\ &P_{\theta|\Gamma_i} = 0 \ \text{in } \breve{H}(\Gamma_i)',\ 1 \le i \le n+1. \end{aligned}$$

Conversely, a vector field solution of the above system is the curl of an axisymmetric singular magnetic field.

Note that, in the course of the proof, it is useful to replace (39) by the same formula, for fields of $\{\mathbf{A} \in \breve{\mathbf{L}}^2(\Omega) \;:\; \mathbf{curl}\,\mathbf{A} \in \breve{\mathbf{H}}^1(\Omega),\ A_{\theta|\Gamma} = 0\}$, as it is necessary to remove the divergence-free condition on the test fields.

On the meridian half-plane, we use the operator Δ^-, defined at (13). The second characterization of elements of $\mathbf{curl}\,\breve{\mathcal{W}}_S$ follows.

Corollary 5.14 *Let $P_\theta = p/r$: $p \in L^2_{-1}(\omega)$ can be characterized as the solution to*

$$\begin{aligned} &\Delta^- p = 0 \ \text{in } \omega, \\ &p_{|\gamma_i} = 0,\ 1 \le i \le n+1, \\ &\frac{p}{r} \in \mathcal{C}^\infty(\overline{\omega} \setminus \mathcal{V}_b),\ \text{for any neighborhood } \mathcal{V}_b \text{ of } \gamma_b. \end{aligned}$$

Remark 5.2 The smoothness of p/r in the neighborhood of any segment included in γ_a yields $p(M_a) = 0$, for all points M_a of γ_a. ■

Proof : The fact that p is described by the set of above equations is clear, after one notices that $\Delta \mathbf{P} = 1/r\,\Delta^- p\,\mathbf{e}_\theta$.
In order to prove the reciprocal assertion, the following method can be used. Let $\mathbf{P} =$

$p/r\mathbf{e}_\theta$: $\mathbf{P}$ belongs to $\breve{\mathbf{L}}^2(\Omega)$ by construction. Let us show that $\Delta\mathbf{P} = 0$. For that, let $\mathbf{v} \in \breve{\mathbf{D}}(\Omega)$ and $\mathcal{V}_b$ be a neighborhood of Γ such that $\mathrm{supp}(\mathbf{v}) \cap \mathcal{V}_b = \emptyset$.

$$\begin{aligned} < \Delta\mathbf{P}, \mathbf{v} > &= < \mathbf{P}, \Delta\mathbf{v} >= \int_\Omega \mathbf{P}\cdot\Delta\mathbf{v}\, d\Omega = 2\pi \int_\omega \frac{p}{r}\Delta^-(rv_\theta)\, d\omega \\ &= 2\pi \int_{\omega\setminus\mathcal{V}_b} \frac{p}{r}\Delta^-(rv_\theta)\, d\omega = 2\pi \int_{\omega\setminus\mathcal{V}_b} \frac{1}{r}\Delta^- p(rv_\theta)\, d\omega = 0. \end{aligned}$$

The double integration by parts is justified by the smoothness of p/r and rv_θ in $\omega \setminus \mathcal{V}_b$. There is no boundary term, as both p and rv_θ vanish on $\gamma_a \cap \mathcal{V}_b$.
In particular, $\mathbf{P}$ belongs to $D(\Delta,\Omega)^3$. This allows to define its trace on Γ_i in $\breve{H}(\Gamma_i)'$, and the condition $p_{|\gamma_i} = 0$ finally leads to $P_{\theta|\Gamma_i} = 0$. ■

By studying the properties of the operator Δ^- (cf. a forthcoming paper), one finds that the dimension of the vector space $\mathbf{curl}\,\breve{\mathcal{W}}_S$, and thus that of $\breve{\mathcal{W}}_S$, is equal to

the number of reentrant edges.

5.3 The mixed case

Here, we simply address the problem of the regularity of elements of $\mathbb{H}$. It is fairly simple to check whether they belong to $\breve{\mathbf{Z}}_R$ or not. As a matter of fact, recall that a basis of $\mathbb{H}$ is of the form $(\mathbf{grad}\, p_k)_{1\le k\le K_m-1}$, with p_k defined at (31). Now, $\mathbf{grad}\, p_k$ can be (locally) replaced by $\mathbf{grad}\,(p_k - 1)$, since $1 \in \mathcal{C}^\infty$, and $(p_k - 1)_{|\Gamma_1^k} = 0$. Consequently, the regularity problem amounts (locally) to that of the solution of an homogeneous mixed Laplace problem. Therefore, $\mathbf{grad}\, p_k$ is in $\mathbf{H}^1(\Omega)$ (cf. [13]) if and only if

1 For any dihedral angle π/α interior to $(\Gamma_1^k)_k$ or to $(\Gamma_2^l)_l$, $\alpha \ge 1$,

2 For any dihedral angle π/α' on $\partial\Gamma$, $\alpha' \ge 2$,

3 The conical angle π/β is such that $\beta > \beta_-$ (case of a Dirichlet boundary condition).

6 The time-dependent Maxwell equations

Given $T > 0$, let us recall Maxwell's equations in time. If we let c and ε_0 be respectively the speed of light and the dielectric permittivity, they read, in $\Omega\times]0,T[$

$$\frac{\partial\mathcal{E}}{\partial t} - c^2\,\mathbf{curl}\,\mathcal{B} = -\frac{1}{\varepsilon_0}\mathcal{J}, \tag{40}$$

$$\frac{\partial\mathcal{B}}{\partial t} + \mathbf{curl}\,\mathcal{E} = 0, \tag{41}$$

$$\mathrm{div}\,\mathcal{E} = \frac{\rho}{\varepsilon_0}, \tag{42}$$

$$\mathrm{div}\,\mathcal{B} = 0, \tag{43}$$

where ρ and $\mathcal{J}$ are the charge and current densities. They satisfy the charge conservation equation (a consequence of equations (40) and (42))

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,\mathcal{J} = 0. \tag{44}$$

These equations are supplied with appropriate bounday conditions: in our case, as the domain Ω is enclosed in a perfectly conducting material, they are, in $\Gamma \times]0,T[$,

$$\mathcal{E} \times \mathbf{n} = 0, \tag{45}$$

$$\mathcal{B} \cdot \mathbf{n} = 0. \tag{46}$$

Last, initial conditions are provided to close the system of equations,

$$\mathcal{E}(\cdot,0) = \mathcal{E}_0, \tag{47}$$

$$\mathcal{B}(\cdot,0) = \mathcal{B}_0, \tag{48}$$

with an *ad hoc* initial value $(\mathcal{E}_0, \mathcal{B}_0)$ of the electromagnetic field.

In order to prove the existence and uniqueness of the electromagnetic field under suitable assumptions on the data and the initial conditions, one can use for instance the semi-group theory to get the

Theorem 6.1 *Assume that $(\mathcal{E}_0, \mathcal{B}_0)$ belongs to $\mathbf{H}_0(\mathbf{curl}\,,\Omega) \times \mathbf{H}(\mathbf{curl}\,,\Omega)$, and that $\mathcal{J} \in \mathcal{C}^1(0,T;\mathbf{L}^2(\Omega))$. Then, there exists one and only one solution to the time dependent problem (40)-(41), (45), (47)-(48), such that*

$$\begin{aligned} &\mathcal{E} \in \mathcal{C}^0(0,T;\mathbf{H}_0(\mathbf{curl}\,,\Omega)) \cap \mathcal{C}^1(0,T;\mathbf{L}^2(\Omega)), \\ &\mathcal{B} \in \mathcal{C}^0(0,T;\mathbf{H}(\mathbf{curl}\,,\Omega)) \cap \mathcal{C}^1(0,T;\mathbf{L}^2(\Omega)). \end{aligned} \tag{49}$$

Assume moreover that ρ belongs to $\mathcal{C}^0(0,T;L^2(\Omega))$ and that the initial data satisfy

$$\operatorname{div} \mathcal{E}_0 = \frac{\rho(\cdot,0)}{\varepsilon_0},\ \operatorname{div} \mathcal{B}_0 = 0,\ \mathcal{B}_0 \cdot \mathbf{n}_{|\Gamma} = 0.$$

Consequently, (42) and (43) are fulfilled, and in addition to (49),

$$\begin{aligned} &\mathcal{E} \in \mathcal{C}^0(0,T;\mathcal{X}), \\ &\mathcal{B} \in \mathcal{C}^0(0,T;\mathcal{Y}) \cap \mathcal{C}^1(0,T;\mathbf{H}(\operatorname{div}\,,\Omega)). \end{aligned} \tag{50}$$

(The proof of the first part of the Theorem is a standard application of the semi-group theory, whereas the second part can be obtained through some simple verifications.)

Provided that Ω is axisymmetric, if the data and initial conditions are axisymmetric, the solution of (40)-(48) is also axisymmetric:

Proposition 6.2 *If ρ, $\mathcal{J}$ and $(\mathcal{E}_0, \mathcal{B}_0)$ are axisymmetric, so is the solution to (40)-(48). And, provided that $\mathcal{J} \in \mathcal{C}^1(0,T;\breve{\mathbf{L}}^2(\Omega))$ and that $\rho \in \mathcal{C}^0(0,T;\breve{L}^2(\Omega))$, there holds*

$$\begin{aligned} &\mathcal{E} \in \mathcal{C}^0(0,T;\breve{\mathcal{X}}), \\ &\mathcal{B} \in \mathcal{C}^0(0,T;\breve{\mathcal{W}}). \end{aligned} \tag{51}$$

The consequence of these results, and of the decomposition (35), (37) of the spaces $\breve{\mathcal{X}}$ and $\breve{\mathcal{W}}$ is that it is possible to decompose the electromagnetic field into regular and singular parts continuously, with respect to time, i.e.

Corollary 6.3 *Assume that $(\mathcal{E}, \mathcal{B})$ belongs to $\mathcal{C}^0(0,T;\breve{\mathcal{X}} \times \breve{\mathcal{W}})$: one can write*

$$\mathcal{E}(\cdot,t) = \mathcal{E}_R(\cdot,t) + \mathcal{E}_S(\cdot,t), \quad (\mathcal{E}_R, \mathcal{E}_S) \in \mathcal{C}^0(0,T;\breve{\mathcal{X}}_R \times \breve{\mathcal{X}}_S), \tag{52}$$

$$\mathcal{B}(\cdot,t) = \mathcal{B}_R(\cdot,t) + \mathcal{B}_S(\cdot,t), \quad (\mathcal{B}_R, \mathcal{B}_S) \in \mathcal{C}^0(0,T;\breve{\mathcal{W}}_R \times \breve{\mathcal{W}}_S). \tag{53}$$

Conclusion

We have presented new results concerning Maxwell's equations in an axisymmetric domain, with axisymmetric data (and axisymmetric initial conditions).
In particular, we proved that, in space, the regular subspaces, $\breve{\mathcal{X}}_R$, $\breve{\mathcal{W}}_R$, are closed in $\breve{\mathcal{X}}$ and $\breve{\mathcal{W}}$ respectively, with the exception of one value of the conical angle (equal to π/β_-) in the electric case. This lead to the decomposition of the solution of either the static or the time-dependent Maxwell equations, into a regular part and a singular part, the so-called Singular Complement Method. In addition, we proved that the singular subspaces $\breve{\mathcal{X}}_S$ and $\breve{\mathcal{W}}_S$ are finite-dimensional.
This suggests that one can use the Singular Complement Method for numerical applications in axisymmetric geometries, as already done in [5, 4] in two-dimensional cartesian domains. The regular part of the solution is approximated *via* the P1 Lagrange Finite Element, whereas the singular part is computed after a suitable discretization of the basis of $\breve{\mathcal{X}}_S$ and $\breve{\mathcal{W}}_S$ has been carried out. These issues will be addressed in forthcoming papers.

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7 Annex

7.1 Operators in cylindrical coordinates

In *cylindrical coordinates*, the gradient, divergence and curl operators read

$$\begin{aligned}
\mathbf{grad}\, f &= \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z && \text{(i)}\\
\operatorname{div} \mathbf{v} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_r\right) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} && \text{(ii)}\\
\mathbf{curl}\, \mathbf{v} &= \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\, v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z. && \text{(iii)}
\end{aligned}$$

The expression of the Jacobian of any vector field $\mathbf{v}$ is

$$\nabla \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{pmatrix}. \tag{iv}$$

7.2 Operators in spherical coordinates

In the *non-standard spherical coordinates*, the first order differential operators are

$$\begin{aligned}
\mathbf{grad}\, f &= \frac{\partial f}{\partial \rho}\,\mathbf{e}_\rho + \frac{1}{\rho\,\sin\phi}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi && \text{(v)}\\
\operatorname{div} \mathbf{v} &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\, v_\rho\right) + \frac{1}{\rho\,\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\rho\,\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\, v_\phi\right) && \text{(vi)}\\
\mathbf{curl}\, \mathbf{v} &= \frac{1}{\rho\,\sin\phi}\left(\frac{\partial v_\phi}{\partial \theta} - \frac{\partial}{\partial \phi}\left(\sin\phi\, v_\theta\right)\right)\mathbf{e}_\rho + \frac{1}{\rho}\left(\frac{\partial v_\rho}{\partial \phi} - \frac{\partial}{\partial \rho}\left(\rho\, v_\phi\right)\right)\mathbf{e}_\theta \\
&\quad + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho v_\theta\right) - \frac{1}{\sin\phi}\frac{\partial v_\rho}{\partial \theta}\right)\mathbf{e}_\phi && \text{(vii)}
\end{aligned}$$

$$\nabla \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_\rho}{\partial \rho} & \dfrac{1}{\rho\,\sin\phi}\dfrac{\partial v_\rho}{\partial \theta} - \dfrac{v_\theta}{\rho} & \dfrac{1}{\rho}\left(\dfrac{\partial v_\rho}{\partial \phi} - v_\phi\right) \\ \dfrac{\partial v_\theta}{\partial \rho} & \dfrac{1}{\rho\,\sin\phi}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{\cot\phi}{\rho}v_\phi + \dfrac{v_\rho}{\rho} & \dfrac{1}{\rho}\dfrac{\partial v_\theta}{\partial \phi} \\ \dfrac{\partial v_\phi}{\partial \rho} & \dfrac{1}{\rho\,\sin\phi}\dfrac{\partial v_\phi}{\partial \theta} - \dfrac{\cot\phi}{\rho}v_\theta & \dfrac{1}{\rho}\left(\dfrac{\partial v_\phi}{\partial \phi} + v_\rho\right) \end{pmatrix}. \tag{viii}$$