

A few techniques to achieve invisibility in waveguides

Lecture 4: A spectral problem characterizing zero reflection

Lucas Chesnel

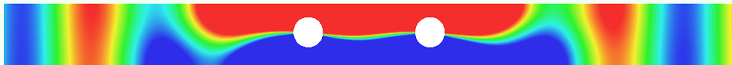
| Idefix team, EDF/Ensta/Inria, France



TOULOUSE, 27/06/2025

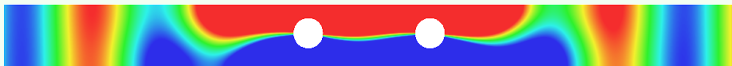
Lecture 3

- ♠ We explained how to exploit the **Fano resonance phenomenon** together with **symmetry** to obtain **zero reflection** for **large** obstacles.



Lecture 3

- ♠ We explained how to exploit the **Fano resonance phenomenon** together with **symmetry** to obtain **zero reflection** for **large** obstacles.



Lecture 4 : Two distinct goals

- 1 A simple example of large invisible defect in acoustics

ASYMPTOTIC ANALYSIS:

k is given, we construct simple examples of Ω such that $T = 1$.

- 2 A spectral approach to determine non reflecting wavenumbers

SPECTRAL THEORY:

Ω is given, we explain how to **find non reflecting** k by solving an unusual **spectral problem**.

Outline of the talk

- 1 A simple example of large invisible defect in acoustics

ASYMPTOTIC ANALYSIS:

k is given, we construct simple examples of Ω such that $T = 1$.

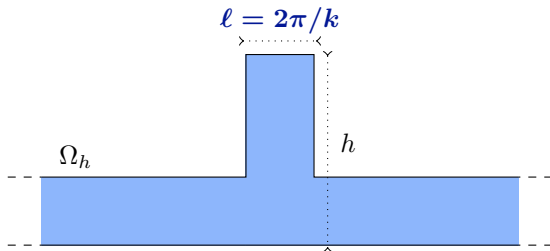
- 2 A spectral approach to determine non reflecting wavenumbers

SPECTRAL THEORY:

Ω is given, we explain how to find non reflecting k by solving an unusual spectral problem.

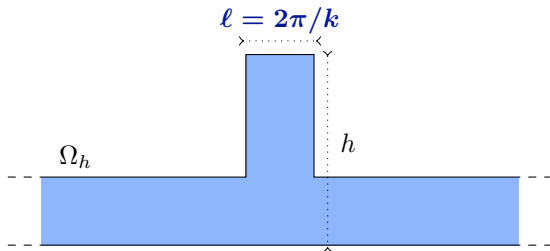
Geometrical setting

- Let us work in the geometry

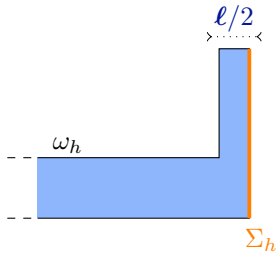


Geometrical setting

- Let us work in the geometry

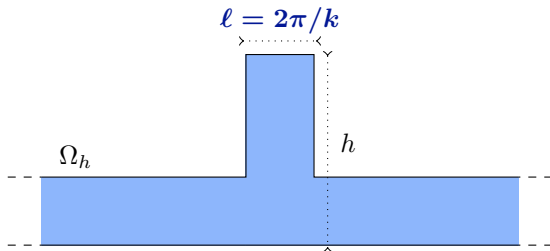


- Introduce the two **half-waveguide** problems

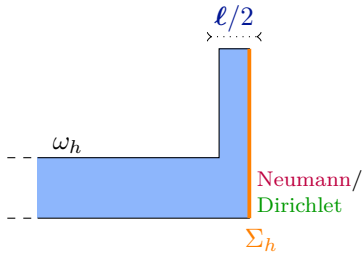


Geometrical setting

- Let us work in the geometry

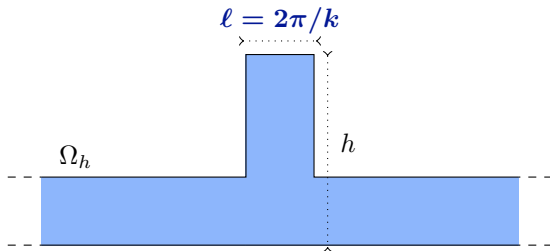


- Introduce the two **half-waveguide** problems

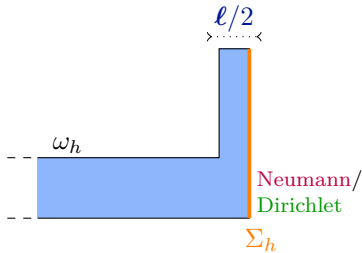


Geometrical setting

- Let us work in the geometry



- Introduce the two **half-waveguide** problems

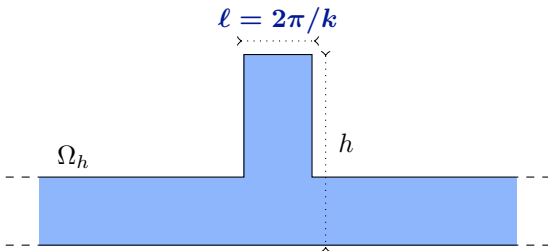


$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \omega_h \\ \partial_n u = 0 & \text{on } \partial\omega_h \end{cases}$$

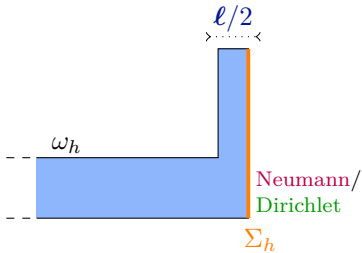
$$\begin{cases} \Delta U + k^2 U = 0 & \text{in } \omega_h \\ \partial_n U = 0 & \text{on } \partial\omega_h \setminus \Sigma_h \\ U = 0 & \text{on } \Sigma_h. \end{cases}$$

Geometrical setting

- Let us work in the geometry



- Introduce the two **half-waveguide** problems



$$\left\{ \begin{array}{ll} \Delta u + k^2 u = 0 & \text{in } \omega_h \\ \partial_n u = 0 & \text{on } \partial\omega_h \end{array} \right. \text{Neumann B.C.}$$

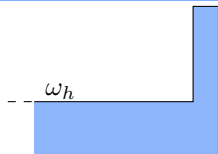
$$\left\{ \begin{array}{ll} \Delta U + k^2 U = 0 & \text{in } \omega_h \\ \partial_n U = 0 & \text{on } \partial\omega_h \setminus \Sigma_h \\ U = 0 & \text{on } \Sigma_h. \end{array} \right. \text{Mixed B.C.}$$

Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

$$u = w^+ + \textcolor{red}{R}^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + \textcolor{green}{R}^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$

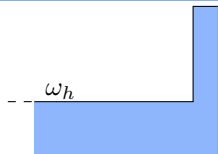


Relations for the scattering coefficients

- ▶ Half-waveguide problems admit the solutions

$$u = w^+ + \textcolor{red}{R}^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + \textcolor{green}{R}^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$



- ▶ Due to conservation of energy, one has

$$|\textcolor{red}{R}^N| = |\textcolor{green}{R}^D| = 1.$$

Relations for the scattering coefficients

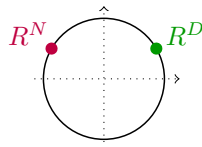
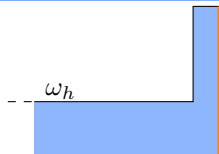
- ▶ Half-waveguide problems admit the solutions

$$u = w^+ + R^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + R^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$

- ▶ Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$

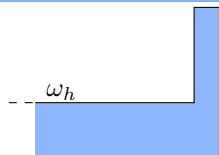


Relations for the scattering coefficients

- ▶ Half-waveguide problems admit the solutions

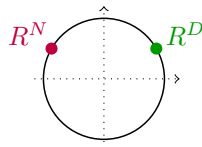
$$u = w^+ + R^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + R^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$



- ▶ Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



- ▶ Using symmetry considerations, one can show that

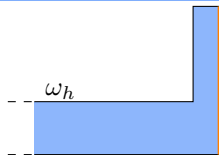
$$R = \frac{R^N + R^D}{2} \quad \text{and} \quad T = \frac{R^N - R^D}{2}$$

Relations for the scattering coefficients

- ▶ Half-waveguide problems admit the solutions

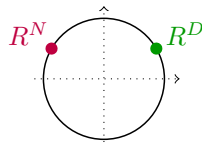
$$u = w^+ + R^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + R^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$



- ▶ Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



- ▶ Using symmetry considerations, one can show that

$$R = \frac{R^N + R^D}{2}$$

and

$$T = \frac{R^N - R^D}{2}$$

Perfect invisibility

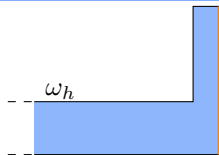
$$\Leftrightarrow [R^N = 1, R^D = -1]$$

Relations for the scattering coefficients

- ▶ Half-waveguide problems admit the solutions

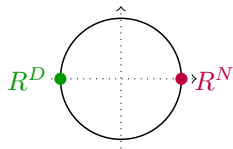
$$u = w^+ + R^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + R^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$



- ▶ Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



- ▶ Using symmetry considerations, one can show that

$$R = \frac{R^N + R^D}{2}$$

and

$$T = \frac{R^N - R^D}{2}$$

Perfect invisibility

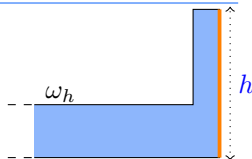
$$\Leftrightarrow [R^N = 1, R^D = -1]$$

Relations for the scattering coefficients

- ▶ Half-waveguide problems admit the solutions

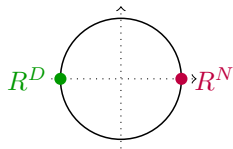
$$u = w^+ + R^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + R^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$



- ▶ Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



- ▶ Using symmetry considerations, one can show that

$$R = \frac{R^N + R^D}{2}$$

and

$$T = \frac{R^N - R^D}{2}$$

Perfect invisibility

$$\Leftrightarrow [R^N = 1, R^D = -1]$$

Crucial point: in this **particular geometry** ω_h ,

$u = w^+ + w^- = 2 \cos(kx)$ solves the Neum. pb.

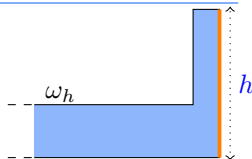
$$\Rightarrow R^N = 1, \forall h > 1.$$

Relations for the scattering coefficients

- ▶ Half-waveguide problems admit the solutions

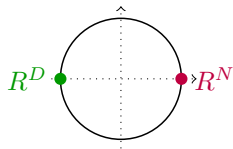
$$u = w^+ + R^N w^- + \tilde{u}, \quad \text{with } \tilde{u} \in H^1(\omega_h)$$

$$U = w^+ + R^D w^- + \tilde{U}, \quad \text{with } \tilde{U} \in H^1(\omega_h).$$



- ▶ Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



- ▶ Using symmetry considerations, one can show that

$$R = \frac{R^N + R^D}{2} \quad \text{and} \quad T = \frac{R^N - R^D}{2}$$

Perfect invisibility

$$\Leftrightarrow [R^N = 1, R^D = -1]$$

Crucial point: in this **particular geometry** ω_h ,

$u = w^+ + w^- = 2 \cos(kx)$ solves the Neum. pb.

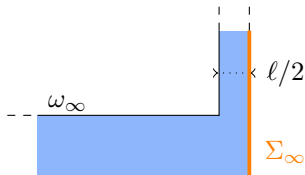
$$\Rightarrow R^N = 1, \forall h > 1.$$

→ It remains to study the behaviour of $R^D = R^D(h)$ as $h \rightarrow +\infty$.

Asymptotics of R^D as $h \rightarrow +\infty$



Depends on the nb. of propagating modes in the vertical branch of ω_∞

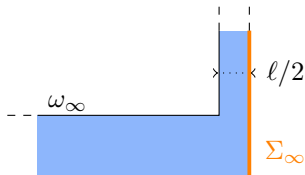


$$(\mathcal{P}^D) \left| \begin{array}{ll} \Delta\varphi + k^2\varphi & = 0 \quad \text{in } \omega_\infty \\ \partial_n\varphi & = 0 \quad \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = 0 \quad \text{on } \Sigma_\infty. \end{array} \right.$$

Asymptotics of R^D as $h \rightarrow +\infty$



Depends on the nb. of **propagating modes** in the **vertical branch** of ω_∞



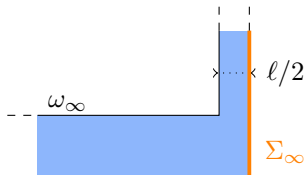
$(\mathcal{P}^D)$	$\begin{aligned} \Delta\varphi + k^2\varphi &= 0 && \text{in } \omega_\infty \\ \partial_n\varphi &= 0 && \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi &= 0 && \text{on } \Sigma_\infty. \end{aligned}$
-------------------	--

- For $\ell = 2\pi/k$, **2 modes** can **propagate** in the vertical branch of ω_∞ .

Asymptotics of R^D as $h \rightarrow +\infty$



Depends on the nb. of **propagating modes** in the **vertical branch** of ω_∞



$$(\mathcal{P}^D) \left| \begin{array}{ll} \Delta\varphi + k^2\varphi &= 0 \quad \text{in } \omega_\infty \\ \partial_n\varphi &= 0 \quad \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi &= 0 \quad \text{on } \Sigma_\infty. \end{array} \right.$$

- For $\ell = 2\pi/k$, **2 modes** can **propagate** in the vertical branch of ω_∞ .

► Using asymptotic analysis, one shows that when $h \rightarrow +\infty$,

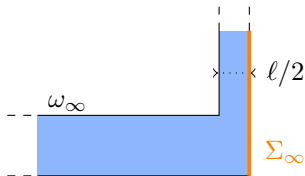
$$|R^D(h) - R_{\text{asy}}^D(h)| \leq Ce^{-ch}$$

where $R_{\text{asy}}^D(h)$ runs periodically on the unit circle $\mathcal{C}$.

Asymptotics of R^D as $h \rightarrow +\infty$



Depends on the nb. of **propagating modes** in the **vertical branch** of ω_∞



$(\mathcal{P}^D)$	$\Delta\varphi + k^2\varphi = 0$	in ω_∞
	$\partial_n\varphi = 0$	on $\partial\omega_\infty \setminus \Sigma_\infty$
	$\varphi = 0$	on Σ_∞ .

- For $\ell = 2\pi/k$, **2 modes** can **propagate** in the vertical branch of ω_∞ .

► Using asymptotic analysis, one shows that when $h \rightarrow +\infty$,

$$|R^D(h) - R_{\text{asy}}^D(h)| \leq Ce^{-ch}$$

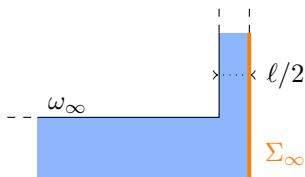
where $R_{\text{asy}}^D(h)$ runs periodically on the unit circle $\mathcal{C}$.

► Additionally one can prove that $h \mapsto R^D(h)$ runs **continuously** on $\mathcal{C}$.

Asymptotics of R^D as $h \rightarrow +\infty$



Depends on the nb. of **propagating modes** in the **vertical branch** of ω_∞



$$(\mathcal{P}^D) \left| \begin{array}{ll} \Delta\varphi + k^2\varphi &= 0 \quad \text{in } \omega_\infty \\ \partial_n\varphi &= 0 \quad \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi &= 0 \quad \text{on } \Sigma_\infty. \end{array} \right.$$

- For $\ell = 2\pi/k$, **2 modes** can **propagate** in the vertical branch of ω_∞ .

► Using asymptotic analysis, one shows that when $h \rightarrow +\infty$,

$$|R^D(h) - R_{\text{asy}}^D(h)| \leq Ce^{-ch}$$

where $R_{\text{asy}}^D(h)$ runs periodically on the unit circle $\mathcal{C}$.

► Additionally one can prove that $h \mapsto R^D(h)$ runs **continuously** on $\mathcal{C}$.

⇒ There is a sequence (h_n) with $h_n \rightarrow +\infty$ such that $R^D(h_n) = -1$.

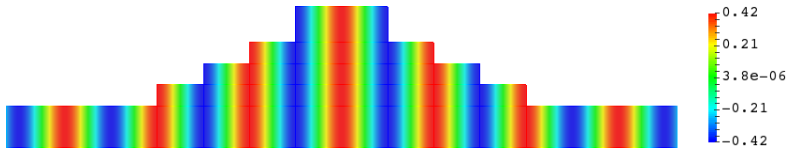
Conclusion

THEOREM: There is an unbounded sequence (h_n) such that for $h = h_n$, we have $T = 1$ (perfect invisibility).

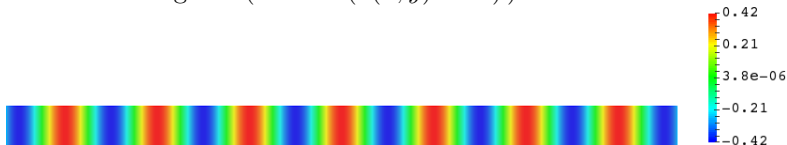
Numerical results

- ▶ Works also in the geometry below. When we vary h , the height of the **central branch**, T runs exactly on the circle $\mathcal{C}(1/2, 1/2)$.
→ Numerically, we simply **sweep** in h and extract the h such that $T(h) = 1$.

- ▶ **Perfectly invisible** defect ($t \mapsto \Re(v(x, y)e^{-i\omega t})$)

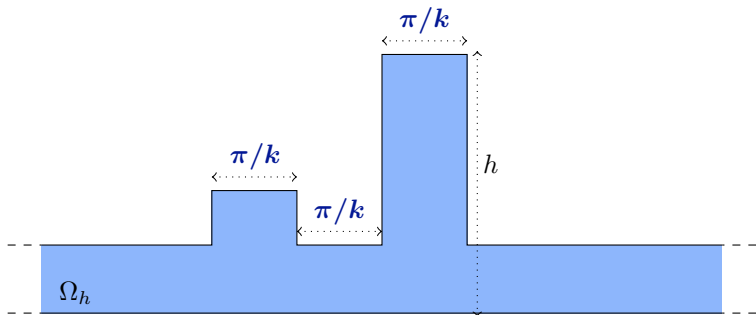


- ▶ Reference waveguide ($t \mapsto \Re(v(x, y)e^{-i\omega t})$)



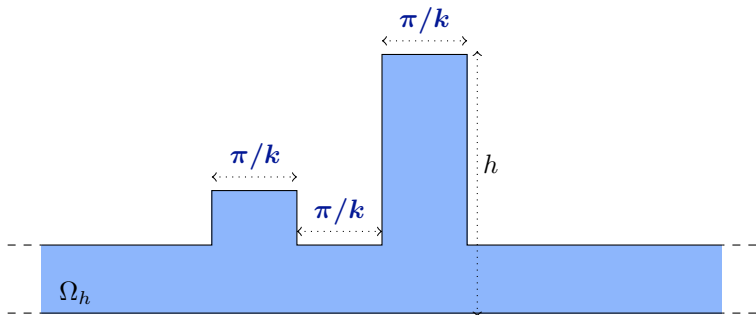
Remark

- Actually Ω does not have to be symmetric and we can work in the following geometry:



Remark

- Actually Ω **does not have to be symmetric** and we can work in the following geometry:



- In this Ω_h , we can show that there holds $R + T = 1$.
- With the **identity of energy** $|R|^2 + |T|^2 = 1$, this guarantees that T must be on the circle $\mathcal{C}(1/2, 1/2)$.
- Finally, with asy. analysis, we show that T goes through 1 as $h \rightarrow +\infty$.

Outline of the talk

- 1 A simple example of large invisible defect in acoustics

ASYMPTOTIC ANALYSIS:

k is given, we construct simple examples of Ω such that $T = 1$.

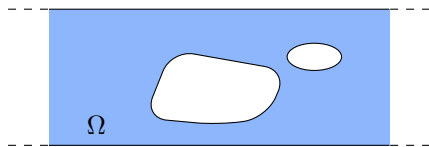
- 2 A spectral approach to determine non reflecting wavenumbers

SPECTRAL THEORY:

Ω is given, we explain how to **find non reflecting** k by solving an unusual **spectral problem**.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$

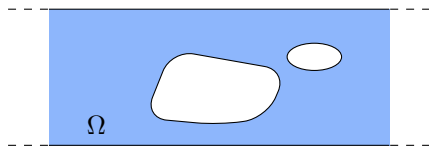


Find $v = v_i + v_s$ s. t.

$$\begin{aligned}\Delta v + k^2 v &= 0 && \text{in } \Omega, \\ \partial_n v &= 0 && \text{on } \partial\Omega, \\ v_s &\text{ is outgoing.}\end{aligned}$$

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



Find $v = v_i + v_s$ s. t.

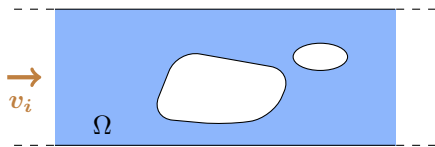
$$\begin{aligned}\Delta v + k^2 v &= 0 && \text{in } \Omega, \\ \partial_n v &= 0 && \text{on } \partial\Omega, \\ v_s &\text{ is outgoing.}\end{aligned}$$

- For this problem, the **modes** are

Propagating	$w_n^\pm(x, y) = e^{\pm i\beta_n x} \cos(n\pi y), \beta_n = \sqrt{k^2 - n^2\pi^2}, n \in \llbracket 0, N-1 \rrbracket$
Evanescent	$w_n^\pm(x, y) = e^{\mp \beta_n x} \cos(n\pi y), \beta_n = \sqrt{n^2\pi^2 - k^2}, n \geq N.$

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



Find $v = v_i + v_s$ s. t.
$$\Delta v + k^2 v = 0 \quad \text{in } \Omega,$$
$$\partial_n v = 0 \quad \text{on } \partial\Omega,$$
$$v_s \text{ is outgoing.}$$

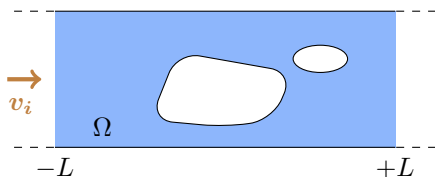
- For this problem, the **modes** are

Propagating	$\left \begin{array}{l} w_n^\pm(x, y) = e^{\pm i\beta_n x} \cos(n\pi y), \quad \beta_n = \sqrt{k^2 - n^2\pi^2}, \quad n \in \llbracket 0, N-1 \rrbracket \\ w_n^\pm(x, y) = e^{\mp \beta_n x} \cos(n\pi y), \quad \beta_n = \sqrt{n^2\pi^2 - k^2}, \quad n \geq N. \end{array} \right.$
Evanescent	

- Set $v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



Find $v = v_i + v_s$ s. t.
 $\Delta v + k^2 v = 0$ in Ω ,
 $\partial_n v = 0$ on $\partial\Omega$,
 v_s is outgoing.

- For this problem, the modes are

Propagating	$w_n^\pm(x, y) = e^{\pm i\beta_n x} \cos(n\pi y), \quad \beta_n = \sqrt{k^2 - n^2\pi^2}, \quad n \in \llbracket 0, N-1 \rrbracket$
Evanescent	$w_n^\pm(x, y) = e^{\mp \beta_n x} \cos(n\pi y), \quad \beta_n = \sqrt{n^2\pi^2 - k^2}, \quad n \geq N.$

- Set $v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

- v_s is outgoing $\Leftrightarrow$

$$v_s = \sum_{n=0}^{+\infty} \gamma_n^\pm w_n^\pm \quad \text{for } \pm x \geq L, \text{ with } (\gamma_n^\pm) \in \mathbb{C}^{\mathbb{N}}.$$

Goal of the section

DEFINITION: v is a non reflecting mode if v_s is expo. decaying for $x \leq -L$
 $\Leftrightarrow \gamma_n^- = 0, n \in \llbracket 0, N-1 \rrbracket \Leftrightarrow$ energy is completely transmitted.

GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v .

Goal of the section

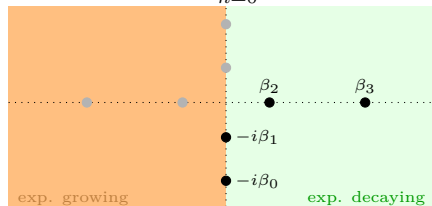
DEFINITION: v is a **non reflecting mode** if v_s is **expo. decaying** for $x \leq -L$
 $\Leftrightarrow \gamma_n^- = 0, n \in \llbracket 0, N-1 \rrbracket \Leftrightarrow$ **energy is completely transmitted.**

GOAL

For a **given geometry**, we present a method to find **values of k** such that there is a **non reflecting mode** v .

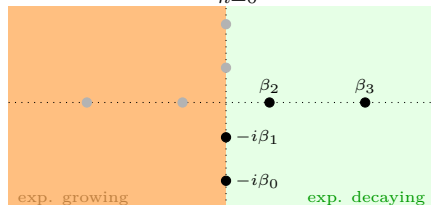
→ Note that **non reflection** occurs for **particular v_i** to be computed.

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \pm x \geq L.$

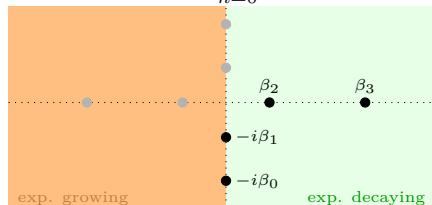


Modal exponents for v_s ($x \leq -L$)

► For $\theta \in (0; \pi/2)$, consider the **complex change of variables**

$$\mathcal{I}_\theta(x) = \begin{cases} -L + (x + L) e^{i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L) e^{i\theta} & \text{for } x \geq L. \end{cases}$$

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)

- For $\theta \in (0; \pi/2)$, consider the **complex change of variables**

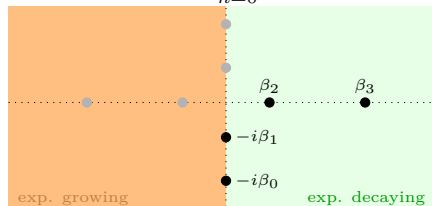
$$\mathcal{I}_\theta(x) = \begin{cases} -L + (x + L) e^{i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L) e^{i\theta} & \text{for } x \geq L. \end{cases}$$

- Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y).$

1) $v_\theta = v_s$ for $|x| < L.$

2) v_θ is exp. decaying at infinity.

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^\pm e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^\pm e^{\mp \beta_n x} \cos(n\pi y), \quad \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)

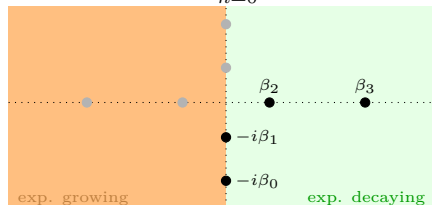
$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^\pm e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^\pm e^{\mp \tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

► Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y).$

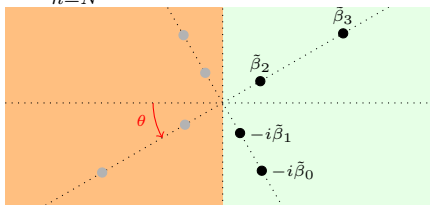
1) $v_\theta = v_s$ for $|x| < L.$

2) v_θ is exp. decaying at infinity.

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)



Modal exponents for v_θ ($x \leq -L$)

$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^{\pm} e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^{\pm} e^{\mp \tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

► Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y).$

1) $v_\theta = v_s$ for $|x| < L.$

2) v_θ is exp. decaying at infinity.

► v_θ solves

$$(*) \quad \left\{ \begin{array}{ll} \alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v_\theta}{\partial x} \right) + \frac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta = 0 & \text{in } \Omega \\ \partial_n v_\theta = -\partial_n v_i & \text{on } \partial\Omega. \end{array} \right.$$

► v_θ solves

$$(*) \quad \left\{ \begin{array}{ll} \alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v_\theta}{\partial x} \right) + \frac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta = 0 & \text{in } \Omega \\ \partial_n v_\theta = -\partial_n v_i & \text{on } \partial\Omega. \end{array} \right.$$

$$\alpha_\theta(x) = 1 \text{ for } |x| < L$$

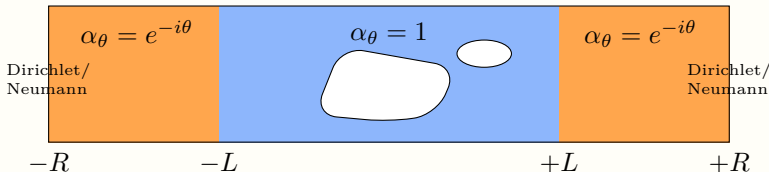
$$\alpha_\theta(x) = e^{-i\theta} \text{ for } |x| \geq L$$

► v_θ solves (*)

$$\left\{ \begin{array}{l} \alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v_\theta}{\partial x} \right) + \frac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta = 0 \quad \text{in } \Omega \\ \partial_n v_\theta = -\partial_n v_i \quad \text{on } \partial\Omega. \end{array} \right.$$

$$\alpha_\theta(x) = 1 \text{ for } |x| < L \quad \alpha_\theta(x) = e^{-i\theta} \text{ for } |x| \geq L$$

- Numerically we solve (*) in the truncated domain



⇒ We obtain a good approximation of v_s for $|x| < L$.

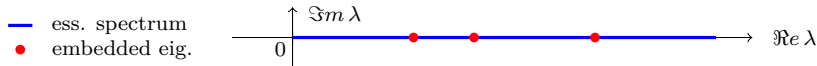
- This is the method of **Perfectly Matched Layers** (PMLs).

Spectral analysis

- Define the operators A , A_θ of $L^2(\Omega)$ such that

$$Av = -\Delta v, \quad A_\theta v = -\left(\alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial y^2}\right) + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\text{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain **embedded eigenvalues** in the essential spectrum.

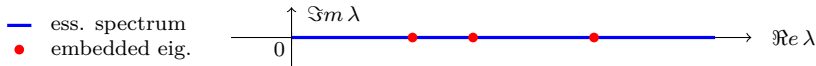


Spectral analysis

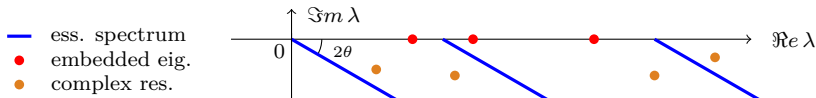
- Define the operators A , A_θ of $L^2(\Omega)$ such that

$$Av = -\Delta v, \quad A_\theta v = -\left(\alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial y^2}\right) + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\text{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain **embedded eigenvalues** in the essential spectrum.

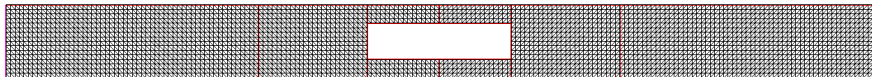


- A_θ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho e^{i\gamma}, \rho \geq 0, \gamma \in [-2\theta; 0]\}$.
- $\sigma_{\text{ess}}(A_\theta) = \cup_{n \in \mathbb{N}} \{n^2 \pi^2 + t e^{-2i\theta}, t \geq 0\}$.
- **real eigenvalues** of $A_\theta = \text{real eigenvalues of } A$.



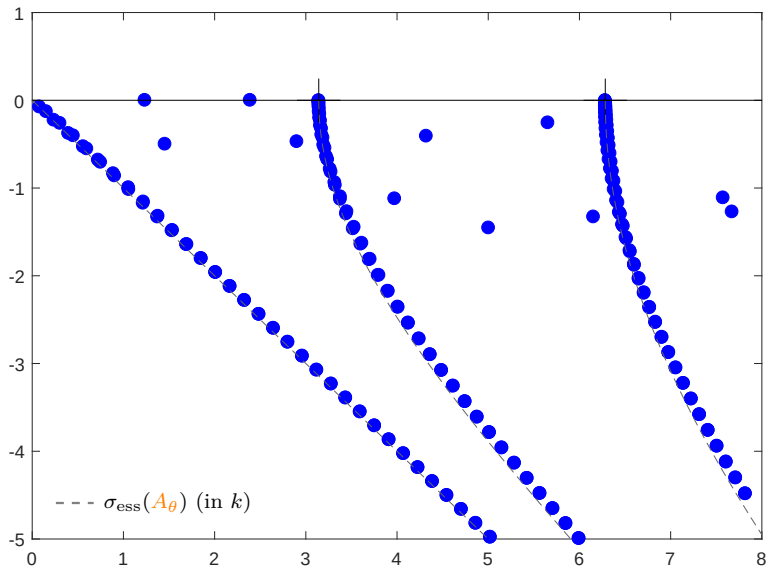
Numerical results

- We work in the geometry



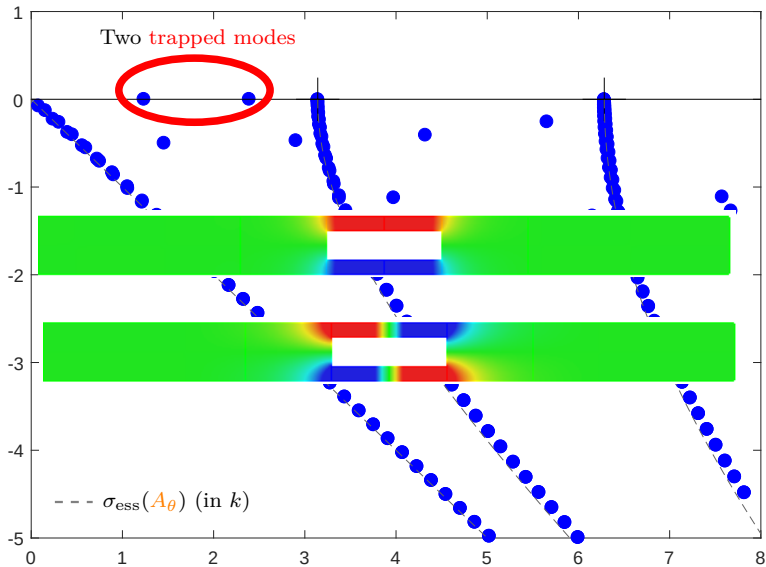
Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and **outgoing** at $+\infty$.

IMPORTANT REMARK: **general** v decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

IMPORTANT REMARK: **non reflecting** v decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

IMPORTANT REMARK: non reflecting v decompose as

$$v = \sum_{n=0}^{N-1} \alpha_n w_n^+ + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

- In other words, non reflecting v are

ingoing at $-\infty$ and outgoing at $+\infty$.

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

IMPORTANT REMARK: non reflecting v decompose as

$$v = \sum_{n=0}^{N-1} \alpha_n w_n^+ + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

- In other words, non reflecting v are

ingoing at $-\infty$ and outgoing at $+\infty$.



Let us change the sign of the complex scaling at $-\infty$!

A new complex spectrum for non reflecting v

- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{J}_\theta(x) = \begin{cases} -L + (x + L) e^{-i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L) e^{+i\theta} & \text{for } x \geq L. \end{cases}$$

A new complex spectrum for non reflecting v

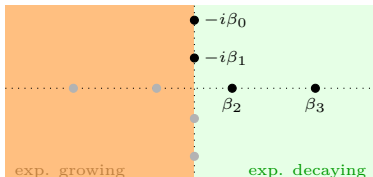
- For $\theta \in (0; \pi/2)$, consider the **complex change of variables**

$$\mathcal{J}_\theta(x) = \begin{cases} -L + (x + L) e^{-i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L) e^{+i\theta} & \text{for } x \geq L. \end{cases}$$

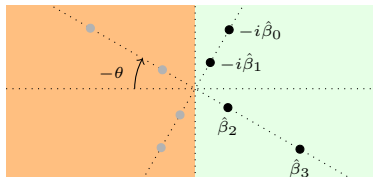
- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

1) $u_\theta = v$ for $|x| < L$.

2) u_θ is exp. decaying at infinity.



Modal exponents for v ($x \leq -L$)



Modal exponents for u_θ ($x \leq -L$)

A new complex spectrum for non reflecting v

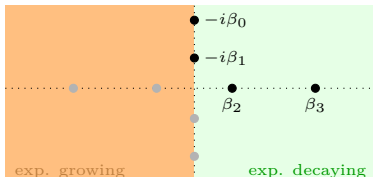
- For $\theta \in (0; \pi/2)$, consider the **complex change of variables**

$$\mathcal{J}_\theta(x) = \begin{cases} -L + (x + L) e^{-i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L) e^{+i\theta} & \text{for } x \geq L. \end{cases}$$

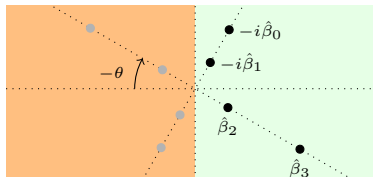
- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

1) $u_\theta = v$ for $|x| < L$.

2) u_θ is exp. decaying at infinity.



Modal exponents for v ($x \leq -L$)



Modal exponents for u_θ ($x \leq -L$)

- u_θ solves $(*) \left| \begin{aligned} \beta_\theta \frac{\partial}{\partial x} \left(\beta_\theta \frac{\partial u_\theta}{\partial x} \right) + \frac{\partial^2 u_\theta}{\partial y^2} + k^2 u_\theta &= 0 & \text{in } \Omega \\ \partial_n u_\theta &= 0 & \text{on } \partial\Omega. \end{aligned} \right.$

A new complex spectrum for non reflecting v

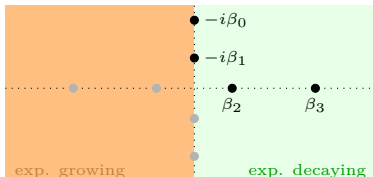
- For $\theta \in (0; \pi/2)$, consider the **complex change of variables**

$$\mathcal{J}_\theta(x) = \begin{cases} -L + (x + L) e^{-i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L) e^{+i\theta} & \text{for } x \geq L. \end{cases}$$

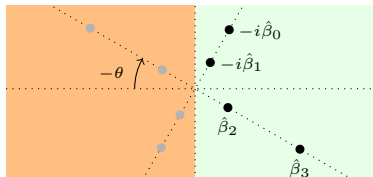
- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

1) $u_\theta = v$ for $|x| < L$.

2) u_θ is exp. decaying at infinity.



Modal exponents for v ($x \leq -L$)



Modal exponents for u_θ ($x \leq -L$)

- u_θ solves $(*) \left| \begin{aligned} \beta_\theta \frac{\partial}{\partial x} \left(\beta_\theta \frac{\partial u_\theta}{\partial x} \right) + \frac{\partial^2 u_\theta}{\partial y^2} + k^2 u_\theta &= 0 & \text{in } \Omega \\ \partial_n u_\theta &= 0 & \text{on } \partial\Omega. \end{aligned} \right.$

$$\beta_\theta(x) = 1 \text{ for } |x| < L, \quad \beta_\theta(x) = e^{i\theta} \text{ for } x \leq -L, \quad \beta_\theta(x) = e^{-i\theta} \text{ for } x \geq L$$

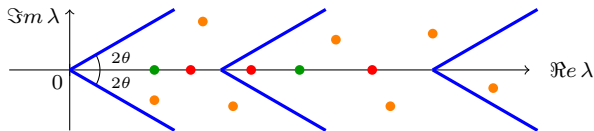
Spectral analysis

- Define the operator B_θ of $L^2(\Omega)$ such that

$$B_\theta v = -\left(\beta_\theta \frac{\partial}{\partial x} \left(\beta_\theta \frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial y^2}\right) \quad + \partial_n v = 0 \text{ on } \partial\Omega.$$

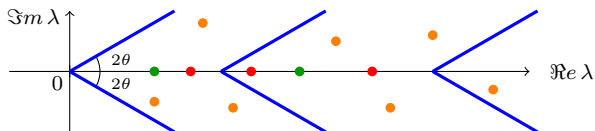
- B_θ is not selfadjoint. $\sigma(B_\theta) \subset \{\rho e^{i\gamma}, \rho \geq 0, \gamma \in [-2\theta; 2\theta]\}$.
- $\sigma_{\text{ess}}(B_\theta) = \cup_{n \in \mathbb{N}} \{n^2\pi^2 + t e^{-2i\theta}, t \geq 0\} \cup \{n^2\pi^2 + t e^{2i\theta}, t \geq 0\}$.
- **real eigenvalues** of B_θ = **real eigenvalues** of A + **non reflecting** k^2 .

- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



Remarks

- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



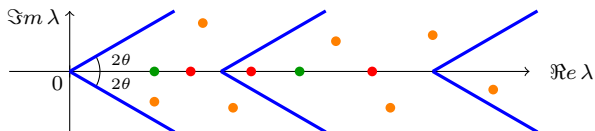
1) • ? eig. correspond to solutions of the Helmholtz equation which are **exp. growing** at one side of Ω , **exp. decaying** at the other.

Different from **complex resonances** for which the eigenfunctions are **exp. growing** both at $\pm\infty$...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is **discrete**.

Remarks

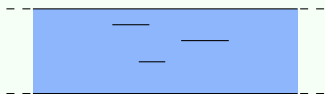
- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



1) • ? eig. correspond to solutions of the Helmholtz equation which are **exp. growing** at one side of Ω , **exp. decaying** at the other.

Different from **complex resonances** for which the eigenfunctions are **exp. growing** both at $\pm\infty$...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is **discrete**.

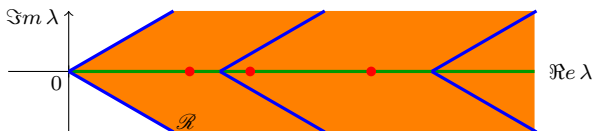


→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathcal{R}$.

Remarks

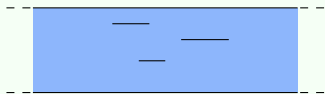
- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



1) • ? eig. correspond to solutions of the Helmholtz equation which are **exp. growing** at one side of Ω , **exp. decaying** at the other.

Different from **complex resonances** for which the eigenfunctions are **exp. growing** both at $\pm\infty$...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is **discrete**.



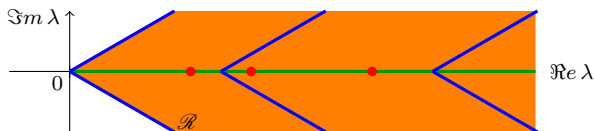
→ **Not true in general!**



$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathcal{R}$.

Remarks

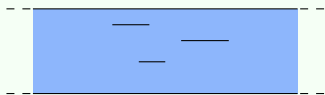
- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



1) • ? eig. correspond to solutions of the Helmholtz equation which are **exp. growing** at one side of Ω , **exp. decaying** at the other.

Different from **complex resonances** for which the eigenfunctions are **exp. growing** both at $\pm\infty$...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is **discrete**.



→ **Not true in general!**

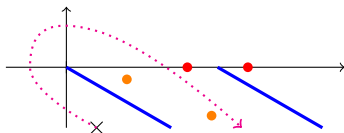


$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathcal{R}$.

→ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is **not connected** $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

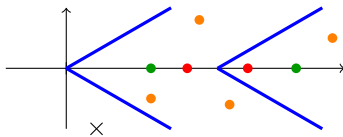
Remarks

sem $\lambda \uparrow$



$A_\theta - z\text{Id}$ invertible

Usual PMLs



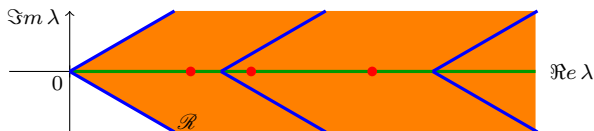
$B_\theta - z\text{Id}$ invertible

Conjugated PMLs

$\rightarrow \mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is **not connected** $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

Remarks

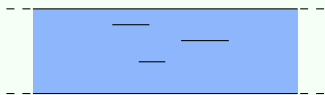
- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



1) • ? eig. correspond to solutions of the Helmholtz equation which are **exp. growing** at one side of Ω , **exp. decaying** at the other.

Different from **complex resonances** for which the eigenfunctions are **exp. growing** both at $\pm\infty$...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is **discrete**.



→ **Not true in general!**

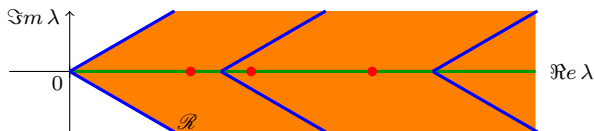


$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathcal{R}$.

→ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is **not connected** $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

Remarks

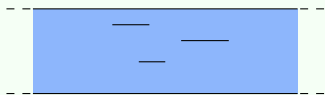
- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



1) • ? eig. correspond to solutions of the Helmholtz equation which are **exp. growing** at one side of Ω , **exp. decaying** at the other.

Different from **complex resonances** for which the eigenfunctions are **exp. growing** both at $\pm\infty$...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is **discrete**.



→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathcal{R}$.

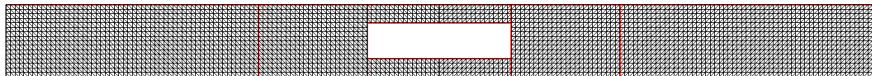
→ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is **not connected** $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

→ A compact perturbation can change drastically the spectrum (B_θ is **not selfadjoint**).

Numerical consequences?

Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

$$\mathcal{P}v(x, y) = v(-x, y) \quad \text{and} \quad \mathcal{T}v(x, y) = \overline{v(x, y)}.$$

PROP.: For **symmetric** $\Omega = \{(-x, y) \mid (x, y) \in \Omega\}$, B_θ is $\mathcal{PT}$ symmetric:

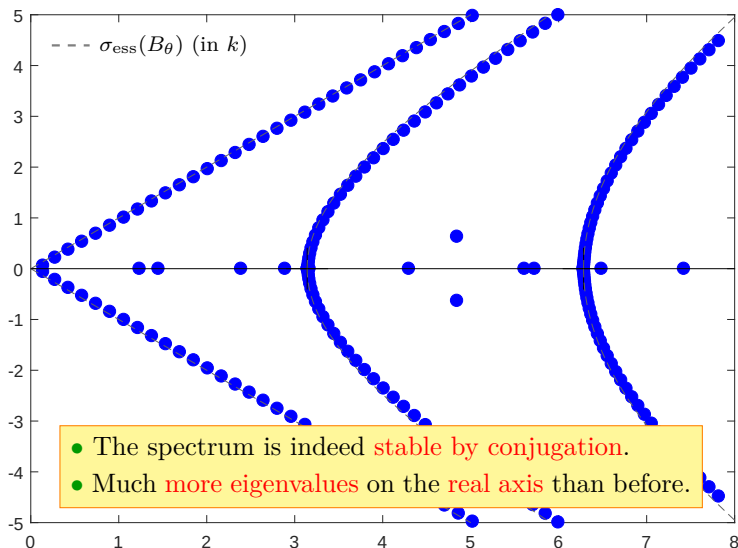
$$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$

As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

$\Rightarrow$ If λ is an “**isolated**” eigenvalue located **close to the real axis**, then $\lambda \in \mathbb{R}$!

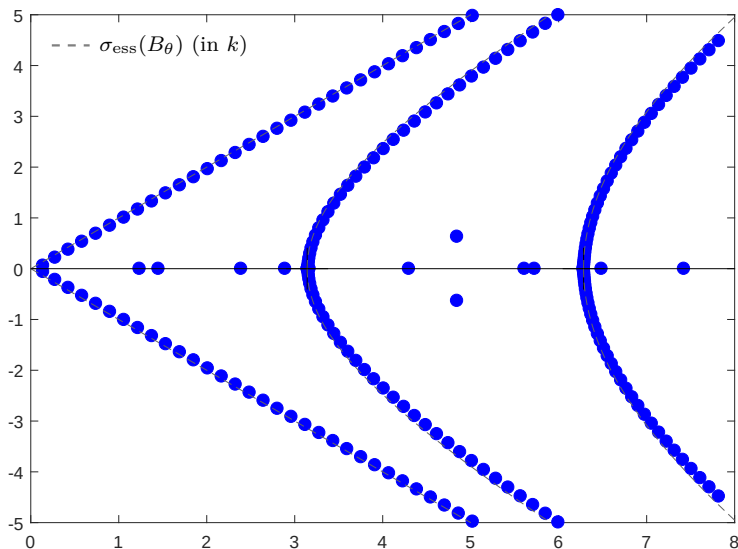
Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



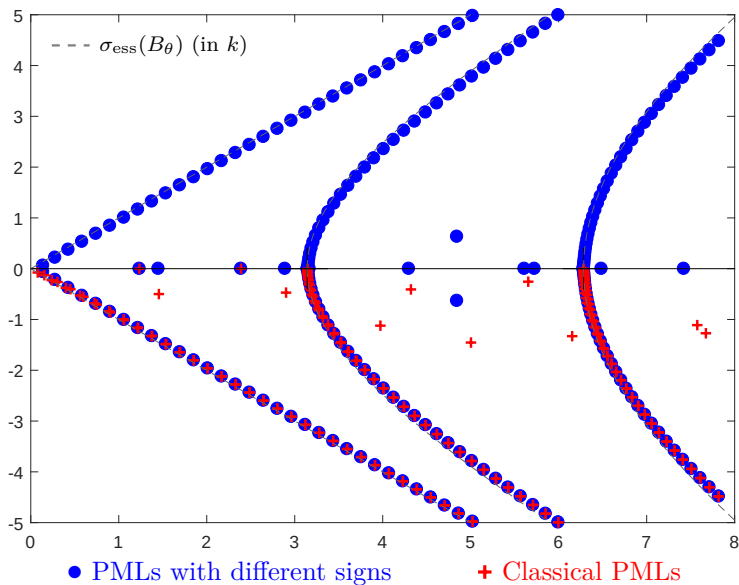
Numerical results

- **Discretized** spectrum in k (not in k^2). We take $\theta = \pi/4$.



Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



Numerical results

- We display the eigenmodes for the **ten first real eigenvalues** in the whole computational domain (including PMLs).



Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode

$$k = 1.2355\dots$$



Second trapped mode

$$k = 2.3897\dots$$



First non reflecting mode

$$k = 1.4513\dots$$

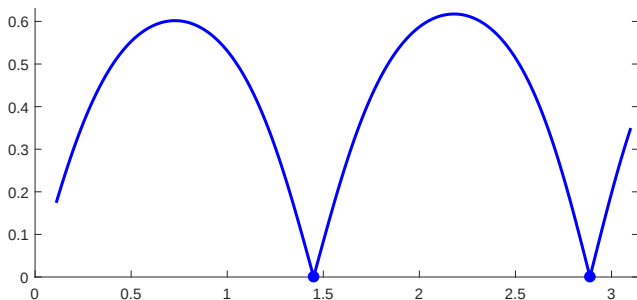


Second non reflecting mode

$$k = 2.8896\dots$$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



First non reflecting mode

$$k = 1.4513\dots$$

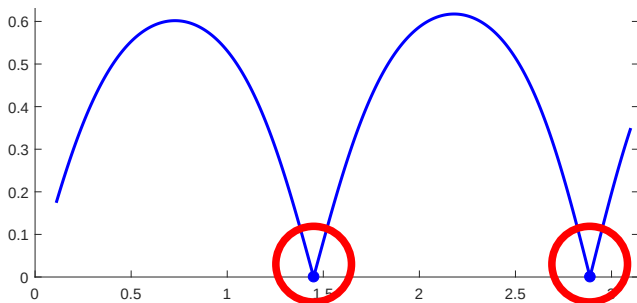


Second non reflecting mode

$$k = 2.8896\dots$$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



First non reflecting mode

$$k = 1.4513\dots$$



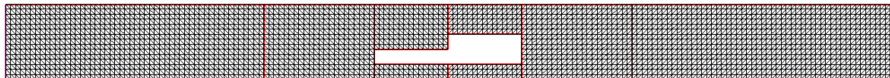
Second non reflecting mode

$$k = 2.8896\dots$$

There is perfect agreement!

Numerical results

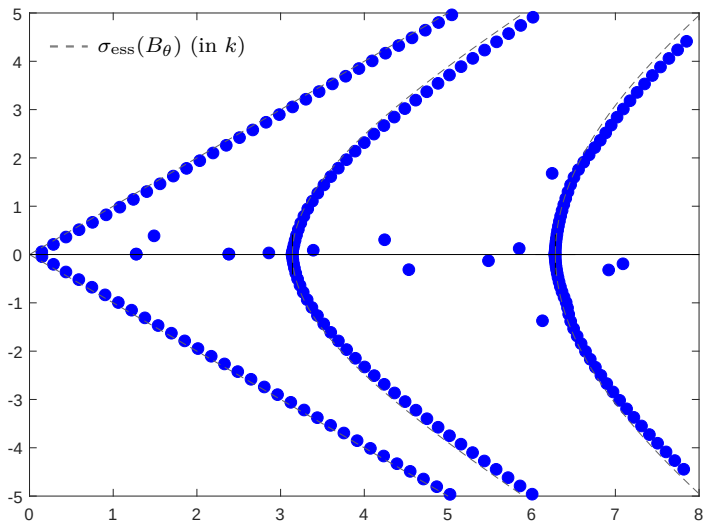
- Now the geometry is **not symmetric** in x nor in y :



- The operator B_θ is **no longer $\mathcal{PT}$ -symmetric** and we expect:
 - No trapped modes
 - No invariance of the spectrum by complex conjugation.

Numerical results

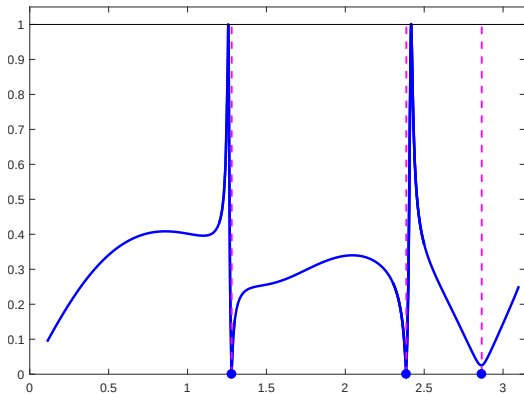
- **Discretized** spectrum of B_θ in k (not in k^2). We take $\theta = \pi/4$.



- Indeed, the spectrum is **not symmetric** w.r.t. the real axis.

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



$$k = 1.28 + 0.0003i$$



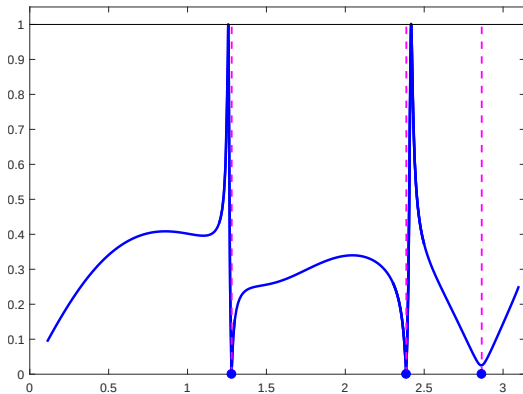
$$k = 2.3866 + 0.0005i$$



$$k = 2.8647 + 0.0243i$$

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



$$k = 1.28 + 0.0003i$$



$$k = 2.3866 + 0.0005i$$



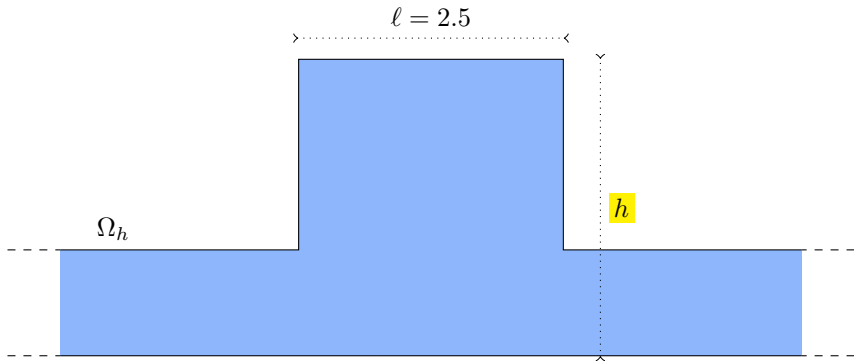
$$k = 2.8647 + 0.0243i$$



Complex eigenvalues also contain information on **almost no reflection**.

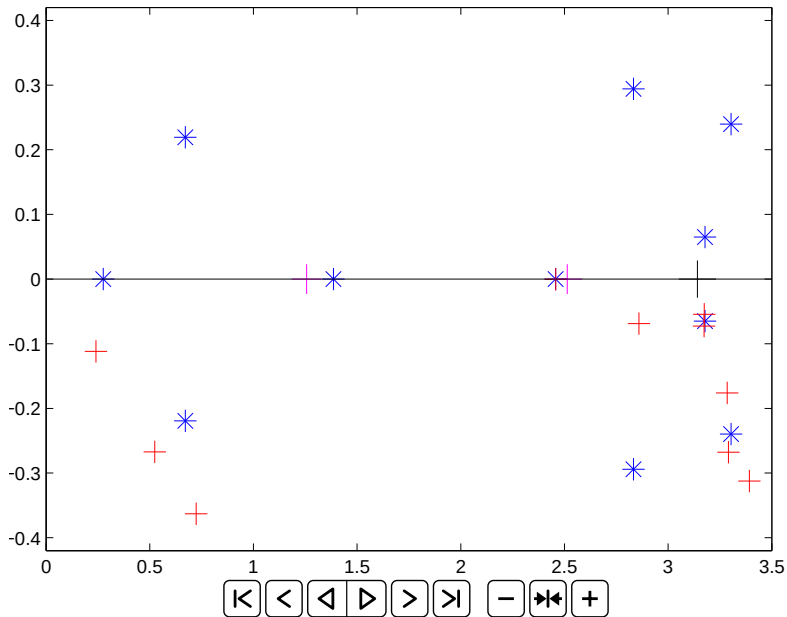
Spectra for a changing geometry

- ▶ Two series of computations: one with PMLs with different sign, one with classical PMLs. We compute the spectra for $h \in (1.3; 8)$.



- ▶ The magenta marks on the real axis correspond to $k = \pi/\ell$ & $k = 2\pi/\ell$. For $k = 2\pi/\ell$, trapped modes and $T = 1$ should occur for certain h .
- ▶ We zoom at the region $0 < \Re k < \pi$.

$$h = 1.3335$$



Conclusion

Part I

- ♠ We explained how to find simple examples of Ω where $T = 1$ for the **Neumann** problem.
- 1) This can be adapted to construct geometries supporting **trapped modes** for the **Neumann** problem.
- 2) However this approach **does not work** for the Dirichlet problem.

Part II

- ♠ **Spectral approach** to compute **non reflecting** k ($R = 0$) for a **given** Ω .
- 1) Can we find a **spectral approach** to compute **completely reflecting** or **completely invisible** k ?
- 2) Can we prove **existence** of **non reflecting** k for the $\mathcal{PT}$ -symmetric pb?

Bibliography

► Part I



L. Chesnel, V. Pagneux. Simple examples of perfectly invisible and trapped modes in waveguides, Quart. J. Mech. Appl. Math., vol. 71, 3:297-315, 2018.

► Part II



A.-S. Bonnet-Ben Dhia, L. Chesnel, V. Pagneux. Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem. PRSA, vol. 474, 2018.



H. Hernandez-Coronado, D. Krejčířík, P. Siegl. Perfect transmission scattering as a PT-symmetric spectral problem. Phys. Lett. A, 375(22):2149-2152, 2011.



W.R. Sweeney, C.W. Hsu, A.D. Stone. Theory of reflectionless scattering modes. Phys. Rev. A, vol. 102, 6:063511, 2020.

Conclusion of the course

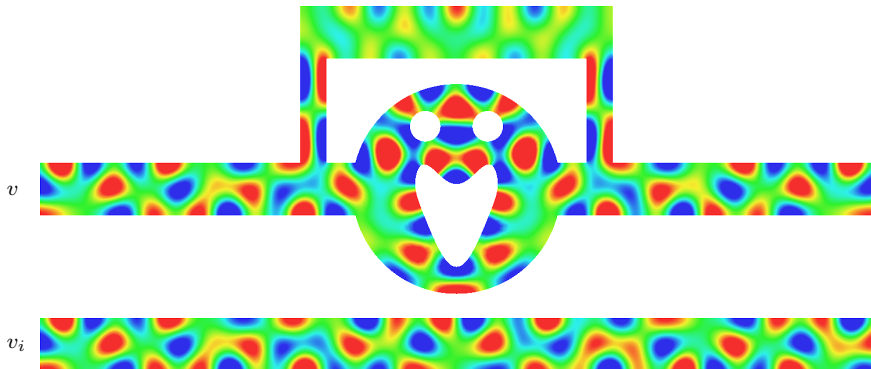
What we did

Lecture 1. We presented rudiments of **scattering theory** in waveguides.

Lecture 2, 3 & 4. We used tools of **asymptotic analysis** and **spectral theory** to identify situations of invisibility:

- Construction of **small amplitude** invisible obstacles.
- Construction of **large amplitude** non reflecting obstacles using **complex resonances**.
- We presented a **spectral** problem characterizing **zero reflection**.

→ **To be continued...**



Thank you for your attention!