

A few techniques to achieve invisibility in waveguides

Lecture 3: Playing with resonances to achieve invisibility

Lucas Chesnel

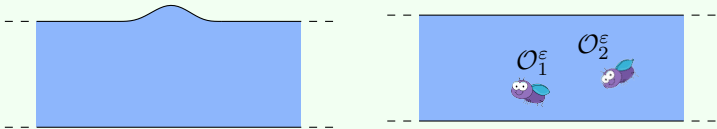
| Idefix team, EDF/Ensta/Inria, France



TOULOUSE, 26/06/2025

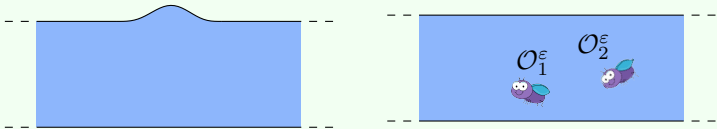
Lecture 2

- ♠ We explained how to construct **small** non reflecting or invisible obstacles by working with **perturbative techniques**.



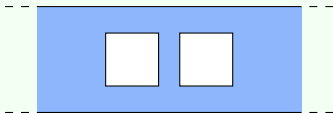
Lecture 2

- ♠ We explained how to construct **small** non reflecting or invisible obstacles by working with **perturbative techniques**.



Lecture 3

- ♠ We wish to obtain **non reflection** or **invisibility** for **large** obstacles by working with **resonant phenomena**.



Outline of lecture 3

- 1 Construction of non reflecting obstacles using Fano resonance
- 2 Cloaking of given obstacles in acoustics using resonant ligaments

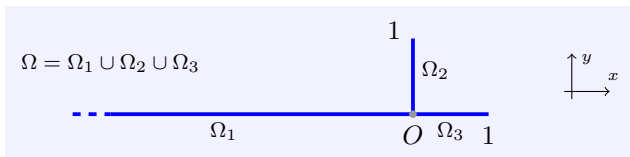
- ① Construction of non reflecting obstacles using Fano resonance
 - A 1D toy problem
 - The Fano resonance in 2D waveguides
 - Non reflection and complete reflection
 - Numerical experiments

- ② Cloaking of given obstacles in acoustics using resonant ligaments

A 1D toy problem

- Fano resonance phenomenon appears in many fields in physics. First, we illustrate it for a simple 1D problem.

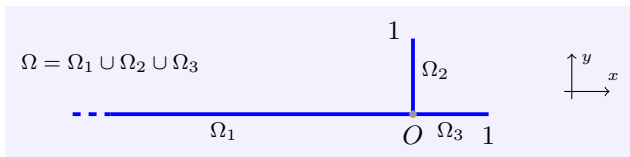
A 1D toy problem



► Consider the **scattering** problem

$$\varphi'' + k^2 \varphi = 0 \text{ in } \Omega, \quad \left\{ \begin{array}{l} \varphi_1 = \varphi_2 = \varphi_3 \text{ at } O \\ \varphi'_1 = \varphi'_2 + \varphi'_3 \text{ at } O \\ \varphi'_2 = \varphi'_3 = 0 \text{ on } \partial\Omega \end{array} \right. \quad \text{with } \underbrace{\varphi_1 = e^{ikx} + R e^{-ikx}}_{\text{radiation condition}}, \quad R \in \mathbb{C}.$$

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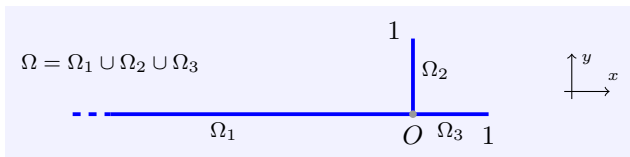


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- **Well-posedness** $\Leftrightarrow$ invertibility of a 3×3 system $\mathbb{M}\Phi = F$.
- **Uniqueness** $\Leftrightarrow k \notin (2\mathbb{N} + 1)\pi/2$. **Existence** for all $k \in \mathbb{R}$ ($F \in \ker {}^t\mathbb{M}^\perp$)

$$R = \frac{\cos(k) + 2i \sin(k)}{\cos(k) - 2i \sin(k)}.$$

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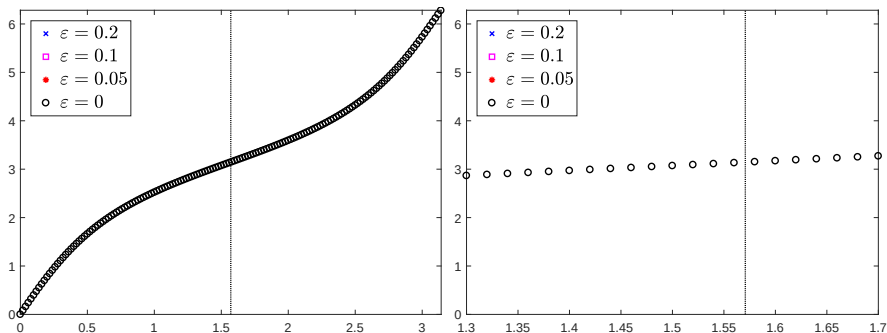


Figure: $k \mapsto \theta^\varepsilon(k)$ for several ε (non uniqueness for $\varepsilon = 0$, $k = \pi/2$).

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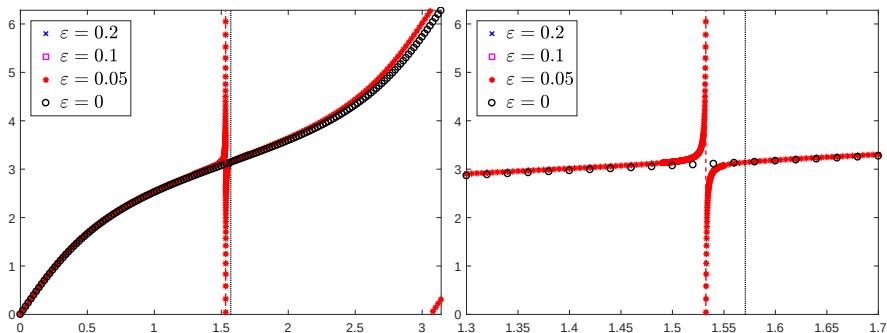


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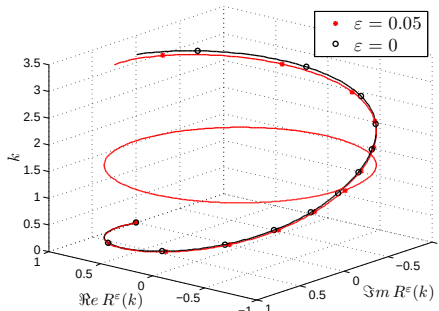
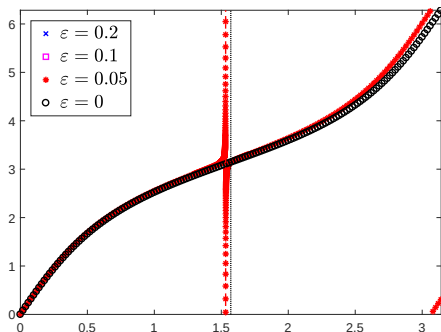


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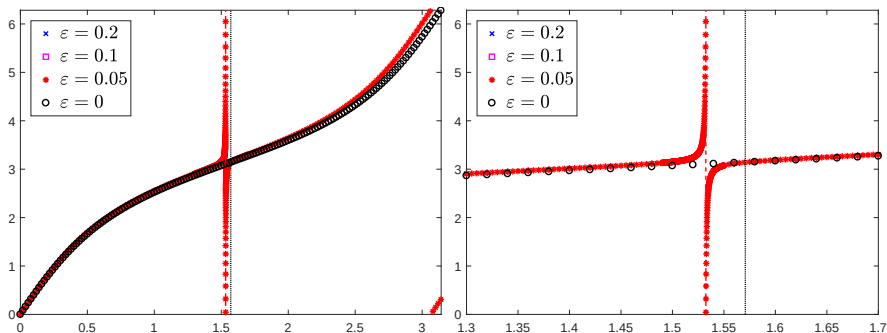


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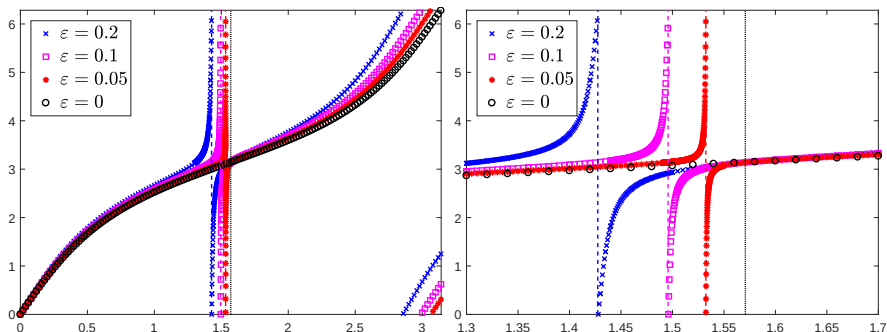
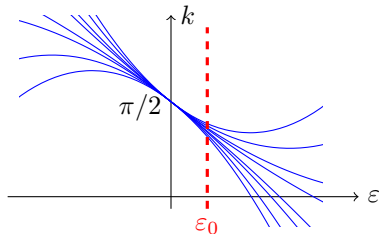
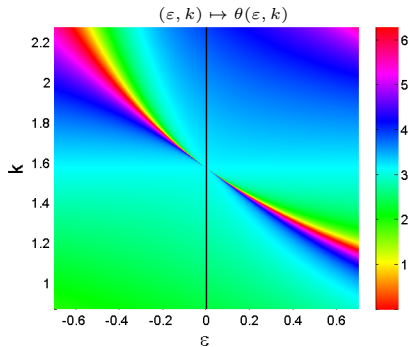


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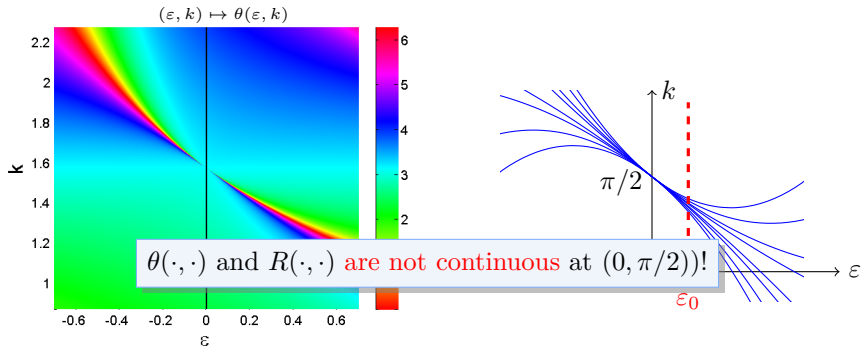
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- Set $R(\varepsilon, k) = e^{i\theta(\varepsilon, k)}$ (functions of two variables).



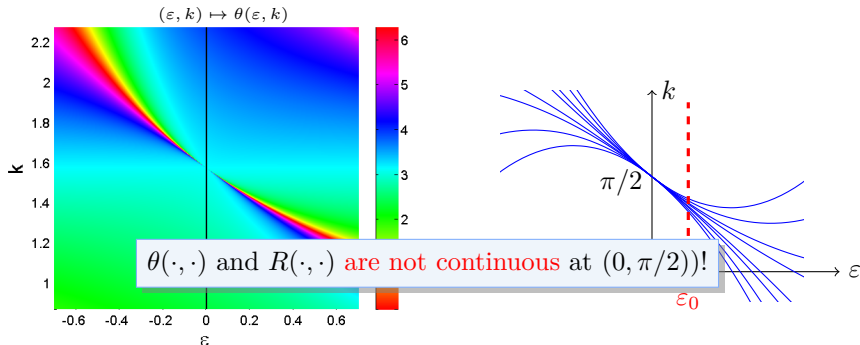
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Next steps

- 1) Prove a similar **Fano resonance** phenomenon for a **2D waveguide**.
- 2) Use it to provide examples of **non reflection** and **complete reflection**.

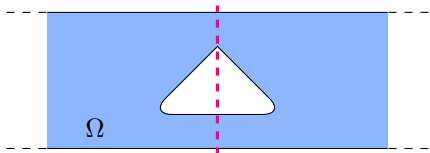
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Setting

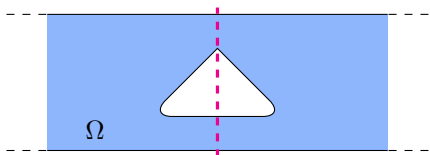
- Scattering in **time-harmonic** regime in a **symmetric** (to simplify) acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



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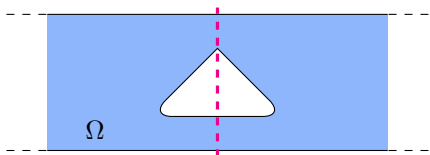
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- We assume that **trapped modes** exist for $\lambda = \lambda^0 \in (0; \pi^2)$:

$u_{\text{tr}} \in H^1(\Omega) \setminus \{0\}$ satisfies $(*)$ for $\lambda = \lambda^0$ (**non uniqueness**).

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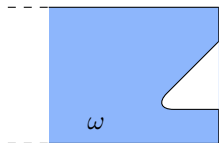


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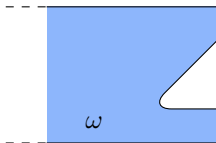
- Due to **symmetry**, u_{tr} is also a trapped mode for the **half waveguide pb.**



$$\left\{ \begin{array}{ll} \Delta v + \lambda v &= 0 \quad \text{in } \omega, \\ \partial_n v &= 0 \quad \text{on } \partial\omega \cap \partial\Omega, \\ \text{ABC}(v) = v/\partial_n v &= 0 \quad \text{on } \partial\omega \setminus \partial\Omega. \end{array} \right.$$

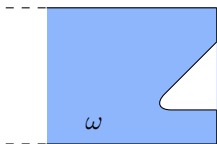
(depends on the sym.)

Scattering problem in the half waveguide



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$$w_{\pm}(x, y) = e^{\pm i k x}.$$

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$$v = w_+ + R w_- + \tilde{v},$$

where $R \in \mathbb{C}$ and $\tilde{v}$ is expo. decaying (**uniqueness** $\Leftrightarrow$ abs. of trapped modes).

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- R is **uniquely defined** (even for $\lambda = \lambda^0$) and $|R| = 1$ (cons. of energy).

Small perturbation of the geometry

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry



Locally $\partial\omega^\varepsilon$ coincides with the graph of $x \mapsto 1 + \varepsilon H(x)$,
where $H \in \mathcal{C}_0^\infty(\mathbb{R})$ is a given **profile function**.

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→ One proves that R is **not continuous** at $(0, \lambda^0)$ (one approach: work with the **augmented scattering matrix** which is **continuous** at $(0, \lambda^0)$).

The Fano resonance

PROPOSITION: There is $\lambda'_p > 0$ such that

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Here a, b, c are some constants that one can characterize.

→ When $\mu \in \mathbb{R}$, the quantity $R + \frac{a}{ib\mu - c}$ runs on the whole unit circle.

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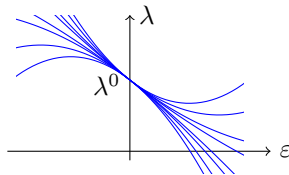
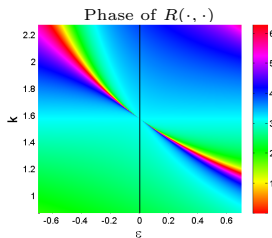
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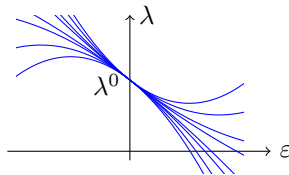
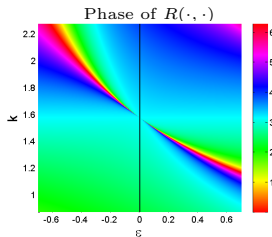
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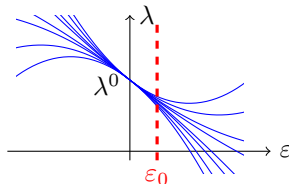
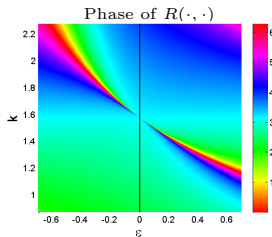
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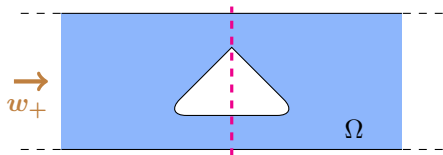
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Exploiting symmetry

- We come back to the problem in the **total waveguide** Ω



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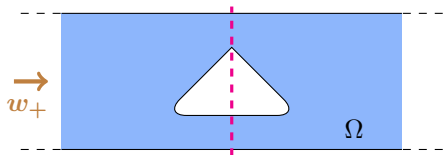
- $(*)$ admits the solution

$$v = \begin{cases} e^{ikx} + R e^{-ikx} + \tilde{v}, & x < 0 & \text{(reflection)} \\ T e^{-ikx} + \tilde{v}, & x > 0 & \text{(transmission)} \end{cases}$$

with $R, T \in \mathbb{C}$ and $\tilde{v} \in H^1(\Omega)$. We have $|R|^2 + |T|^2 = 1$.

Exploiting symmetry

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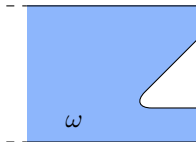
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- Introduce the two **half-waveguide** problems



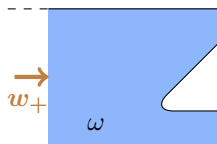
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$$\left\{ \begin{array}{ll} \Delta U + \lambda U = 0 & \text{in } \omega \\ \partial_n U = 0 & \text{on } \partial\omega \setminus \partial\Omega \\ U = 0 & \text{on } \partial\omega \cap \partial\Omega. \end{array} \right.$$

Exploiting symmetry

- Half-waveguide problems admit the solutions

$$\begin{aligned} u &= e^{ikx} + R_N e^{-ikx} + \tilde{u}, & \text{with } \tilde{u} \in H^1(\omega) \\ U &= e^{ikx} + R_D e^{-ikx} + \tilde{U}, & \text{with } \tilde{U} \in H^1(\omega). \end{aligned}$$



Exploiting symmetry

- Half-waveguide problems admit the solutions

$$\begin{aligned} u &= e^{ikx} + \textcolor{red}{R}_N e^{-ikx} + \tilde{u}, & \text{with } \tilde{u} \in H^1(\omega) \\ U &= e^{ikx} + \textcolor{green}{R}_D e^{-ikx} + \tilde{U}, & \text{with } \tilde{U} \in H^1(\omega). \end{aligned}$$



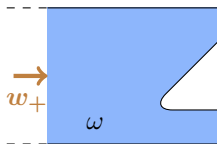
- Due to conservation of energy, one has

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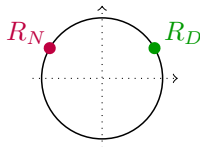
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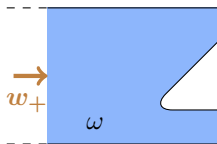
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Exploiting symmetry

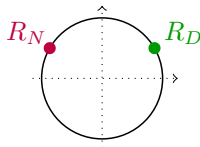
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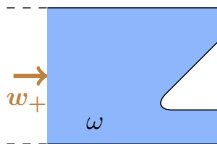
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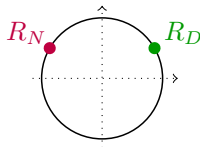
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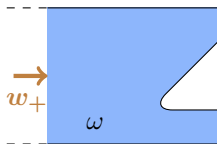
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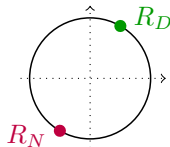
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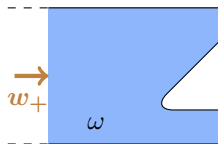
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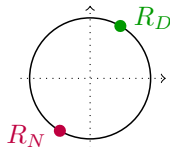
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Non reflection $R = 0$

$$\Leftrightarrow R_N = -R_D$$

Perfect reflection $T = 0$

$$\Leftrightarrow R_N = R_D$$

Non reflection and perfect reflection

$$R = \frac{R_N + R_D}{2}$$

$$T = \frac{R_N - R_D}{2}$$

- To set ideas, we assume that u_{tr} is **symmetric** w.r.t. (Oy) .
 $\Rightarrow u_{\text{tr}}$ is a trapped mode for the pb with **Neumann** B.Cs.

i) No trapped modes for the **Dirichlet** pb at $\lambda = \lambda^0$. This implies

$$|R_D(\varepsilon, \lambda^0 + \varepsilon \lambda'_p + \varepsilon^2 \mu) - R_D(0, \lambda^0)| \leq C \varepsilon, \quad \forall \varepsilon \in (0; \varepsilon_0], \mu \in [-c\varepsilon^{-1}; c\varepsilon].$$

ii) $\mu \mapsto R_N(\varepsilon, \lambda^0 + \varepsilon \lambda'_p + \varepsilon^2 \mu)$ **rushes on the unit circle** for $\mu \in [-c\varepsilon^{-1}; c\varepsilon]$.

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PROPOSITION:

- $\exists \lambda_\varepsilon$, with $\lambda_\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $R(\varepsilon, \lambda_\varepsilon) = 0$ (**non reflection**).
- $\exists \tilde{\lambda}_\varepsilon$, with $\tilde{\lambda}_\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $T(\varepsilon, \tilde{\lambda}_\varepsilon) = 0$ (**perfect reflection**).

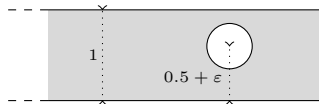
1 Construction of non reflecting obstacles using Fano resonance

- A 1D toy problem
- The Fano resonance in 2D waveguides
- Non reflection and complete reflection
- Numerical experiments

2 Cloaking of given obstacles in acoustics using resonant ligaments

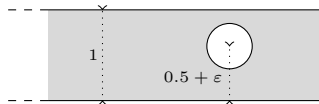
The Fano resonance

- ▶ Numerics using **FE methods** (Freefem++) with **DtN maps** or **PMLs**.
- ▶ Left: domain ω^ε . Right: u_{tr} (trapped mode) for $\varepsilon = 0$.



The Fano resonance

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- Since $|R^\varepsilon| = 1$ (conservation of energy), $\exists \theta^\varepsilon \in]-\pi; \pi]$ s.t. $R^\varepsilon = e^{i\theta^\varepsilon}$.

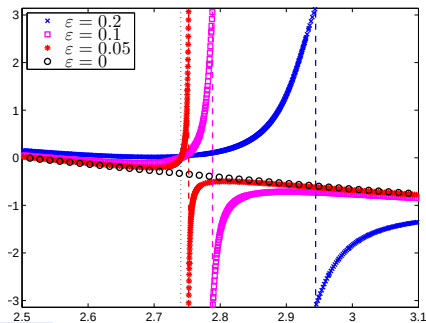
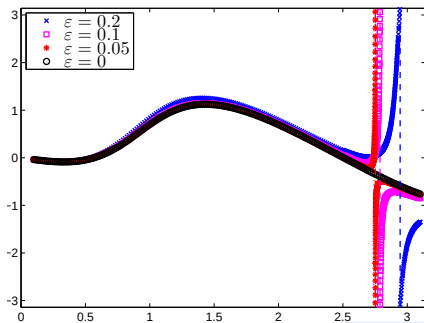
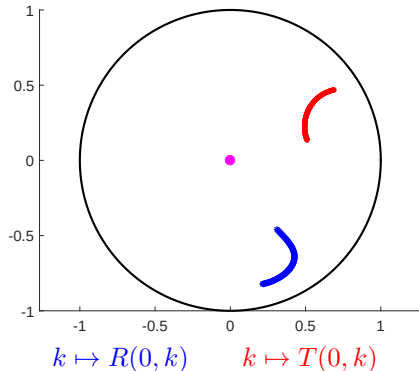


Figure: $k \mapsto \theta^\varepsilon(k)$ for several ε (non uniqueness for $\varepsilon = 0$, $k = 2.7403$).

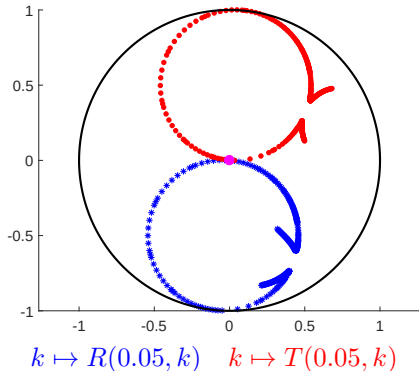
Non reflection/perfect reflection

- Scattering coefficients for $k \in (2.5; 3.1)$.

No shift ($\varepsilon = 0$)



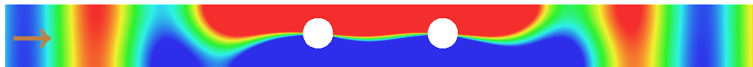
Small shift ($\varepsilon > 0$)



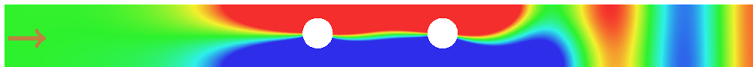
Non reflection/perfect reflection

- ▶ Example of setting where $R(\varepsilon, \lambda^\varepsilon) = 0$ (non reflection).

$\Re v$

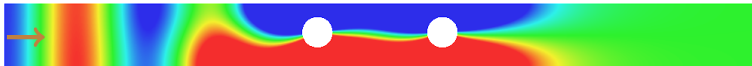


$\Re(v - v_i)$



- ▶ Example of setting where $T(\varepsilon, \lambda^\varepsilon) = 0$ (perfect reflection).

$\Re v$



Frequency behaviour

No shift ($\varepsilon = 0$)

Small shift ($\varepsilon > 0$)

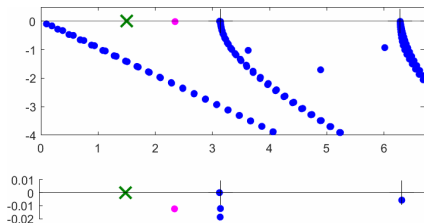
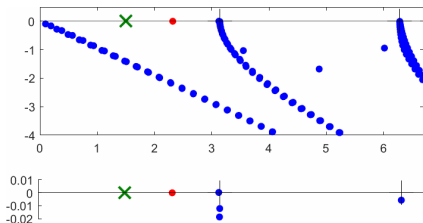
► $k \mapsto \Re v(k)$



► Complex spectrum computed with PMLs (we zoom at the real axis).

• Trapped mode

• Complex resonance



What we did

- ♠ We illustrated the **Fano resonance phenomenon** in a 2D waveguide.
*If trapped modes exist for $(\varepsilon, \lambda) = (0, \lambda^0)$, then for $\varepsilon > 0$ small, $\lambda \mapsto R(\varepsilon, \lambda)$ has a **quick variation** at λ^0 . **Symmetry is not needed**.*
- ♠ We use it to show examples of **non reflection** and **perfect reflection**.
Symmetry is essential.
- ♠ The phenomenon appears with **other B.C.** (Dirichlet, ...), **other kinds of perturbation** (penetrable obstacles, ...), in **any dimension**.

Comments

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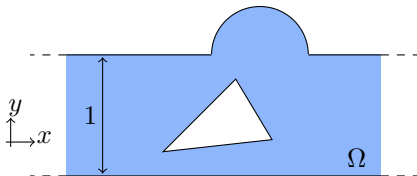
Other directions

- 1) **Without symmetry**, one can show that T still passes through zero.
- 2) Is there **non reflection/perfect reflection** for $k > \pi$ (monomode regime was essential in the mechanism)?
- 3) What happens if λ^0 is **not a simple** eigenvalue?

- 1 Construction of non reflecting obstacles using Fano resonance
- 2 Cloaking of given obstacles in acoustics using resonant ligaments

Setting

- We consider the **propagation of waves** in a 2D **acoustic** waveguide with an obstacle (also relevant in optics, microwaves, water-waves theory,...).

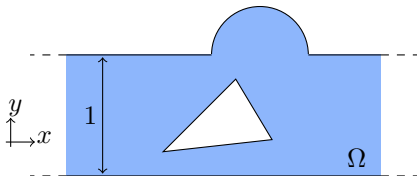


$$(\mathcal{P}) \left| \begin{array}{ll} \Delta u + k^2 u & = 0 \quad \text{in } \Omega, \\ \partial_n u & = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

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- We fix $k \in (0; \pi)$ so that only the plane waves $e^{\pm ikx}$ can propagate.
- The scattering of these waves leads us to consider the solutions of $(\mathcal{P})$ with the decomposition

$$u_+ = \begin{cases} e^{ikx} + R_+ e^{-ikx} + \dots \\ T e^{+ikx} + \dots \end{cases} \quad u_- = \begin{cases} T e^{-ikx} + \dots & x \rightarrow -\infty \\ e^{-ikx} + R_- e^{+ikx} + \dots & x \rightarrow +\infty \end{cases}$$

$R_{\pm}, T \in \mathbb{C}$ are the **scattering coefficients**, the ... are expon. decaying terms.

Goal

We wish to slightly **perturb the walls** of the guide to obtain $R_{\pm} = 0$, $T = 1$ in the new geometry (as if there were no obstacle) $\Rightarrow$ **cloaking at “infinity”**.

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Difficulty: the scattering coefficients have a **not explicit** and **not linear** dependence wrt the geometry.

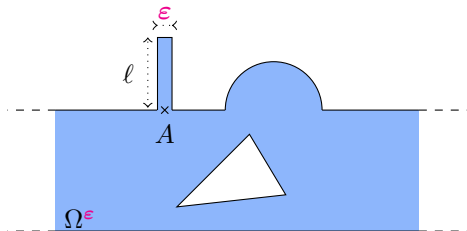
We wish to cloak **big obstacles** and **not only small perturbations**.

- 1 Construction of non reflecting obstacles using Fano resonance
- 2 Cloaking of given obstacles in acoustics using resonant ligaments
 - Asymptotic analysis in presence of thin resonators
 - Almost zero reflection
 - Cloaking

Setting



Main ingredient of our approach: **outer resonators** of width $\epsilon \ll 1$.



$$(\mathcal{P}^\epsilon) \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega^\epsilon, \\ \partial_n u = 0 & \text{on } \partial\Omega^\epsilon \end{cases}$$

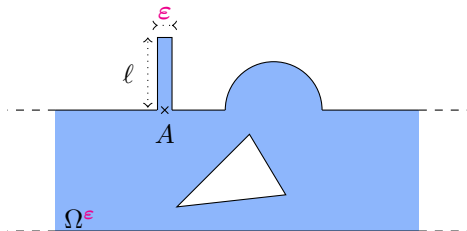
► In this geometry, we have the scattering solutions

$$u_+^\epsilon = \begin{cases} e^{ikx} + R_+^\epsilon e^{-ikx} + \dots \\ T^\epsilon e^{+ikx} + \dots \end{cases} \quad u_-^\epsilon = \begin{cases} T^\epsilon e^{-ikx} + \dots & x \rightarrow -\infty \\ e^{-ikx} + R_-^\epsilon e^{+ikx} + \dots & x \rightarrow +\infty \end{cases}$$

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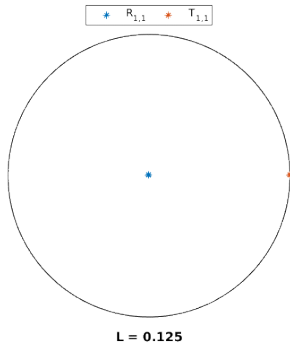
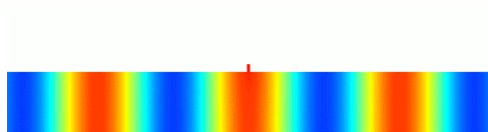
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In general, the thin ligament has only a **weak influence** on the scattering coefficients: $R_\pm^\epsilon \approx R_\pm$, $T^\epsilon \approx T$. But **not always** ...

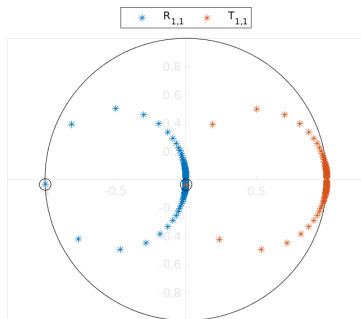
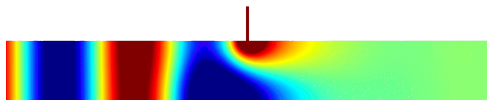
Numerical experiment

- We vary the length of the ligament:



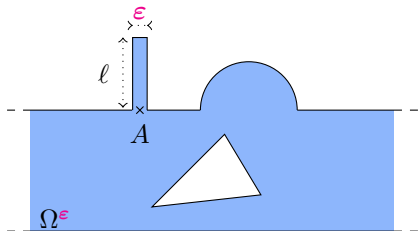
Numerical experiment

- For one particular length of the ligament, we get a **standing mode** (zero transmission):



Asymptotic analysis

To understand the phenomenon, we compute an **asymptotic expansion** of u_+^ε , R_+^ε , T^ε as $\varepsilon \rightarrow 0$.



$$(\mathcal{P}^\varepsilon) \left| \begin{array}{ll} \Delta u_+^\varepsilon + k^2 u_+^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u_+^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{array} \right.$$

$$u_+^\varepsilon = \left| \begin{array}{l} e^{ikx} + R_+^\varepsilon e^{-ikx} + \dots \\ T^\varepsilon e^{+ikx} + \dots \end{array} \right.$$

► To proceed we use techniques of **matched asymptotic expansions** (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin & Zhang 17, 18, Brandao, Holley, Schnitzer 20, ...).

Asymptotic analysis

- We work with the **outer expansions**

$$u_+^\varepsilon(x, y) = u^0(x, y) + \dots \quad \text{in } \Omega,$$

$$u_+^\varepsilon(x, y) = \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots \quad \text{in the resonator.}$$

- Considering the restriction of $(\mathcal{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous **1D** problem

$$(\mathcal{P}_{1D}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1 + \ell) \\ v(1) = \partial_y v(1 + \ell) = 0. \end{array} \right.$$

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The features of $(\mathcal{P}_{1D})$ play a key role in the **physical phenomena** and in the **asymptotic analysis**.

Asymptotic analysis

- ▶ We work with the **outer expansions**

$$\begin{aligned}u_+^\varepsilon(x, y) &= u^0(x, y) + \dots && \text{in } \Omega, \\u_+^\varepsilon(x, y) &= \varepsilon^{-1}v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.}\end{aligned}$$

- ▶ Considering the restriction of $(\mathcal{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous **1D** problem

$$(\mathcal{P}_{1D}) \quad \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1 + \ell) \\ v(1) = \partial_y v(1 + \ell) = 0. \end{array} \right.$$



The features of $(\mathcal{P}_{1D})$ play a key role in the **physical phenomena** and in the **asymptotic analysis**.

- ▶ We denote by ℓ_{res} (**resonance lengths**) the values of ℓ , given by

$$\ell_{\text{res}} := \pi(m + 1/2)/k, \quad m \in \mathbb{N},$$

such that $(\mathcal{P}_{1D})$ admits the **non zero** solution $v(y) = \sin(k(y - 1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \rightarrow 0$, we get

$$u_{\pm}^{\varepsilon}(x, y) = u_{\pm} + o(1) \quad \text{in } \Omega,$$

$$u_{\pm}^{\varepsilon}(x, y) = u_{\pm}(A) v_0(y) + o(1) \quad \text{in the resonator,}$$

$$R_{\pm}^{\varepsilon} = R_{\pm} + o(1), \quad T^{\varepsilon} = T + o(1).$$

Here $v_0(y) = \cos(k(y-1)) + \tan(k(y-\ell)) \sin(k(y-1))$.

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The thin resonator has no influence at order ε^0 .

→ **Not interesting for our purpose** because we want $\left| \begin{array}{l} R_{\pm}^{\varepsilon} = 0 + \dots \\ T^{\varepsilon} = 1 + \dots \end{array} \right.$

Asymptotic analysis – Resonant case

► Now assume that $\ell = \ell_{\text{res}}$. Then we find $v^{-1}(y) = a \sin(k(y-1))$ for some a to determine.

Asymptotic analysis – Resonant case

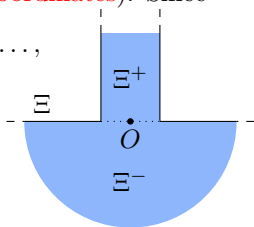
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► Inner expansion. Set $\xi = \varepsilon^{-1}(x - A)$ (stretched coordinates). Since

$$(\Delta_x + k^2)u_+^\varepsilon(\varepsilon^{-1}(x - A)) = \varepsilon^{-2}\Delta_\xi u^\varepsilon(\xi) + \dots,$$

when $\varepsilon \rightarrow 0$, we are led to study the problem

$$(\star) \quad \left\{ \begin{array}{ll} -\Delta_\xi Y = 0 & \text{in } \Xi \\ \partial_\nu Y = 0 & \text{on } \partial\Xi. \end{array} \right.$$



Asymptotic analysis – Resonant case

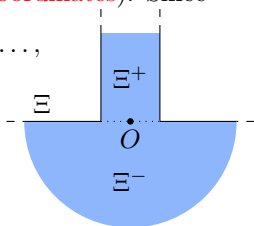
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► Problem $(\star)$ admits a solution Y^1 (up to a constant) with the expansion

$$Y^1(\xi) = \begin{cases} \xi_y + C_\Xi + O(e^{-\pi\xi_y}) & \text{as } \xi_y \rightarrow +\infty, \quad \xi \in \Xi^+ \\ \frac{1}{\pi} \ln \frac{1}{|\xi|} + O\left(\frac{1}{|\xi|}\right) & \text{as } |\xi| \rightarrow +\infty, \quad \xi \in \Xi^-. \end{cases}$$

Asymptotic analysis – Resonant case

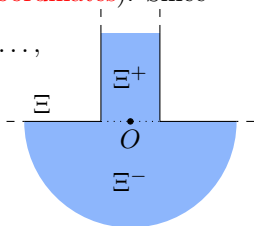
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$$u_+^\varepsilon(x) = C^A Y^1(\xi) + c^A + \dots \quad (c^A, C^A \text{ constants to determine}).$$

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Asymptotic analysis – Resonant case

- In the ansatz $u_+^\varepsilon = u^0 + \dots$ in Ω , we deduce that we must take

$$u_0 = u_+ + ak\gamma$$

where γ is the outgoing **Green function** such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega. \end{array} \right.$

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- ▶ Matching the **constant** behaviour in the resonator, we obtain

$$v^0(1) = u_+(A) + ak(\Gamma + \pi^{-1} \ln |\varepsilon| + C_\Xi).$$

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- ▶ This is a Fredholm problem with a non zero **kernel**. A solution exists iff the **compatibility condition** is satisfied. This sets

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1} \ln |\varepsilon| + C_\Xi}$$

and **ends the calculus of the first terms.**

Asymptotic analysis – Resonant case

► Finally for $\ell = \ell_{\text{res}}$, when $\varepsilon \rightarrow 0$, we obtain

$$u_+^\varepsilon(x, y) = u_+(x, y) + ak\gamma(x, y) + o(1) \quad \text{in } \Omega,$$

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This time the thin resonator **has an influence at order ε^0**

Asymptotic analysis – Resonant case

► Similarly for $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, by modifying only the last step with the compatibility relation, when $\varepsilon \rightarrow 0$, we obtain

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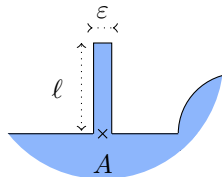
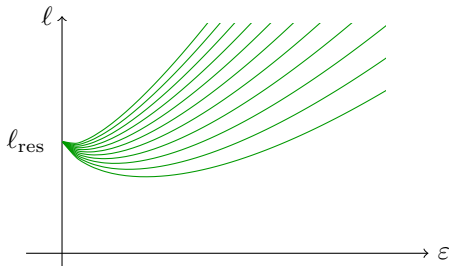


This time the thin resonator **has an influence at order ε^0** and it depends on the choice of η !

Asymptotic analysis – Resonant case

- Below, for several $\eta \in \mathbb{R}$, we display the paths

$$\{(\varepsilon, \ell_{\text{res}} + \varepsilon(\eta - \pi^{-1}|\ln \varepsilon|)), \varepsilon > 0\} \subset \mathbb{R}^2.$$

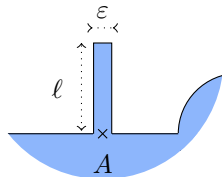
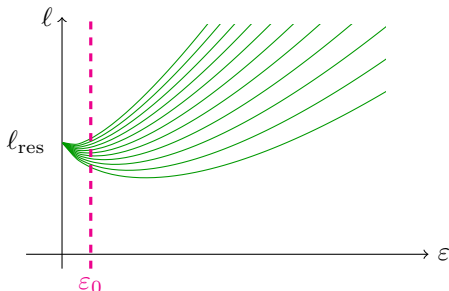


According to η , the limit of the scattering coefficients along the path as $\varepsilon \rightarrow 0^+$ is **different**.

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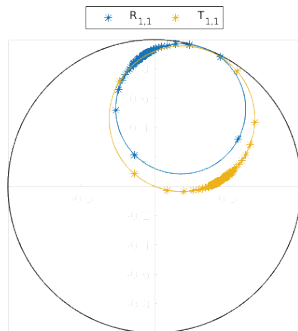
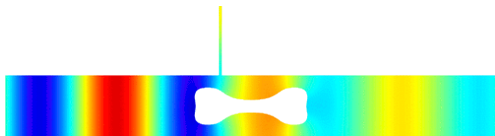
- For a **fixed small** ε_0 , the scattering coefficients have a **rapid variation** for ℓ varying in a neighbourhood of the resonance length.

- 1 Construction of non reflecting obstacles using Fano resonance
- 2 Cloaking of given obstacles in acoustics using resonant ligaments
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 - Almost zero reflection
 - Cloaking

Almost zero reflection



From this expansion, we find that asymptotically, when the length of the resonator is perturbed **around** ℓ_{res} , R_+^ε , T^ε run on **circles** whose **features depend on the choice for A** .

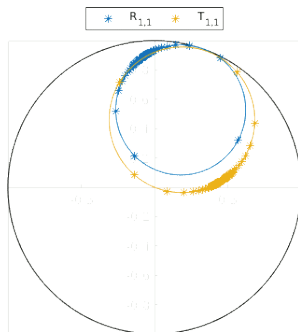
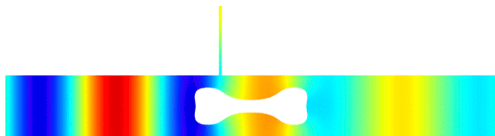


slit[0].x: -0.500000

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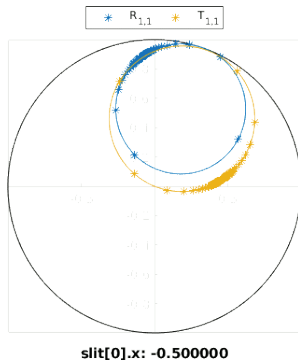
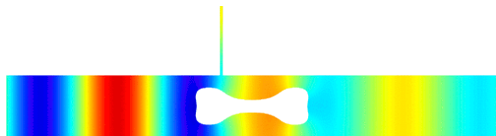
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PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta) \mid \eta \in \mathbb{R}\}$ passes **through zero**.

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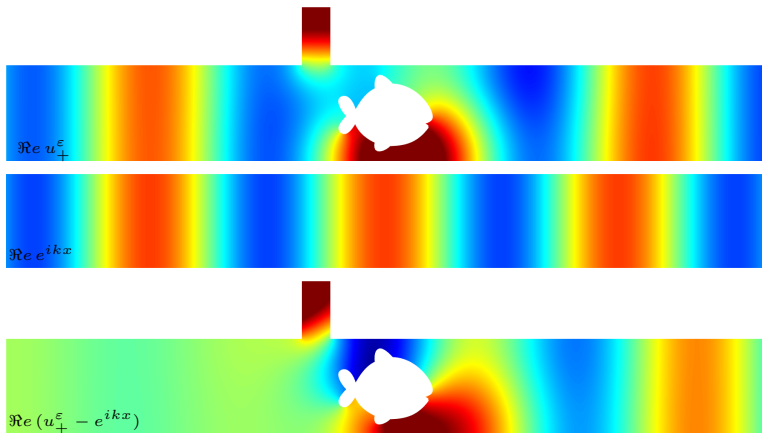


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PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta) \mid \eta \in \mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

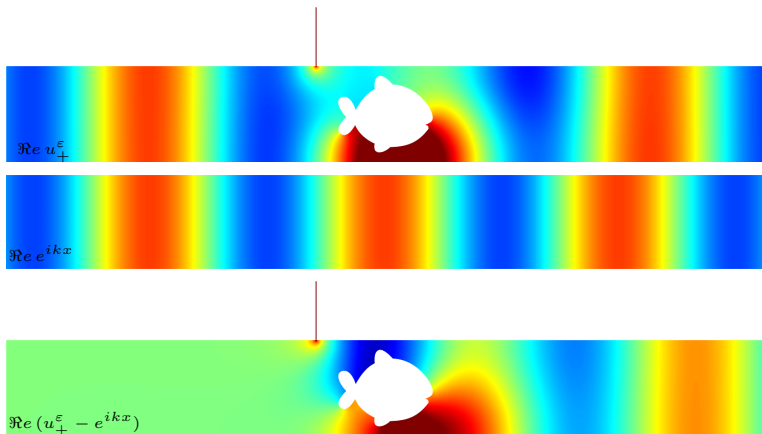
- Example of situation where we have almost zero reflection ($\varepsilon = 0.3$).



→ Simulations realized with the **Freefem++** library.

Almost zero reflection

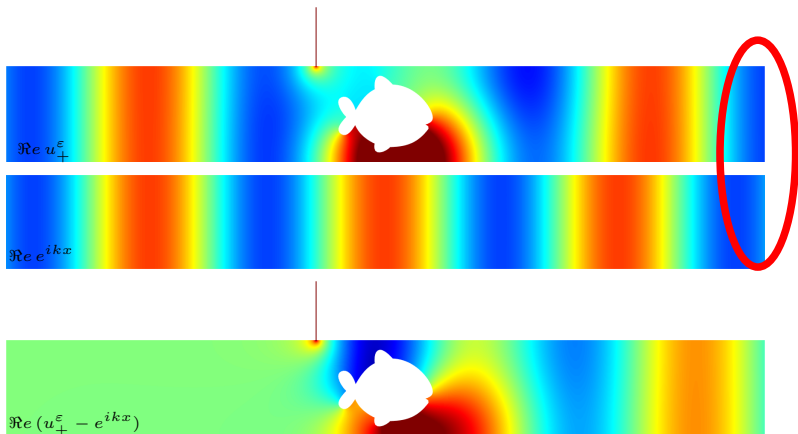
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Almost zero reflection

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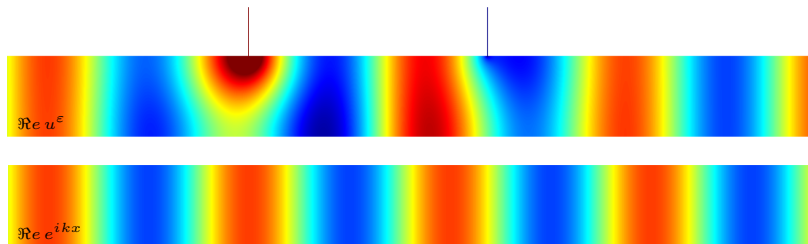
→ Simulations realized with the **Freefem++** library.

To cloak the object, it remains to compensate the **phase shift**!

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Phase shifter

- ▶ Working with **two resonators**, we can create **phase shifters**, that is devices with **almost zero reflection** and any **desired phase**.



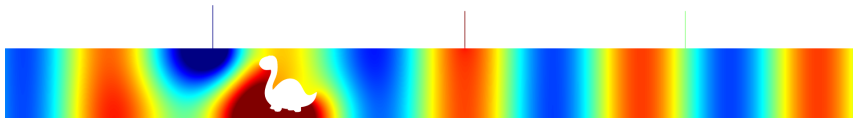
- ▶ Here the device is designed to obtain a **phase shift** approx. equal to $\pi/4$.

Cloaking with three resonators

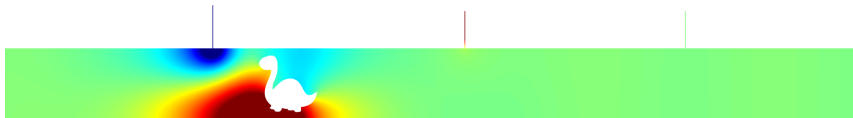
- Gathering the two previous results, we can cloak any object with **three resonators**.



$\Re u_+$



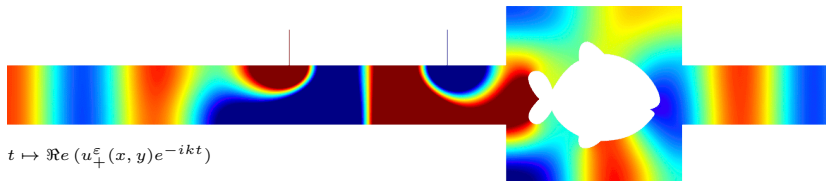
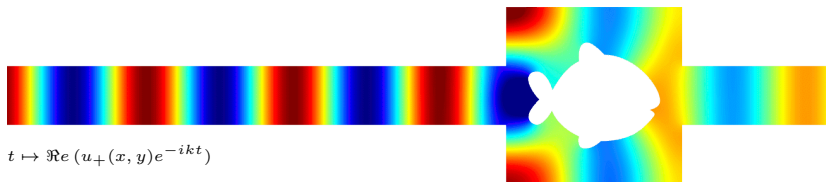
$\Re u_+^\epsilon$



$\Re(u_+^\epsilon - e^{ikx})$

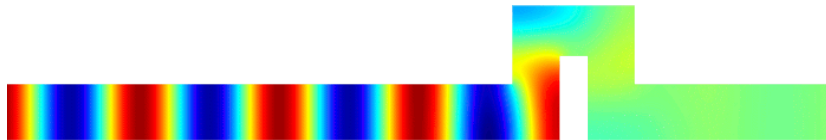
Cloaking with two resonators

- Working a bit more, one can show that **two resonators** are enough to cloak any object.

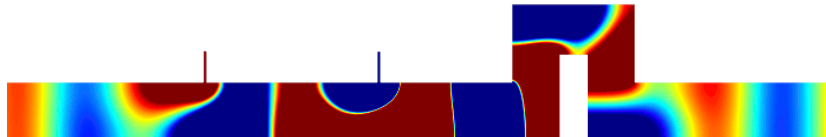


Cloaking with two resonators

- ▶ Another example



$$t \mapsto \Re(u_+(x, y)e^{-ikt})$$



$$t \mapsto \Re(u_+^\varepsilon(x, y)e^{-ikt})$$



$$t \mapsto \Re(e^{ik(x-t)})$$

Recap of the cloaking strategy

What we did

- ♠ We explained how to **approximately cloak** any object in **monomode regime** using **thin resonators**. Two main ingredients:
 - Around **resonant lengths**, effects of **order ε^0** with perturb. of **width ε** .
 - The **1D limit problems** in the resonator provide a rather **explicit** dependence wrt to the geometry.

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Possible extensions and open questions

- 1) We can similarly hide **penetrable obstacles** or work in **3D**.
- 2) We can do cloaking at a **finite number** of wavenumbers (thin structures are **resonant at one wavenumber** otherwise act at order ε).
- 3) With **Dirichlet BCs**, other ideas must be found.
- 4) Can we realize **exact cloaking** ($T = 1$ exactly)? This question is also related to **robustness** of the device.

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► Part I



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► Part II



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