

A few techniques to achieve invisibility in waveguides

Lecture 2: Invisible perturbations of the reference geometry

Lucas Chesnel

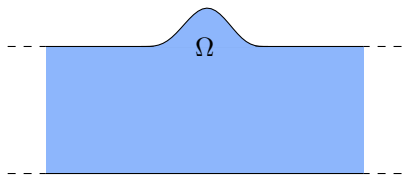
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Waveguide problem

- Scattering in **time-harmonic** regime of a **plane wave** in the **acoustic** waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.

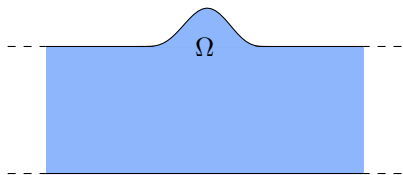


Find $u = u_i + u_s$ s. t.

$$\begin{aligned}\Delta u + k^2 u &= 0 && \text{in } \Omega, \\ \partial_n u &= 0 && \text{on } \partial\Omega, \\ u_s &\text{ is outgoing.}\end{aligned}$$

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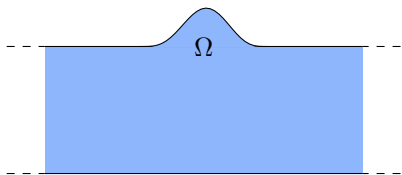
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- For this problem, the **modes** are

Propagating		$w_n^\pm(x, y) = e^{\pm i\beta_n x} \cos(n\pi y), \beta_n = \sqrt{k^2 - n^2\pi^2}, n \in \llbracket 0, N-1 \rrbracket$
Evanescent		$w_n^\pm(x, y) = e^{\mp \beta_n x} \cos(n\pi y), \beta_n = \sqrt{n^2\pi^2 - k^2}, n \geq N.$

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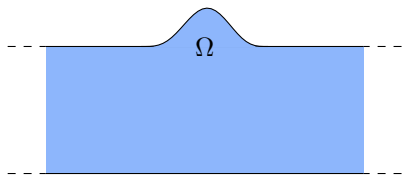


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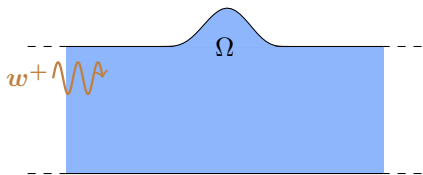
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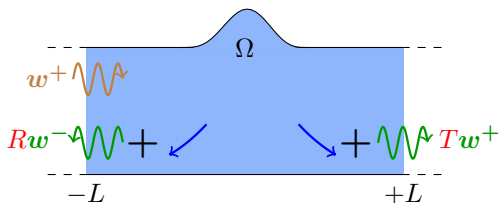
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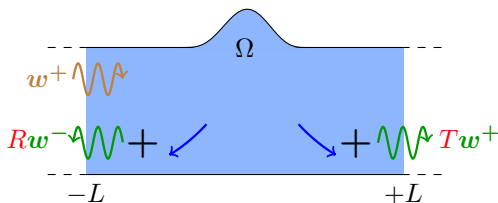
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- We have

$$u = \begin{cases} w_+ + R w_- + \dots & \text{for } x \leq -L \\ T w_+ + \dots & \text{for } x \geq +L \end{cases}$$

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DEFINITION: $R, T \in \mathbb{C}$ are the **reflection** and **transmission** coefficients.

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- ▶ At infinity, one measures only R and/or T (other terms are too small).
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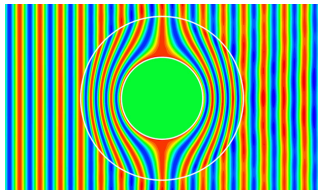
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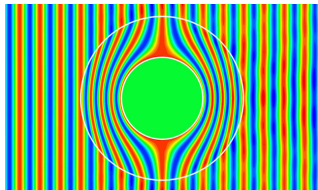
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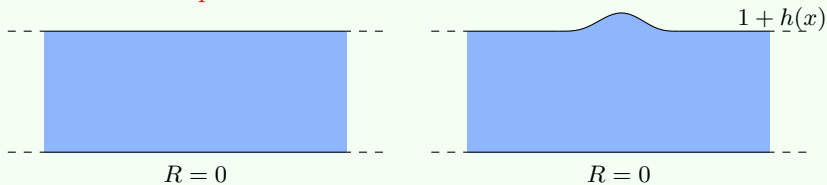


GOAL

We explain how to use **perturbative techniques** to construct geometries such that $R = 0$ or $T = 1$.

General picture

- **Perturbative** technique: we construct small non reflecting defects using variants of the **implicit functions theorem**.



Outline of lecture 2

- 1 A few notions of asymptotic analysis
- 2 Invisible smooth perturbations of the reference geometry
- 3 Non smooth invisible perturbations of the reference geometry

1 A few notions of asymptotic analysis

- Perturbation in the equation
- Smooth perturbation of the domain

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Perturbation of the Poisson's problem

- We study a **first simple example** with a **perturbation in the equation**. For Ω a bounded Lipschitz domain and $f \in L^2(\Omega)$, consider the problem

$$(\mathcal{P}_\varepsilon) \quad \left| \begin{array}{rcl} -\Delta u_\varepsilon + \varepsilon u_\varepsilon & = & f \quad \text{in } \Omega \\ u_\varepsilon & = & 0 \quad \text{on } \partial\Omega. \end{array} \right.$$

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GENERAL PROCEDURE:

Step I: we propose an **expansion** (ansatz) and identify the terms of this expansion.

Step II: we prove **error estimates**.

Step I - ansatz

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► Consider the ansatz

$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

where the terms $u_0, u_1, u_2, \dots$ have to be determined.

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- Each of these problems admits a unique solution in $H_0^1(\Omega)$.
→ This defines the expansion.

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$$\int_{\Omega} |\nabla u_{\varepsilon}|^2 + \varepsilon |u_{\varepsilon}|^2 dx = \int_{\Omega} f u_{\varepsilon} dx.$$

From the Poincaré inequality

$$\|\varphi\|_{L^2(\Omega)} \leq C_P \|\varphi\|_{H_0^1(\Omega)} := \|\nabla \varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega),$$

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*“The solution of $(\mathcal{P}_{\varepsilon})$ is controlled **uniformly** (C_P is independent of ε , f) by the source term.”*

2) Consistency results. Set $\hat{u}_\varepsilon := \sum_{n=0}^N \varepsilon^n u_n \in H_0^1(\Omega)$.

Inserting the **error** $u_\varepsilon - \hat{u}_\varepsilon$ in $(\mathcal{P}_\varepsilon)$, we obtain the **discrepancy**

$$(-\Delta + \varepsilon)(u_\varepsilon - \hat{u}_\varepsilon) = f - \left(-\sum_{n=0}^N \varepsilon^n \Delta u_n + \sum_{n=1}^{N+1} \varepsilon^n u_{n-1} \right) = -\varepsilon^{N+1} u_N.$$

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Using this **consistency result** in the **stability estimate** (*), we find

$$\|u_\varepsilon - \hat{u}_\varepsilon\|_{H_0^1(\Omega)} \leq C_P \varepsilon^{N+1} \|u_N\|_{L^2(\Omega)}.$$

Noting that $\|u_N\|_{L^2(\Omega)} \leq C_P \|u_N\|_{H_0^1(\Omega)} \leq C_P^3 \|u_{N-1}\|_{H_0^1(\Omega)}$, finally we get:

PROPOSITION: We have the error estimate

$$\|u_\varepsilon - \hat{u}_\varepsilon\|_{H_0^1(\Omega)} \leq C_P^{2N+2} \varepsilon^{N+1} \|f\|_{L^2(\Omega)}.$$

1 A few notions of asymptotic analysis

- Perturbation in the equation
- Smooth perturbation of the domain

2 Invisible smooth perturbations of the reference geometry

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Smooth perturbation of the domain

- ▶ We **perturb slightly** ($\varepsilon \geq 0$ is **small**) the geometry



Locally $\partial\Omega_\varepsilon$ coincides with the graph of $x \mapsto \varepsilon h(x)$,
where $h \in \mathcal{C}_0^\infty(-1; 1)$ is a given **profile function**.

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What is the dependence of u_ε with respect to ε ?

→ This question has been extensively studied in **shape optimization**.

A first formal approach

► Let $\mathcal{O}$ be a **fixed** neighbourhood of the perturbation. To simplify, we assume that $f \in L^2(\Omega_\varepsilon)$ is zero in $\mathcal{O}$. In Ω_0 , we consider the ansatz

$$u_\varepsilon = u_0 + \varepsilon u_1 + \dots$$

where the terms u_0, u_1 **have to be determined**.

► Observing that at the limit $\varepsilon \rightarrow 0$, Ω_ε converges to Ω_0 , we get

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→ Let us see how to justify this **formal** calculus.

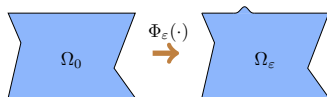


To establish error estimates, we consider a change of variables to work in a **fixed geometry**.

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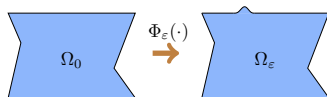


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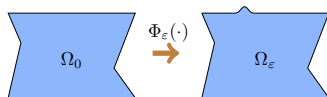


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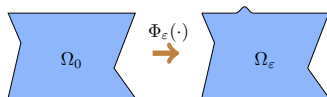


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- ▶ Observe that we have $\Phi_\varepsilon|_{\Omega_0 \setminus \overline{\mathcal{O}}} = \text{Id}$.

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- Thus we obtain the problem

$$\left| \begin{array}{l} \text{Find } U_\varepsilon \in H_0^1(\Omega_0) \text{ such that} \\ -\text{div}(\sigma_\varepsilon \nabla U_\varepsilon) = F J_{\Phi_\varepsilon} \text{ in } \Omega_0 \end{array} \right.$$

with $\left| \begin{array}{l} \sigma_\varepsilon := J_{\Phi_\varepsilon} (\text{Id} + \varepsilon(D\phi))^{-1} (\text{Id} + \varepsilon(D\phi)^\top)^{-1} = \text{Id} + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \dots \\ F J_{\Phi_\varepsilon} = F + \varepsilon h \partial_{x_2} \rho F. \end{array} \right.$



Now the **geometry is fixed** and we have a **perturbation in the equation**.

- Considering the expansion

$$U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots,$$

we can prove the following error estimate with C independent of $\varepsilon \in (0; \varepsilon_0]$

$$\|U_\varepsilon - \sum_{n=0}^N \varepsilon^n U_n\|_{H_0^1(\Omega_0)} \leq C \varepsilon^{N+1} \|f\|_{L^2(\Omega_0)}.$$



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- ▶ Using that
$$\begin{cases} U_0 \circ \Phi_\varepsilon^{-1} + \varepsilon U_1 \circ \Phi_\varepsilon^{-1} = U_0 + \varepsilon (U_1 - \nabla U_0 \cdot \phi) + \dots \\ U_0 = u_0, \quad U_1 - \nabla U_0 \cdot \phi = U_1 - h\rho \partial_{x_2} U_0 = u_1, \end{cases}$$

finally we obtain

$$\|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{H^1(\Omega_0 \setminus \mathcal{O})} \leq C \varepsilon^2 \|f\|_{L^2(\Omega_0)}.$$

Comments

► This is only to give a **flavour**. Much more refined results exist in the literature concerning **shape optimization**.



M. Pierre and A. Henrot. **Shape Variation and Optimization. A Geometrical Analysis**. EMS, 2018.



M.C. Delfour and J.P. Zolésio. **Shapes and geometries: metrics, analysis, differential calculus, and optimization**. Society for Industrial and Applied Mathematics, 2011.

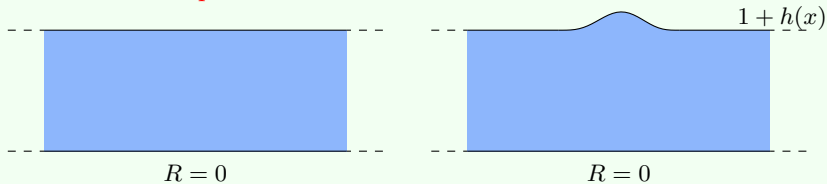
► In particular:

- For this Dirichlet problem, **smoothness assumptions** of the geometry can be considerably relaxed and result exist when Ω_0 is only **measurable**.
- **Higher order terms** can be computed but then **smoothness on f** is required.

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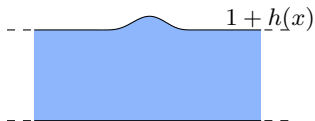
General picture

- **Perturbative** technique: we construct small non reflecting defects using variants of the **implicit functions theorem**.



Sketch of the method

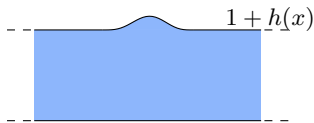
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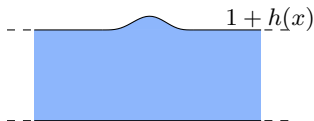
| Note that $R(0) = 0$
(no obstacle leads to null measurements).



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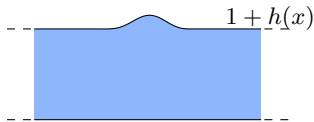
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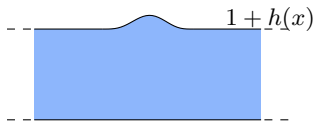


- We look for h of the form $h = \varepsilon \mu$ with $\varepsilon > 0$ small and μ to determine.

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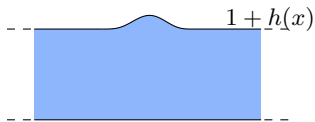


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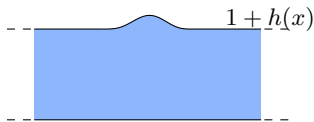
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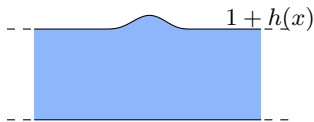
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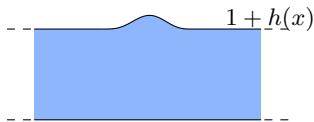
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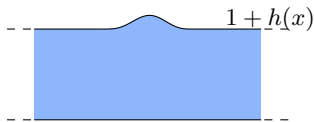
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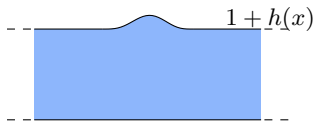
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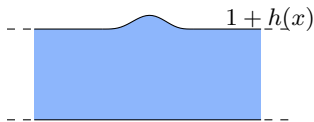
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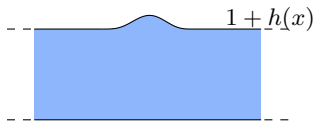
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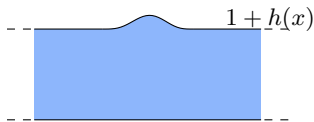
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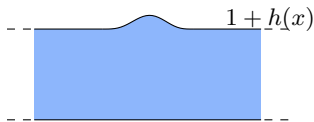
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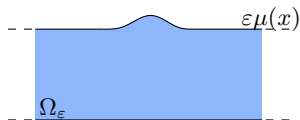
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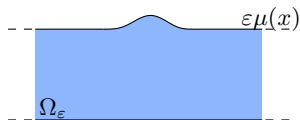
G^ε is a **contraction** $\Rightarrow$ the **fixed-point equation** has a unique solution $\vec{\tau}^{\text{sol}}$.
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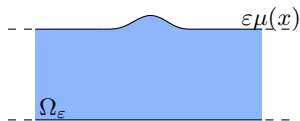


$$(\mathcal{P}_\varepsilon) \left| \begin{array}{lcl} \Delta u_\varepsilon + k^2 u_\varepsilon & = & 0 \quad \text{in } \Omega_\varepsilon \\ u_\varepsilon & = & 0 \quad \text{on } \partial\Omega_\varepsilon \\ u_\varepsilon - w^+ & \text{is outgoing} & \end{array} \right.$$

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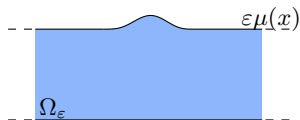
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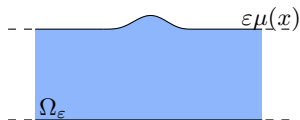
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$\Rightarrow$ Thus we can create geometries Ω_ε where $R_\varepsilon = 0$ for the Dirichlet pb.

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⇒ There exist an **infinite number** of non reflecting geometries.

- ▶ We can show that $|\tau_1^{\text{sol}}| + |\tau_2^{\text{sol}}| = O(\varepsilon)$. Therefore we can **choose** the principal form of the non reflecting perturbation.

- ▶ We can solve the fixed point equation

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- ▶ Pick $\varepsilon > 0$, choose μ_0, μ_1, μ_2 once for all. Set $\vec{\tau}^0 = (0, 0)$ and for $p \in \mathbb{N}$

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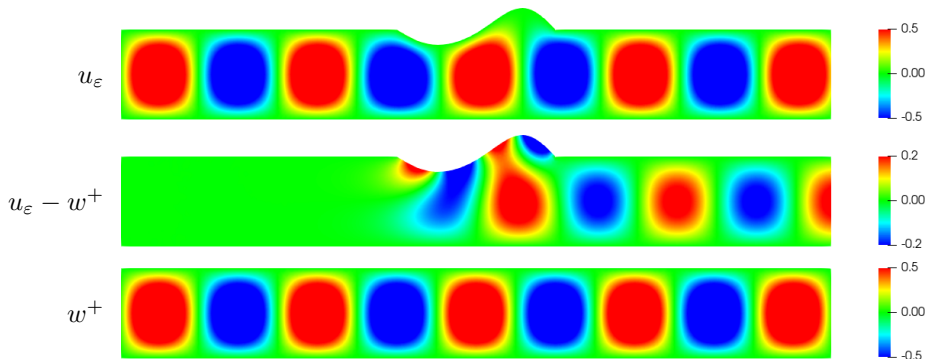
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We have to **mesh** a new domain Ω_p^ε at each step $p \geq 0$.

- An example of **non reflecting** perturbation obtained after 24 iterations ($\varepsilon = 0.2$).



- The algorithm converges though ε is not that small!

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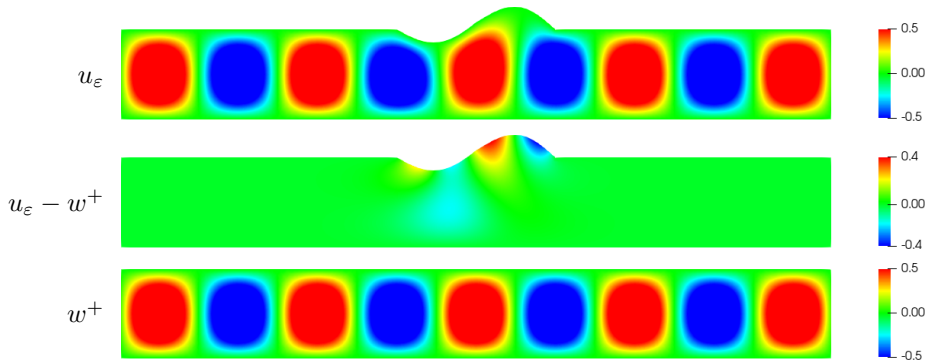
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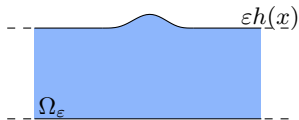
Numerics

- An example of **perfectly invisible** perturbation.



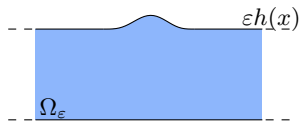
- The scattered field is exponentially decaying **both at $\pm\infty$** and this time there is **no phase shift** for the transmitted field.

- 1 A few notions of asymptotic analysis
- 2 Invisible smooth perturbations of the reference geometry
 - General scheme
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- 3 Non smooth invisible perturbations of the reference geometry



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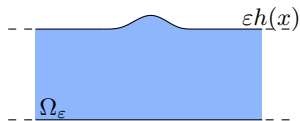


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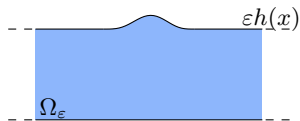
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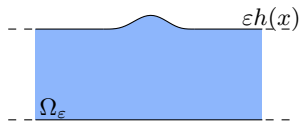
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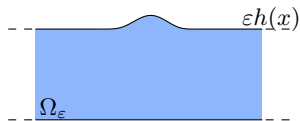
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$$\left| \begin{array}{l} n_\varepsilon = \frac{1}{\sqrt{1 + \varepsilon^2 (h'(x))^2}} \begin{pmatrix} -\varepsilon h'(x) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} -h'(x) \\ 0 \end{pmatrix} + \dots \\ \nabla u_\varepsilon(x, \varepsilon h(x)) = \nabla u_\varepsilon(x, 0) + \varepsilon h(x) \begin{pmatrix} \partial_{xy}^2 u_\varepsilon(x, 0) \\ \partial_{yy}^2 u_\varepsilon(x, 0) \end{pmatrix} + \dots \end{array} \right.$$

We use that $u_0 = w^+ \Rightarrow \partial_{yy}^2 u_0 = 0$

so that we get $0 = n_\varepsilon \cdot \nabla u_\varepsilon(x, \varepsilon h(x)) = \partial_y u_0 + \varepsilon (\partial_y u_1 - \varepsilon h'(x) \partial_x u_0) + \dots$

- ▶ We have $u_0 = w_+$ and u_1 is uniquely defined.
- ▶ Set $\Sigma_{\pm L} = \{\pm L\} \times (-1; 0)$ for L large enough. From the known formula

$$2ikR(\varepsilon\mu) = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^+ - u_\varepsilon \partial_n w^+ d\sigma, \quad \text{where } \partial_n = \pm \partial_x \text{ at } x = \pm L,$$

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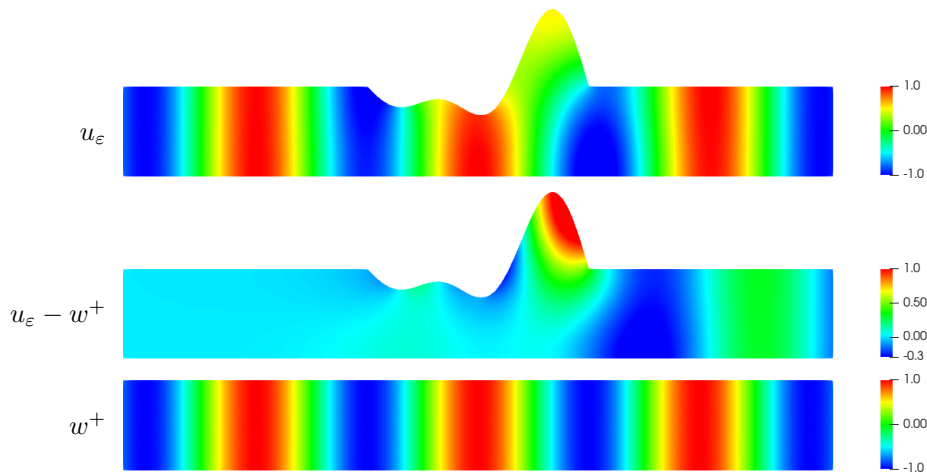
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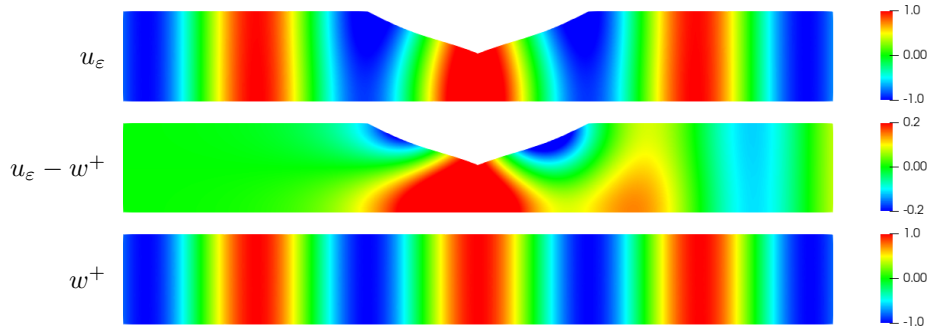
Thus we can construct geometries Ω_ε where $R_\varepsilon = 0$ for the Neumann pb.

- An example of **non reflecting** perturbation obtained after 15 iterations ($\varepsilon = 0.4$).

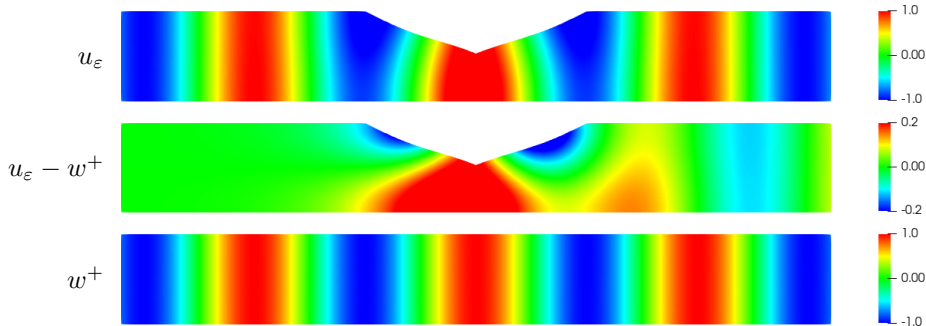


- Again, the algorithm converges though ε is not that small.

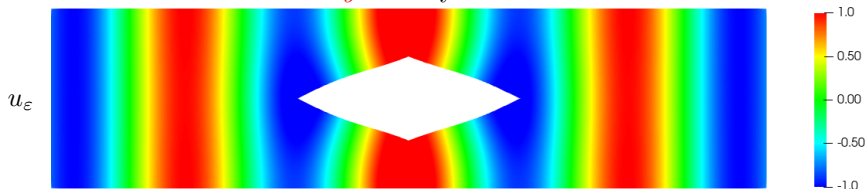
- Other example **non reflecting** perturbation.



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- The defect lies **below the line $y = 1$** . Symmetrisation:



$T = 1$ for the Neumann problem?

PROPOSITION: We have

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$dT(0)$ is **null** $\Rightarrow$ the approach fails to impose $T = 1$.

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An example of singularly perturbed problem

- For $a > 0$, $a \neq 1$, consider the 1D problem

$$(\mathcal{P}_\varepsilon) \quad \left\{ \begin{array}{l} \varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) - a = 0 \text{ in } \Omega := (0; 1) \\ u_\varepsilon(0) = 0, \quad u_\varepsilon(1) = 1. \end{array} \right.$$

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But since $\|u_\varepsilon(x) - \hat{u}_0(x)\|_{L^\infty(\bar{\Omega})} = |1 - a|$, (u_ε) does not cv to $\hat{u}_0$ in $H^1(\Omega)$.

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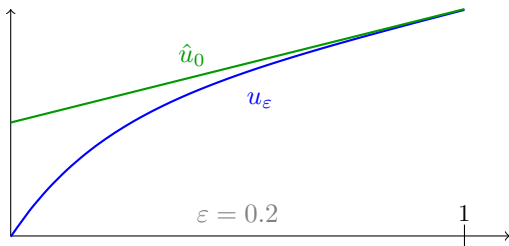
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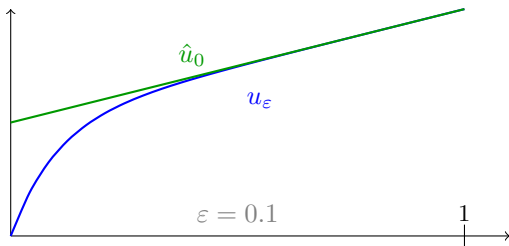


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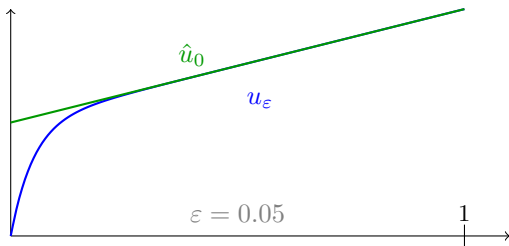
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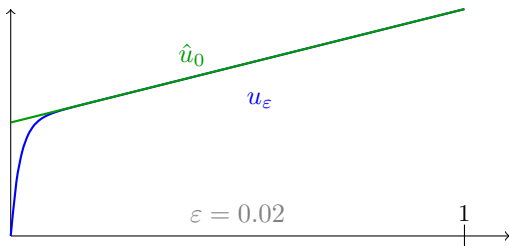


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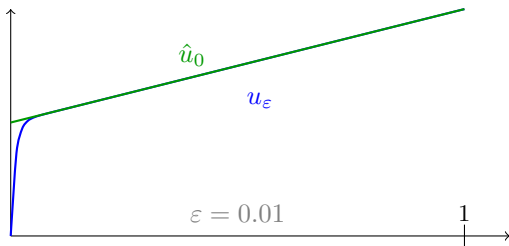
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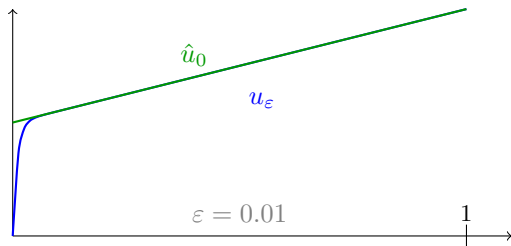


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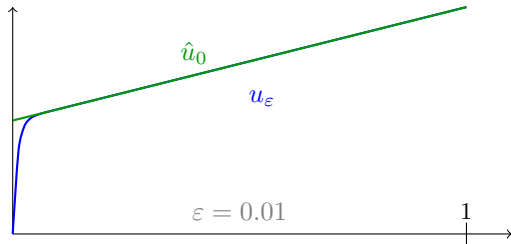
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- Our expansion fails to provide a good representation of u_ε due to this **boundary layer phenomenon**. We say that $(\mathcal{P}_\varepsilon)$ is a **singularly perturbed problem**.
- To approximate correctly u_ε **near the origin**, we will have to incorporate terms which depend on the **rapid variable** x/ε .

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One small obstacle

Can one hide a small **Dirichlet** obstacle centered at M_1 ?

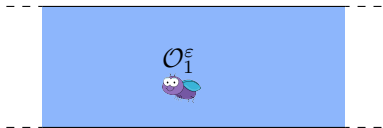
► Set $\mathcal{O}_1^\varepsilon := M_1 + \varepsilon\mathcal{O}$ where $M_1 \in \mathbb{R} \times \omega$ and $\mathcal{O}$ is a bounded Lipschitz domain. We consider the problem

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Non zero terms!
($\operatorname{cap}(\mathcal{O}) > 0$)

⇒ One single small obstacle **cannot** be **non reflecting**.

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$$u_\varepsilon = u_0 + \zeta(x) v_0(\varepsilon^{-1}(x - M)) + \varepsilon \left(u_1 + \zeta(x) v_1(\varepsilon^{-1}(x - M)) \right) + \dots$$

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- v_0 serves to impose Dirichlet BC on $\partial\mathcal{O}^\varepsilon$ at order ε^0 . For $x \in \partial\mathcal{O}^\varepsilon$,
 $u_0(x) = u_0(M) + (x - M) \cdot \nabla u_0(M) + \dots$ (note that $x - M$ is of order ε).

Therefore we impose $v_0 = -u_0(M)$ on $\partial\mathcal{O}$.

- Introduce the fast variable $\xi = \varepsilon^{-1}(x - M)$. In a vicinity of M , we have

$$\begin{aligned} & (\Delta_x + k^2 \text{Id}) \left(v_0(\varepsilon^{-1}(x - M)) + \varepsilon v_1(\varepsilon^{-1}(x - M)) + \dots \right) \\ &= \varepsilon^{-2} \boxed{\Delta_\xi v_0(\xi)} + \varepsilon^{-1} \Delta_\xi v_1(\xi) + \dots \end{aligned}$$

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- As $|\xi| \rightarrow +\infty$, we have

$$W(\xi) = \frac{\text{cap}(\mathcal{O})}{|\xi|} + \vec{q} \cdot \nabla \Phi(\xi) + O(|\xi|^{-3}),$$

where $\Phi := \xi \mapsto -1/(4\pi|\xi|)$ is the **Green function** of the Laplacian in $\mathbb{R}^3$, $\text{cap}(\mathcal{O}) > 0$, $\vec{q} \in \mathbb{R}^3$.

- Now, we turn to the terms of order ε in the expansion of u^ε

$$u_\varepsilon = u_0 + \zeta(x) v_0(\varepsilon^{-1}(x - M)) + \varepsilon \left(u_1 + \zeta(x) v_1(\varepsilon^{-1}(x - M)) \right) + \dots$$

- By inserting $u_0 + \zeta(x) v_0(\varepsilon^{-1}(x - M))$ into $(\mathcal{P}_\varepsilon)$ and replacing v_0 by its **main contribution at infinity**, we find that u_1 must solve

$$\left| \begin{array}{ll} -\Delta u_1 - k^2 u_1 = -([\Delta_x, \zeta] + k^2 \zeta \text{Id}) \left(w^+(M) \frac{\text{cap}(\mathcal{O})}{|x - M|} \right) & \text{in } \Omega_0 \\ u_1 = 0 & \text{on } \partial\Omega_0. \end{array} \right.$$

where $[\Delta_x, \zeta]\varphi := \Delta_x(\zeta\varphi) - \zeta\Delta_x\varphi = 2\nabla\varphi \cdot \nabla\zeta + \varphi\Delta\zeta$ (**commutator**).

- Now, we turn to the terms of order ε in the expansion of u^ε

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→ This uniquely defines u_1 .

Asymptotic of the scattering coefficients

- We consider the ansatz

$$R_\varepsilon = R_0 + \varepsilon R_1 + \dots \quad T_\varepsilon = T_0 + \varepsilon T_1 + \dots$$

- Set $\Sigma_{\pm L} = \{\pm L\} \times \omega$ for L large enough. From the known formula

$$2ikR_\varepsilon = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^+ - u_\varepsilon \partial_n w^+ d\sigma, \quad 2ikT_\varepsilon = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^- - u_\varepsilon \partial_n w^- d\sigma,$$

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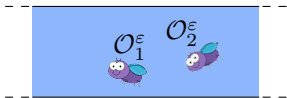
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Integrating by parts, finally we get the final result:

PROPOSITION: We have

$$\begin{aligned} R_\varepsilon &= 0 + \varepsilon \left(4i\pi \operatorname{cap}(\mathcal{O}) w^+(M_1)^2 \right) + O(\varepsilon^2) \\ T_\varepsilon &= 1 + \varepsilon \left(4i\pi \operatorname{cap}(\mathcal{O}) |w^+(M_1)|^2 \right) + O(\varepsilon^2). \end{aligned}$$

Several small obstacles

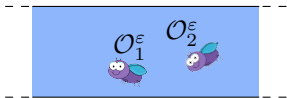


- One small obstacle cannot be non reflecting. Let us try with **TWO**, located at M_1, M_2 .

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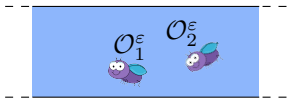


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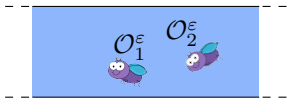
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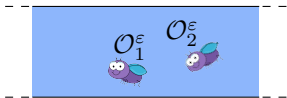
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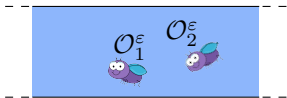


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COMMENTS:

- Hard part is to **justify the asymptotics** for the fixed point problem.
- We **cannot** impose $T_\epsilon = 1$ with this strategy.
- When there are **more propagating waves**, we need **more obstacles**.

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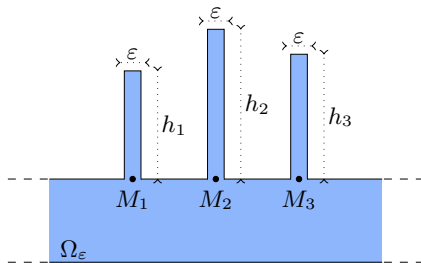


Acting as a **team**, obstacles can become invisible!

- 1 A few notions of asymptotic analysis
- 2 Invisible smooth perturbations of the reference geometry
- 3 Non smooth invisible perturbations of the reference geometry
 - An example of singularly perturbed problem
 - Invisible clouds of small obstacles
 - Perfect invisibility for the Neumann problem

$T = 1$ for the Neumann problem

- ▶ We study the **same problem** in the geometry Ω_ε

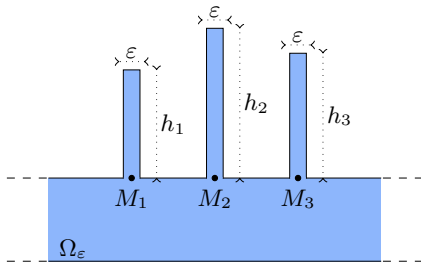


Singular perturbation
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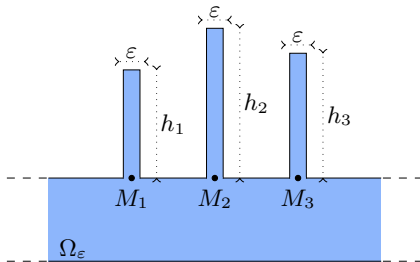
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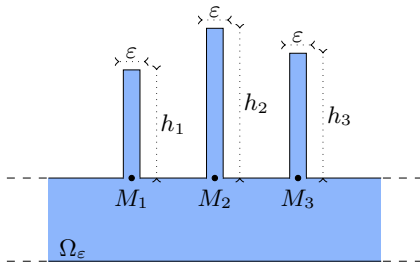
1) We can find M_n, h_n such that $R_\varepsilon = O(\varepsilon^2)$ and $T_\varepsilon = 1 + O(\varepsilon^2)$.

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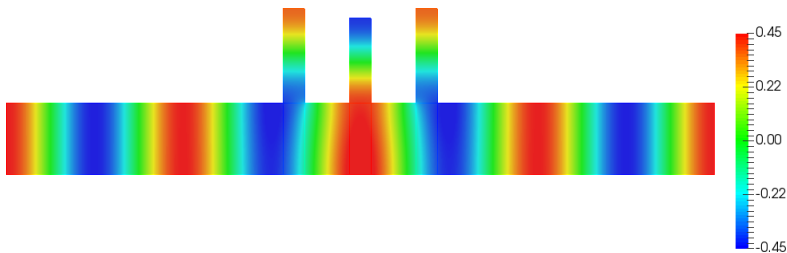
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 - 3) **Energy conservation** + $[T_\varepsilon = 1 + O(\varepsilon)] \Rightarrow T_\varepsilon = 1$.

Numerical results

- ▶ Perturbed waveguide ($\Re(u_\epsilon(x, y)e^{-i\omega t})$)

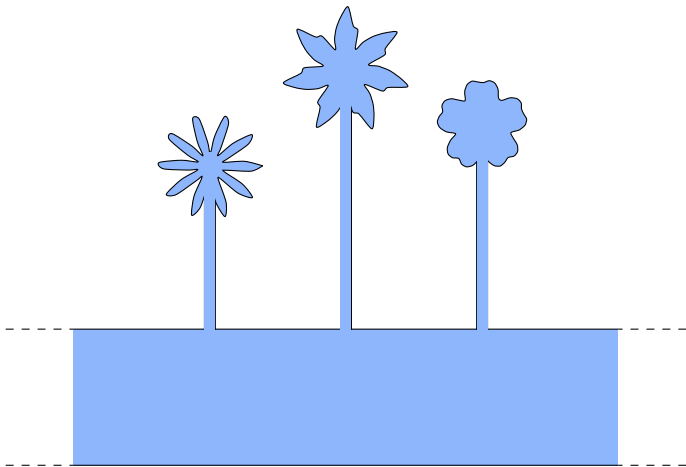


- ▶ Reference waveguide ($\Re(u_i(x, y)e^{-i\omega t})$)



Comments

- We could also have hidden **gardens of flowers!**



- 1 A few notions of asymptotic analysis
- 2 Invisible smooth perturbations of the reference geometry
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Conclusion of lecture 2

What we did

- 1) Perturbation in the **PDE**. Recall the standard scheme

Step I: **ansatz** and identification of the terms of the ansatz;

Step II: **error estimates** (stability estimate + consistency result).

- 2) Smooth perturbation of the **geometry**. Use a change of variable to show error estimates in a **fixed** geometry.
- 3) Construction of **smooth** and **non smooth invisible** defects in waveguides.



Use the **first term** in the asymptotic whose dependence wrt the perturbation is **explicit** and linear to cancel the whole expansion by solving a **fixed point problem**.

Next lecture

- ♠ We will explain how to use **resonant phenomena** to construct **large** invisible defects.

Bibliography

► Part I



V. Maz'Ya, S.A. Nazarov, B.A. Plamenevskij. Asymptotic theory of elliptic boundary value problems in singularly perturbed domains (Vol. 1 and 2). Springer Science, Business Media, 2000.

► Part II



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► Part III



L. Chesnel, S.A. Nazarov. Team organization may help swarms of flies to become invisible, Inverse Problems and Imaging, vol. 10, 4:977-1006, 2016.



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