

A few techniques to achieve invisibility in waveguides

Lecture 1: Rudiments of waveguide theory

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Introduction

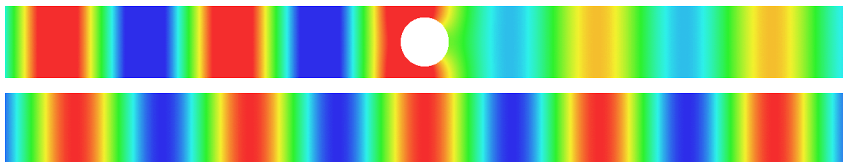
- ▶ **Waveguides** appear in many fields of physics: acoustics, water waves, electromagnetics, classical mechanics, quantum mechanics, ...



- ▶ One can think to musical instruments, loudspeakers, optical fibers, conductive metal pipes, ...

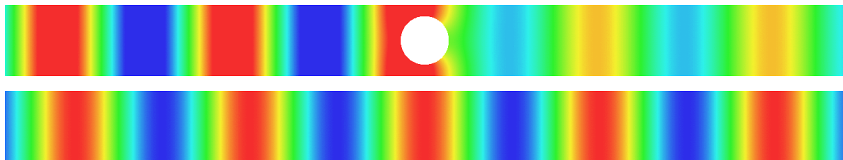
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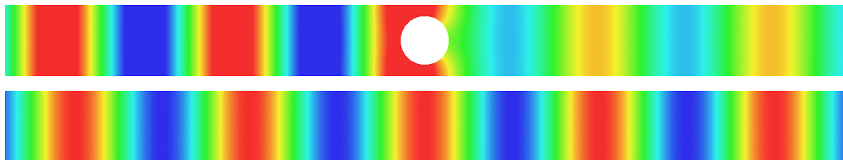
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How to identify situations (geometry, frequency, ...) where waves go through like if there were no defect



- One can wish to have **good energy transmission** through the structure.
- One can wish to **hide objects**.

Goals of the mini course

- 1) To explain how to model **propagation of scalar waves in waveguides** in time-harmonic regime.
- 2) To present different **tools** of applied mathematics to identify situations of **invisibility**:
 - Asymptotic analysis;
 - Spectral theory for self-adjoint and non self-adjoint problems;
 - Finite element methods.

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Structure of the mini course

Lecture 1. Rudiments of waveguide theory.

Lecture 2. Invisible perturbations of the reference geometry.
→ Construction of *small amplitude* invisible obstacles.

Lecture 3. Playing with resonances to achieve invisibility.
→ Construction of *large amplitude* invisible obstacles.

Lecture 4. A spectral problem characterizing zero reflection.
→ Given an obstacle, *find frequencies* such that one has zero reflection.

Lecture 1: Rudiments of waveguide theory

- 1 Setting
- 2 Dirichlet problem for $k < \pi$
- 3 Dirichlet problem for $k > \pi$
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1 Setting

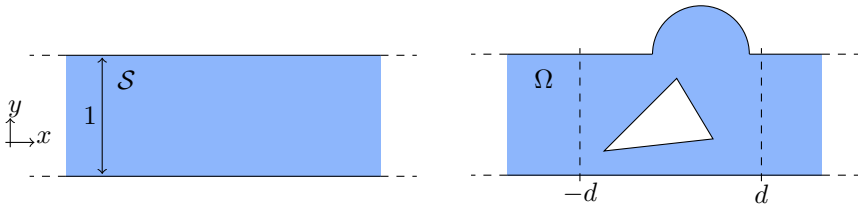
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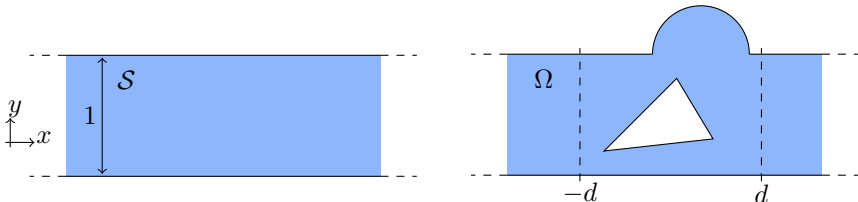
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► Set $I := (0; 1)$, $\mathcal{S} := \mathbb{R} \times I$. Let $\Omega \subset \mathbb{R}^2$ be a **waveguide** which coincides with $\mathcal{S}$ outside of a compact region.



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► Consider the wave equation, for $t \geq 0$,

$$\left\{ \begin{array}{lcl} \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} - \Delta U & = & F \quad \text{in } \Omega \\ U & = & 0 \quad \text{on } \partial\Omega, \end{array} \right.$$

with some initial conditions. Assume that the celerity $c > 0$ is **constant**. This problem appears for example in **electromagnetics**.

Setting

- For **time-harmonic** F , *i.e.* of the form

$$F(x, y, t) = f(x, y)e^{-i\omega t},$$

for some pulsation $\omega > 0$, it is natural to look for solutions of the form

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$$(\mathcal{P}_D) \quad \left| \begin{array}{ll} -\Delta u - k^2 u & = f \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

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Goal of the lecture

To understand the features of $(\mathcal{P}_D)$ according to the **values of $k > 0$** .

- 1 Setting
- 2 Dirichlet problem for $k < \pi$
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- Assume that $f \in L^2(\Omega)$. The **variational formulation** of $(\mathcal{P}_D)$ writes

$$\left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u, v) = \ell(v), \quad \forall v \in H_0^1(\Omega), \end{array} \right.$$

with $H_0^1(\Omega) := \{w \in H^1(\Omega) \mid w = 0 \text{ on } \partial\Omega\}$ and

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- Since $a(\cdot, \cdot)$ is continuous, with **Riesz** we can define the linear bounded operator $A(k) : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

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THEOREM: Pick $k \in (0; \pi)$. The operator $A(k)$ decomposes as

$$A(k) = B + K$$

where $B : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is an **isomorphism**, $K : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is **compact**.

► We deduce that $A(k)$ satisfies the **Fredholm alternative**:

- Either $A(k)$ is **injective** and then is it an **isomorphism**;
- Or $A(k)$ has a **kernel** of finite dim. $\text{span}(u_1, \dots, u_P)$ and then the equation

$$A(k)u = F$$

has a solution (defined up to $\text{span}(u_1, \dots, u_P)$) if and only if F satisfies the **compatibility conditions**

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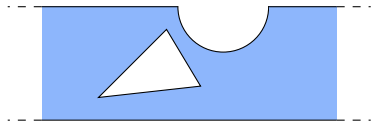
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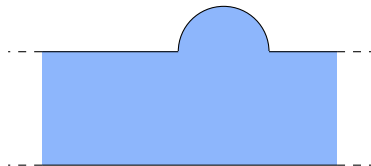
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$A(k)$ isom. for all $k \in (0; \pi)$



$A(k)$ not always isom. for $k \in (0; \pi)$

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Computation of modes

- Modes are defined as the solutions with **separate variables**

$$u(x, y) = \alpha(x)\varphi(y),$$

of $(\mathcal{P}_D)$ in the reference strip $\mathcal{S}$ for $f \equiv 0$.

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- We obtain

$$\alpha''(x)\varphi(y) + \alpha(x)\varphi''(y) + k^2\alpha(x)\varphi(y) = 0$$

which gives

$$\left| \begin{array}{ll} -\varphi''(y) = \lambda \varphi(y) & \text{in } I \\ \varphi(0) = \varphi(1) = 0 \end{array} \right. \quad -\alpha''(x) = (k^2 - \lambda) \alpha(x) \quad \text{in } \mathbb{R}$$

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- ▶ We deduce that

$$\lambda_n = n\pi, \quad \varphi_n(y) = \sqrt{2} \sin(n\pi y), \quad n \in \mathbb{N}^* := \{1, 2, \dots\}.$$

When $k \notin \mathbb{N}\pi$, the **modes** coincide with the family $\{w_n^\pm\}_{n \in \mathbb{N}^*}$ where

$$w_n^\pm(x, y) = e^{\pm i\beta_n x} \varphi_n(y), \quad \beta_n := \sqrt{k^2 - n^2\pi^2}.$$

Comments

- ▶ $\sqrt{\cdot}$ is chosen such that $\Im m \sqrt{\cdot} \geq 0$.
- ▶ Pick $k \in (N\pi; (N+1)\pi)$ for some $N \in \mathbb{N}$. Two families of modes :

★ There are N propagating modes

$$w_n^\pm(x, y) = e^{\pm i \sqrt{k^2 - n^2 \pi^2} x} \varphi_n(y), \quad n = 1, \dots, N.$$

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$$w_n^\pm(x, y) = e^{\mp i\sqrt{n^2\pi^2 - k^2}x} \varphi_n(y), \quad n = N+1, N+2, \dots,$$

which are expon. decaying as $x \rightarrow \pm\infty$ and expon. growing as $x \rightarrow \mp\infty$.

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Ill-posedness in $H_0^1(\Omega)$

DEFINITION: Let $T : X \rightarrow Y$ be a **continuous** linear map between two Hilbert spaces. T is said to be **Fredholm** iff

- i) $\dim(\ker T) < +\infty$ and $\text{range } T$ is closed;
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Message: loss of Fredholmness in $H_0^1(\Omega)$ is due to the existence of **propagating modes**.

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- To model **dissipation**, for $\eta > 0$, work on the problem

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- With the convention $U_\eta(x, y, t) = u_\eta(x, y)e^{-i\omega t}$, this originates from

$$\left| \begin{array}{rcl} \frac{\partial^2 U_\eta}{\partial t^2} + \eta \frac{\partial U_\eta}{\partial t} - \frac{1}{c^2} \Delta U_\eta & = & F \quad \text{in } \Omega, t > 0, \\ U_\eta & = & 0 \quad \text{on } \partial\Omega, t > 0. \end{array} \right. \quad (1)$$

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- Assume that $F \equiv 0$. Multiplying (1) by $\partial_t \bar{U}$ and integrating in Ω gives

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Conclusion: the **energy**

$$E(t) = \frac{1}{2} \int_{\Omega} \left| \frac{\partial U_\eta}{\partial t} \right|^2 + \frac{1}{c^2} |\nabla U_\eta|^2 dx dy$$

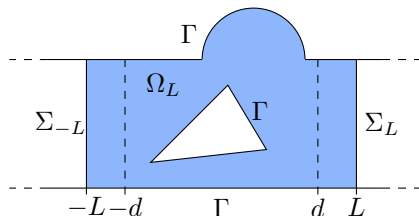
indeed **decreases**, due to the term $\eta \partial_t U_\eta$, when $\eta > 0$.

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► To take the limit $\eta \rightarrow 0$ (**limiting absorption principle**), let us rewrite the problem in a **bounded** domain. For $L > d$, set

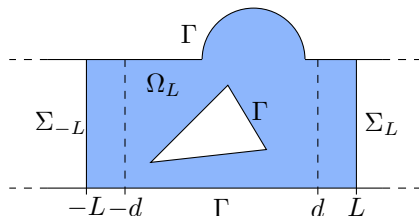
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We must impose *ad hoc* transparent conditions on the artificial boundaries $\Sigma_{\pm}$ that do not create **spurious reflections**.

- Set $\mathcal{S}_{\pm} := \pm(L; +\infty) \times I$ and define the **Dirichlet-to-Neumann** operators

$$\Lambda_{\pm}^{\eta} : H_{00}^{1/2}(\Sigma_{\pm L}) \rightarrow H^{-1/2}(\Sigma_{\pm L})$$
$$\varphi \mapsto \frac{\partial v_{\varphi}}{\partial \nu},$$

where $\pm \partial_{\nu} = \partial_x$ on $\Sigma_{\pm L}$ and $v_{\varphi} \in H^1(\mathcal{S}_{\pm})$ is the function such that

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PROPOSITION: $u_{\eta} \in H_0^1(\Omega)$ solves $(\mathcal{P}_{\eta})$ iff $u_{\eta}|_{\Omega_L}$ satisfies

$$(\mathcal{P}_{\eta}^L) \quad \left| \begin{array}{ll} \text{Find } u_{\eta} \in H_0^1(\Omega_L; \Gamma) \text{ such that} \\ -\Delta u_{\eta} - (k^2 + ik\eta)u_{\eta} &= f \quad \text{in } \Omega_L \\ \frac{\partial u_{\eta}}{\partial \nu} &= \Lambda_{\pm}^{\eta}(u_{\eta}) \quad \text{on } \Sigma_{\pm L} \end{array} \right.$$

where $\partial_{\nu} = \pm \partial_x$ at $x = \pm L$.

- For the $\Lambda_{\pm}^{\eta}$, we have the **explicit representation**

$$\begin{aligned}\Lambda_{\pm}^{\eta}(\varphi) &= \sum_{n=1}^{+\infty} i\beta_n^{\eta}(\varphi, \varphi_n)_{L^2(\Sigma_{\pm L})} e^{\pm i\beta_n^{\eta}(x \mp L)} \varphi_n(y)|_{\Sigma_{\pm L}} \\ &= \sum_{n=1}^{+\infty} i\beta_n^{\eta}(\varphi, \varphi_n)_{L^2(\Sigma_{\pm L})} \varphi_n(y)\end{aligned}$$

where $\beta_n^{\eta} := \sqrt{k^2 + i\eta - n^2\pi^2}$ (the solution decomposes on the **exponentially decaying modes**).

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- Taking the limit $\eta \rightarrow 0$, we define the operators

$$\Lambda_{\pm}(\varphi) = \sum_{n=1}^{+\infty} i\beta_n (\varphi, \varphi_n)_{L^2(\Sigma_{\pm L})} \varphi_n(y)$$

and consider the problem

$$(\mathcal{P}^L) \quad \left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega_L; \Gamma) \text{ such that} \\ -\Delta u - k^2 u = f \quad \text{in } \Omega_L \\ \frac{\partial u}{\partial \nu} = \Lambda_{\pm}(u) \quad \text{on } \Sigma_{\pm L}. \end{array} \right.$$

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- Its variational formulation writes

$$\left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega_L; \Gamma) \text{ such that} \\ a^{\text{out}}(u, v) = \ell(v), \quad \forall v \in H_0^1(\Omega_L; \Gamma), \end{array} \right.$$

with

$$a^{\text{out}}(u, v) = \int_{\Omega_L} \nabla u \cdot \nabla \bar{v} - k^2 u \bar{v} \, dx dy - \langle \Lambda_+(u), v \rangle_{\Sigma_L} - \langle \Lambda_-(u), v \rangle_{\Sigma_{-L}}.$$

► Since $a^{\text{out}}(\cdot, \cdot)$ is continuous, with **Riesz** we can define the linear bounded operator $A^{\text{out}}(k) : H_0^1(\Omega_L; \Gamma) \rightarrow H_0^1(\Omega_L; \Gamma)$ such that

$$(A^{\text{out}}(k)u, v)_{H^1(\Omega)} = a^{\text{out}}(u, v), \quad \forall u, v \in H_0^1(\Omega).$$

THEOREM: For $k \in (N\pi; (N+1)\pi)$, $N \in \mathbb{N}^*$, $A^{\text{out}}(k)$ decomposes as

$$A^{\text{out}}(k) = B + K$$

where $B : H_0^1(\Omega_L; \Gamma) \rightarrow H_0^1(\Omega_L; \Gamma)$ is an **isomorphism** and $K : H_0^1(\Omega_L; \Gamma) \rightarrow H_0^1(\Omega_L; \Gamma)$ is **compact**.

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THEOREM: Fix $k \in (\pi; +\infty) \setminus \{N\pi\}$ and assume that $A^{\text{out}}(k)$ is injective. There is $C > 0$ (independent of η), η_0 such that we have

$$\|u_\eta - u\|_{H^1(\Omega_L)} \leq C\eta \|f\|_{L^2(\Omega_L)}, \quad \forall \eta \in (0; \eta_0].$$

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- ▶ In sequel, we will not work with source terms f but instead consider the scattering of incident waves.
- ▶ To simplify, assume that $k \in (\pi; 2\pi)$ so that only the waves

$$w_{\pm}(x, y) = e^{\pm i\beta_1 x} \varphi_1(y) = e^{\pm i\sqrt{k^2 - \pi^2} x} \varphi_1(y).$$

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► The scattering of the rightgoing wave w_+ coming from the left branch of the waveguide leads us to consider the problem

$(\mathcal{P}_+)$	$\left \begin{array}{l} \text{Find } u_+ \in H_{0,\text{loc}}^1(\Omega) \text{ such that } u_+ - w_+ \text{ is } \text{outgoing} \text{ and} \\ \Delta u_+ + k^2 u_+ = 0 \quad \text{in } \Omega \\ u_+ = 0 \quad \text{on } \partial\Omega. \end{array} \right.$
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The sentence “ $u_+ - w_+$ is **outgoing**” means that we impose

$$u_+ - w_+ = \begin{cases} \sum_{n=1}^{+\infty} \alpha_n^+ e^{i\beta_n x} \varphi_n(y) & \text{for } x > L \\ \sum_{n=1}^{+\infty} \alpha_n^- e^{-i\beta_n x} \varphi_n(y) & \text{for } x < -L \end{cases}, \text{ for some } \alpha_n^{\pm} \in \mathbb{C}.$$

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- Similarly, $(\mathcal{P})$ admits a solutions u_- with the representation

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PROPOSITION: We have $T_+ = T_- =: T$ and the identities

$$|R_{\pm}|^2 + |T|^2 = 1 \quad (\text{conservation of energy}).$$

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Numerical approximation

Computers don't like infinite!

► Ω is **unbounded**. Let us work in Ω_L on the formulation

$$\left| \begin{array}{l} \text{Find } u_+ \in H_0^1(\Omega_L; \Gamma) \text{ such that for all } v \in H_0^1(\Omega_L; \Gamma), \\ \int_{\Omega_L} \nabla u_+ \cdot \nabla \bar{v} - k^2 u \bar{v} \, dx dy - \langle \Lambda_+(u_+), v \rangle_{\Sigma_L} - \langle \Lambda_-(u_+), v \rangle_{\Sigma_{-L}} = -2i\beta_1 \int_{\Sigma_{-L}} w_+ \bar{v} \, dy. \end{array} \right.$$

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- $H_0^1(\Omega_L; \Gamma)$ is of **infinite dimension**. Use **Finite Element spaces**. Finally

$$\left| \begin{array}{l} \text{Find } u_+^h \in V_h \text{ such that for all } v^h \in V_h, \\ \int_{\Omega_L} \nabla u_+^h \cdot \nabla \bar{v}^h - k^2 u_+^h \bar{v}^h \, dx dy - \sum_{n=1}^M i\beta_n (u_+^h, \varphi_n)_{L^2(\Sigma_L)} \overline{(v^h, \varphi_n)_{L^2(\Sigma_L)}} \\ - \sum_{n=1}^M i\beta_n (u_+^h, \varphi_n)_{L^2(\Sigma_{-L})} \overline{(v^h, \varphi_n)_{L^2(\Sigma_{-L})}} = -2i\beta_1 \int_{\Sigma_{-L}} w_+ \bar{v}^h \, dy. \end{array} \right.$$

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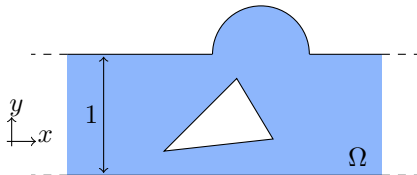


For h small enough, L large enough (one has exponential convergence with respect to L), u_+^h yields a **good approximation** of u_+ .

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Neumann problem

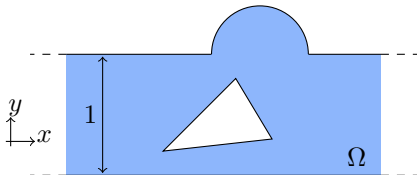
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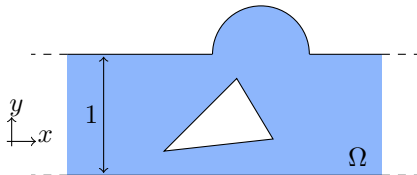
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- For this problem with $k \in (N\pi; (N+1)\pi)$, the **modes** are

$$\begin{array}{l|l} \text{Propagating} & w_n^\pm(x, y) = e^{\pm i\beta_n x} \cos(n\pi y), \quad \beta_n = \sqrt{k^2 - n^2\pi^2}, \quad n \in \llbracket 0, N-1 \rrbracket \\ \text{Evanescent} & w_n^\pm(x, y) = e^{\mp \beta_n x} \cos(n\pi y), \quad \beta_n = \sqrt{n^2\pi^2 - k^2}, \quad n \geq N. \end{array}$$

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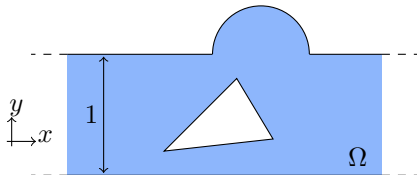
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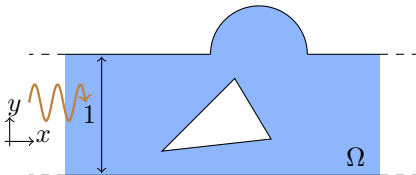


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- We fix $k \in (0; \pi)$ so that only the plane waves $e^{\pm ikx}$ can propagate.
- The scattering of the wave e^{ikx} leads us to consider the solutions of $(\mathcal{P})$ with the decomposition

$$u = \left| \begin{array}{ll} e^{ikx} + R e^{-ikx} + \dots & x \rightarrow -\infty \\ T e^{+ikx} + \dots & x \rightarrow +\infty \end{array} \right.$$

$R, T \in \mathbb{C}$ are the **scattering coefficients**, the $\dots$ are expon. decaying terms.

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Conclusion of lecture 1

What we did

We studied waveguides problems in **time-harmonic** regime.

- Dirichlet BCs with $k < \pi$: $H_0^1(\Omega)$ **ok** because no propagating modes.
- Dirichlet BCs with $k > \pi$: $H_0^1(\Omega)$ **not ok** because propagating modes
→ impose **radiation conditions** (add dissipation and take the limit $\eta \rightarrow 0$).
- Neumann BCs with $k > 0$: $H^1(\Omega)$ **not ok** because propagating modes
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Next lecture

- ♠ We will study questions of **invisibility**. How to create defects which provide the same scattering coefficients as in the reference strip?