

ÉCOLE DOCTORALE DE L'INSTITUT POLYTECHNIQUE DE PARIS

Mémoire présenté en vue de l'obtention du

Diplôme d'Habilitation à Diriger des Recherches

Discipline : Mathématiques

par

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**Analysis of wave propagation phenomena.
From negative materials to invisibility.**



Travail présenté publiquement le 2 novembre 2026 devant le jury composé de :

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REMERCIEMENTS

Je souhaite tout d'abord remercier les personnes qui ont participé à l'évaluation de cette habilitation.

Merci donc à Éric Cancès, Simon Chandler-Wilde et Céline Grandmont d'avoir accepté d'écrire un rapport sur ce mémoire. Ce texte a ceci de particulier que son objectif est double. Il constitue une synthèse d'un long travail de recherche. Mais il a également été rédigé pour mettre en lumière certains points relativement précis. Il présente donc un grain inégal, ce qui n'a pas dû faciliter sa lecture. Je vous remercie donc de l'avoir regardé en détail et plus généralement, d'avoir pris le temps de vous pencher sur mon travail.

Je remercie également les examinatrices et examinateurs Fioralba Cakoni, Clotilde Fermanian, Roland Griesmaier et David Krejčířík de me faire l'honneur d'être membres du jury.

La recherche est une activité collective, les résultats présentés dans ce document sont en bonne partie le fruit des interactions avec mes collègues, je désire les remercier ici.

Je tiens en premier lieu à exprimer ma profonde reconnaissance envers Anne-Sophie Bonnet-Ben Dhia. Certaines personnes rayonnent autour d'elles, sortes de phares dans le brouillard, Anne-Sophie en fait partie. Bien au-delà des considérations scientifiques, tes engagements, tes valeurs, ton énergie, nourrissent au quotidien celles et ceux qui t'accompagnent. C'est une chance pour moi que nos chemins se soient croisés. La vie ne t'a malheureusement pas épargnée, je pense bien fort à Hachmi.

Depuis maintenant presque quinze ans, j'ai beaucoup appris scientifiquement auprès de Sergei Aleksandrovich Nazarov. Virtuose du développement asymptotique, Sergei est quelqu'un d'enthousiaste, curieux, aux plaisanteries parfois mordantes, d'une grande culture, pas seulement mathématique. Je me souviendrai longtemps des bons repas russes préparés par Nadia à la Töölö Tower d'Helsinki. Il est bien dommage que les circonstances politiques ne nous aient pas permis de nous voir plus. Большое спасибо Сергей!

J'en arrive à Sonia Fliss, que je connais depuis mes débuts dans la recherche et que je remercie pour sa générosité, sa gentillesse, mais pas seulement. Sonia, c'est la personne sur qui on peut compter, toujours disponible, malgré un emploi du temps débordant, et qui rassure, en toutes circonstances. Autre caractère, autre fonctionnement, Laurent Bourgeois, avec qui j'ai toujours plaisir à discuter. Une pointe de cynisme par-ci, une pointe d'athlétisme par-là, et une présence quotidienne, qui compte.

Je veux aussi remercier Housseem Haddar et Lorenzo Audibert qui dirigent avec souplesse l'équipe Idefix en nous épargnant bon nombre de tâches fastidieuses. Housseem, toujours arrangeant, m'a

par ailleurs guidé dans mes premiers pas de chercheur au sein de l'équipe Defi. Lorenzo surgit parfois dans mon bureau avec des questions aussi déroutantes que fertiles. C'est toujours agréable de discuter avec lui de mathématiques, de sciences ou bien de nos filles.

Merci à Patrick Ciarlet pour son soutien fidèle depuis l'encadrement de ma thèse, sa bienveillance et ses conseils avisés. Merci à Patrick Joly pour son exigence scientifique, ses explications lumineuses et son humour. Vous avez été des présences importantes tout au long de mon parcours.

Les échanges avec Vincent Pagneux ont été particulièrement féconds pour moi. À son contact, j'ai pu découvrir de nouveaux thèmes de recherche. J'espère que nous poursuivrons notre collaboration à l'avenir.

Je remercie mes coautrices et coauteurs, à savoir Lorenzo Audibert, Antoine Bera, Anne-Sophie Bonnet-Ben Dhia, Laurent Bourgeois, Renata Bunoiu, Camille Carvalho, Patrick Ciarlet, Xavier Claeys, Sonia Fliss, Houssem Haddar, Harri Hakula, Jérémy Heleine, Nuutti Hyvönen, Henrik Kettunen, Hongjie Li, Kevish Napal, Sergei A. Nazarov, Vincent Pagneux, Mahran Rihani, Ari Sihvola, Stratos Staboulis, Jari Taskinen, Henrik Wallén et Jingni Xiao, sans qui les travaux que je présente dans ce mémoire n'auraient pas été les mêmes.

Au cours de ces années, j'ai eu de nombreux liens avec les collègues marseillais, notamment Maxence Cassier, Guillaume Demézy, Boris Gralak, Florian Monteghetti et Thomas Ourmières-Bonafos. Ce fut toujours enrichissant d'échanger avec eux.

Je remercie tous les étudiants avec qui j'ai pu interagir, en particulier Antoine Bera, Camille Carvalho, Mathieu Chamaillard, Thomas Creuset, Jérémy Firozaly, Qingqing Hu, Romain Hoang, Jérémy Heleine, Marwa Mansouri, Kevish Napal, Aurélien Parigaux, Mahran Rihani, Lorenzo Scaglione et Michele Zaccaron.

Nous avons la chance de pouvoir travailler à l'Unité de Mathématiques Appliquées de l'Ensta dans une atmosphère familiale et chaleureuse. Celle-ci ne vient pas de nulle part, et je remercie Marc Lenoir, les membres historiques de l'équipe Poems, c'est-à-dire Éliane Bécache, Anne-Sophie Bonnet-Ben Dhia, Christophe Hazard, Patrick Joly, Éric Lunéville, Jean-François Mercier, ainsi que le directeur actuel de l'UMA, Frédéric Jean, d'avoir œuvré à la réalisation de cette belle structure. Je remercie également les assistantes d'équipe Corinne Chen et Marie Énée dont le rôle est crucial dans le fonctionnement du laboratoire.

L'enseignement est une activité qui me tient à cœur. Je remercie mes collègues de l'UMA de m'avoir fait découvrir cet art dans d'excellentes conditions. Je pense qu'on ne loue pas assez la qualité de l'enseignement mathématique à l'Ensta. Grands mercis donc en particulier à Anne-Sophie Bonnet-Ben Dhia, Laurent Bourgeois, Patrick Ciarlet et au modèle de clarté, Christophe Hazard. Je remercie également mes non-moins investis collègues ou ex-collègues d'enseignement du Département de Mathématiques Appliquées de l'École Polytechnique Benjamin Bogosel, Marie Doumic, Sébastien Impériale, Philippe Moireau et Flore Nabet. Je n'oublie pas François Alouges qui a eu un rôle déterminant pour moi par son appui et son accueil à mes débuts.

Auparavant, plusieurs enseignantes et enseignants ont su me transmettre leur passion et m'ont donné confiance en me soutenant. Je pense ainsi régulièrement à Mme Barlier, M. Barlier, M. Poncin et Denis Matignon.

Une partie des travaux présentés dans ce mémoire a été réalisée lorsque l'équipe Defi était localisée au Centre de Mathématiques Appliquées de l'École Polytechnique de 2014 à 2021. Je

remercie les collègues d'alors avec qui j'ai passé de bons moments : Vincent Bansaye, Charles Bertucci, Maxime Breden, Antonin Chambolle, Lucas Gerin, Igor Kortchemski, Aline Lefebvre-Lepot, Teddy Pichard, Gaël Raoul, Nicole Spillane et Amandine Véber. Durant cette période, j'ai pu participer à la mise en place d'une commission Développement Durable avec Mathilde, Giovanni, Pierre et Solange, ce fut une expérience très intéressante. Je n'oublie pas les grandes heures du Real Matrice et de l'AS Saint Hessienne en compagnie notamment d'Alain, Alex, Aymeric, Étienne, Fedor, Florian, Gustaw, Hélène, Lucas, Manon, Martin, Massil, Mathilde, Othmane, Rémi, Simona, ...

J'ai une pensée également pour les collègues du club de course à pied du CSX avec lesquels nous profitons de la nature de plus en plus malmenée du plateau de Saclay le midi. Mickael, Hakim, Julien, Vincent, Marie-Jean, Manu, Didier, ... à bientôt sur les chemins !

Bien d'autres personnes ont compté pour moi au cours de ces années. Je demande pardon à toutes celles et ceux que je ne mentionne pas.

Pour terminer, je souhaite remercier mes proches pour leur soutien indéfectible. Ma mère, mon père qui me manque, ma sœur Jeanne. Et bien entendu Bérangère ainsi que nos deux filles Suzanne et Bertille.

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GENERAL INTRODUCTION

In this introduction, I wish to describe the evolution of my research themes as well as the connections existing between the different topics.

I started my PhD supervised by Anne-Sophie Bonnet-Ben Dhia and Patrick Ciarlet at the Unité de Mathématiques Appliquées of Ensta in Paris in 2009. The initial objective was to investigate unusual transmission problems between two materials modeled by physical constants with different signs which appear in electromagnetism in the field of plasmonics and negative metamaterials. These problems do not satisfy classical positivity properties that are the basis of many techniques and therefore new methods must be developed for their study.

I was first interested in a scalar version of the problem and worked on a variational approach, the T-coercivity, to analyze it. Thanks to geometric arguments, I obtained new criteria ensuring its well-posedness in the Fredholm sense. I also considered the question of the justification of the convergence of finite element methods to approximate its solution when the latter exists. Then I investigated the full Maxwell's problem, which led me to discover and handle associated tools.

On the other hand, in certain configurations where the interface between the two materials is not smooth (presence of corners in 2D, conical tips, edges,... in 3D), curious black hole singularities can appear. The latter lie just outside of the classical spaces of functions with finite energy and make the problem ill-posed in the Hadamard sense (a continuous inverse operator cannot exist). For corners in 2D or conical tips in 3D, we have shown that it is necessary to take these black hole singularities into account in the functional framework and to impose a radiation condition at the singular point of the interface to restore well-posedness. There are strong connections between this problem and waveguide theory that I have better understood on this occasion. The rigorous justification of the well-posedness in the new setting, due to the change of sign of the coefficients, raises several difficulties. To overcome them, following what was proposed in particular by Monique Dauge during the PhD of Karim Ramdani, I have used the Kondratiev theory. This very powerful tool has been developed at the end of the sixties to describe the singularities of the solutions to elliptic problems in non smooth geometries.

During my PhD, we were contacted one day by Sergei A. Nazarov who proposed to visit our laboratory. At the time we had never met him and only knew him from his books and articles. It was a great opportunity for us because we were precisely reading some of his works explaining how to deal with black hole singularities similar to ours in elasticity and in water waves. Thanks to him, with Xavier Claeys we continued to learn about the Kondratiev theory and the subtleties of weighted Sobolev spaces. Sergei is also an experienced specialist in techniques of asymptotic analysis. He was for me the gateway to this world.

Some of the results of the PhD concerning plasmonics and negative metamaterials led to sev-

eral questions of the ANR project Metamath conducted by Sonia Fliss and which was accepted in 2011. In particular one of the aims was to reconsider the homogenization process at the interface between metamaterials and usual positive media to derive new refined models which are well-posed.

At the same period, Anne-Sophie realized that the problem we were studying shares similarities with the so-called Interior Transmission Problem (ITP) that appears in the justification of the Linear Sampling Method (LSM) in inverse problems. The goal of the LSM is to retrieve the support of defects in a reference medium from the measurement of scattered waves. Roughly speaking, it fails at frequencies such that there is a generalized incident wave which does not scatter. The ITP precisely characterizes those frequencies where one cannot use the LSM. We went to discuss this with Houssein Haddar at the Centre de Mathématiques Appliquées (CMAP) of École Polytechnique. It was through this ITP that I discovered some part of the community of inverse problems. At the end of my PhD, Houssein offered me the opportunity to visit Fioralba Cakoni at the University of Delaware. There, besides finishing writing my manuscript, I continued to learn about the recent developments concerning the ITP which was a very dynamic topic at the time.

I defended my PhD in October 2012 and then moved to Finland for a post-doctoral position at Aalto University in Helsinki to work with Nuutti Hyvönen. Nuutti is a specialist in the mathematics of the Electrical Impedance Tomography (EIT), an imaging technique used in particular in medicine. The goal of EIT is to reconstruct the electrical conductivity of an object from the measurement of potentials resulting from the injection of currents between electrodes placed on its surface. This led me to discover new aspects of inverse problems. During this post-doc, I also continued to collaborate actively with Sergei who was on a long-term stay at the University of Helsinki. I became more and more familiar with techniques of asymptotic analysis. We also started working on waveguide problems with classical materials. It was at this point that I became involved in the questions of invisibility. Finally, at Aalto University, I was able to discuss with Ari Sihvola and his group of physicists/engineers. It was very instructive to have a more concrete perspective on metamaterials.

I returned to France to CMAP as a Hadamard lecturer in October 2014. There I worked on my ongoing projects and continued my collaboration with Houssein Haddar. I obtained a permanent position of researcher at Inria in October 2015. Since that time, I have mainly investigated three lines of research:

- 1) *Problems with sign-changing coefficients;*
- 2) *Propagation of waves in waveguides;*
- 3) *Inverse problems related topics.*

For each of these themes, I have considered different directions. For example in theme 1), as part of Camille Carvalho's PhD, we showed how to use Perfectly Matched Layers at the corner points of non smooth interfaces to approximate the solution involving black hole singularities. Then we studied how to incorporate black hole singularities in the Maxwell framework during the PhD of Mahran Rihani. With Mahran, as well as Renata Bunoiu and Karim Ramdani, we also considered homogenization problems in presence of periodically distributed inclusions of negative material embedded into a dielectric. Besides, with Patrick Ciarlet, we supervised the Ensta 2A internship of Qingqing Hu (four months) concerning the convergence of finite element methods for the scalar problem.

In theme 2), with Sergei, we mainly studied how to use asymptotic analysis tools to explain physical phenomena like the Fano resonance or to produce devices that might be interesting for applications. Thus, during the post-doc of Jérémy Heleine as well as the Polytechnique 3A intern-

ship of Lorenzo Scaglione (four months), we showed for example how with thin resonant ligaments, one can hide defects in acoustic guides in the single-mode regime. Related to this theme 2), I had the chance to meet Vincent Pagneux, who is a researcher at the Laboratoire d'Acoustique de l'Université du Mans. Vincent excels in making connections between the applications that interest physicists and what might be relevant to study mathematically. With him and Anne-Sophie, we proposed a fairly simple method for determining in waveguides frequencies for which there is an incident field which is non reflecting. To proceed, we worked with conjugated Perfectly Matched Layers which leads to consider a non-selfadjoint eigenvalue problem. The latter, which can be $\mathcal{PT}$ -symmetric in certain situations, raises very interesting questions of spectral theory. Besides, with Anne-Sophie, we supervised the M2 internship (four months) and the PhD of Antoine Bera. The goal was to develop continuation methods to build large invisible defects in waveguides starting from the unperturbed situation. Recently, during the M2 internship (four months) and the PhD of Aurélien Parigaux, with Anne-Sophie and Sonia Fliss, we investigated the propagation of electromagnetic waves, governed by Maxwell's equations, in waveguides, in particular in heterogeneous cases where the permittivity and/or the permeability vary in the section. The next step will be to exploit this work to consider questions of invisibility in this context. Finally, with a new colleague in the Idefix team, Zoïs Moitier, we studied the possibility of achieving perfect transmission in 1D periodic waveguides during the Ensta 2A internship (four months) of Thomas Creuset.

Concerning theme 3), our Idefix team benefits from the experience of Lorenzo Audibert who is an engineer at EDF. Lorenzo brings us problems for applications that require to develop new interesting mathematics. In particular with him and Houssein, we studied a method to identify the interior transmission eigenvalues from the features of the eigenvalues of the scattering matrix. Besides, during the PhD of Kevish Napal, we developed a promising method for determining the properties of an unknown medium by working with the eigenvalues on an artificial ITP obtained by modifying the scattered field measurements with *ad hoc* computed quantities. In this topic related to inverse problems, I have worked on various aspects and formulations of the ITP. It was an opportunity to supervise the work of several students, namely the post-doc (five months) of Camille Carvalho, the M2 internship (four months) of Marwa Mansouri and the Ensta 2A internship (four months) of Jérémy Firozaly.

During all that time, I have continued to work regularly with the Kondratiev theory as well as with tools of spectral theory. In particular with Laurent Bourgeois and Sonia Fliss, with whom I have been collaborating more actively since our Idefix team moved from CMAP to UMA at Ensta, we have exploited these approaches to deal with problems appearing in plate theory as well as in the quasi-reversibility method, a technique to solve certain (ill-posed) Cauchy problems.

As I write these lines and take a step back, I realize that if rigorous justification constitutes my daily life and my main interest, applications are quite essential for me. I really like to couple mathematical formalization to the study of physical phenomena, in particular when they are counter-intuitive or surprising.

This work has led to the following publications:

◊ **Preprint:**

[152] L. Chesnel. A few techniques to achieve invisibility in waveguides. *arXiv preprint arXiv:2508.08403*, 2026.

◇ Post-PhD articles:

- [24] L. Audibert, L. Chesnel, and H. Haddar. Transmission eigenvalues with artificial background for explicit material index identification. *C. R. Acad. Sci. Paris, Ser. I*, 356(6):626–631, 2018.
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◇ Articles associated with the PhD:

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During my work, I have had the pleasure to work with the following students:

- Michele Zaccaron (Post-doc, 13 months, 2025)
- Jérémy Heleine (Post-doc, 18 months, 2020)
- Camille Carvalho (Post-doc, 5 months, 2016)
- Mathieu Chamaillard (Post-doc, 4 months, 2016)
- Aurélien Parigaux (PhD started in October 2022)
- Mahran Rihani (PhD started in November 2018, defended in February 2022)
- Antoine Bera (PhD started in October 2016, ended in April 2019)
- Kevish Napal (PhD started in October 2016, defended in December 2019)
- Aurélien Parigaux (M2 internship, 4 months, 2022)
- Marwa Mansouri (M2 internship, 4 months, 2019)
- Antoine Bera (M2 internship, 4 months, 2016)
- Kevish Napal (M2 internship, 4 months, 2016)
- Lorenzo Scaglione (École Polytechnique 3A internship, 4 months, 2020)
- Romain Hoang (CentraleSupélec, stage de césure, 4 months, 2026)
- Thomas Creuset (Ensta 2A internship, 4 months, 2025)
- Qingqing Hu (Ensta 2A internship, 4 months, 2020)
- Jérémy Firozaly (Ensta 2A internship, 4 months, 2012)

PROBLEMS WITH SIGN-CHANGING COEFFICIENTS

1.1 Context

Originally, my research was devoted to the study of transmission problems between positive and negative materials in electromagnetism. A material is characterized by its frequency response to periodic excitations. The dielectric permittivity ε and magnetic permeability μ determine this response. Neglecting dissipation, which is reasonable in certain situations, we say that a material is positive at frequency ω if $\varepsilon(\omega)$ and $\mu(\omega)$ are positive. On the other hand, a material is said negative at ω if at least one of the coefficients $\varepsilon(\omega)$, $\mu(\omega)$ is negative. In nature, the vast majority of materials have physical parameters that remain positive at all frequencies. Let us describe the negative materials we have in mind.

First, it has long been known that metals exhibit negative permittivities, neglecting losses, at optical frequencies. Due to the change of sign of the permittivity at the interface metal-dielectric, resonances called surface plasmons can appear [350, 428]. Over the past decades, surface plasmons attracted a great interest in guiding or confining light in nano-photonic devices [34, 240, 477]. However metals always have positive permeabilities. To obtain negative permeabilities, physicists and engineers, benefiting from advances in miniaturization, have started in the last twenty years to design negative metamaterials. These are artificial composite structures, made of periodic arrangements of resonators which are small compared to the wavelength (see Figure 1.1). In this activity, the main challenge is to choose cleverly the structure so that after a homogenization process, the resulting macroscopic material behaves as desired, for example as a material with both negative $\varepsilon(\omega)$ and $\mu(\omega)$ (see *e.g.* [94, 92, 91] for the mathematical justification).

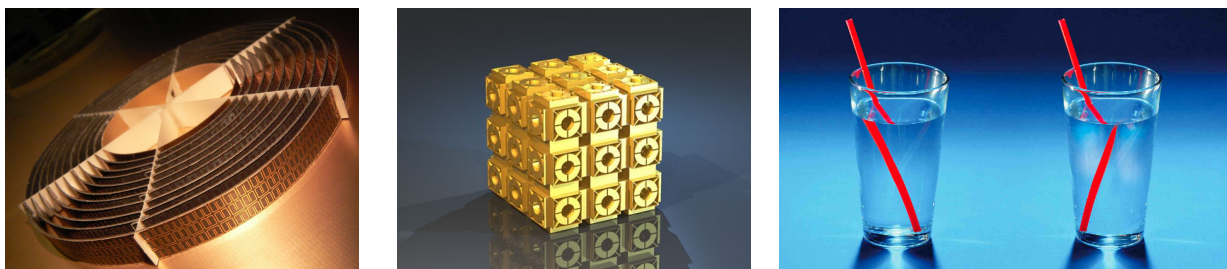


Figure 1.1: Left and center: examples of metamaterials. Pictures: D. Schurig, Duke University (left) and C. Soukoulis, Ames Laboratory (center). Right: illustration of the negative refraction phenomenon. The glass on the left is filled with conventional water, while the one on the right contains some imaginary liquid whose refractive index would be negative.

Structures that behave as if their permittivity and permeability were both negative possess the

original property of having a negative refraction index. At the interface between a conventional material and such a doubly-negative metamaterial, as imagined by Veselago in his pioneering article [467], an unusual phenomenon of negative refraction occurs: for a planar interface, a plane wave impinging from one side is refracted to the other side in a direction which is opposite to the standard one. The latter can be easily understood by writing the Snell-Descartes law that governs the path of light rays and by exploiting the change of sign of the index of refraction. This is the so-called negative refraction which may be useful to create perfect lenses [416] and which has given the name “negative index materials” (see an illustration with Figure 1.1 right).

Potential applications for negative materials abound. However they appear in complicated contexts where one often works at nanometric scales, at least when one wants to handle light. For the latter reason, the experimental development of these devices is very costly and there is a crucial need for numerical methods. Of course, it is extremely expensive to perform calculations on the real material considered in all its complexity. As a first approximation, one can replace the metamaterial by its homogenized version.

In practice, in order to create useful devices, it is important to combine negative materials and classical ones. Indeed, all interesting phenomena appear at interfaces. Mathematically, this leads to study unusual transmission problems with sign changing coefficients that constituted the core of my PhD. Let us mention that the Poems team has been a precursor in this field. My work came after the PhD of Karim Ramdani [432] defended in 1999 and the one of Carlo Maria Zwölf [480] defended in 2008. Apart from that, the corresponding references [80, 84, 77, 78] and the articles [198, 407, 206, 429, 408], the mathematical literature concerning these problems was not very developed at the time.

1.2 Variational tools for scalar problems

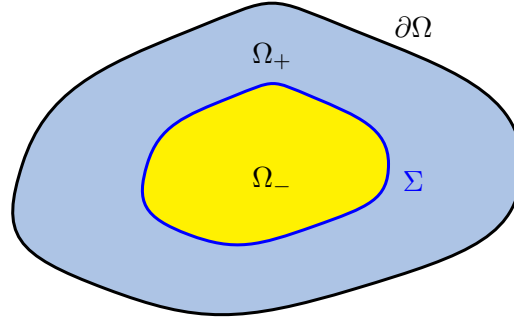


Figure 1.2: Domain Ω .

When Maxwell’s equations are invariant with respect to one variable, their study reduces to that of a 2D scalar problem of the form

$$\left\{ \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ -\operatorname{div}(\mu^{-1} \nabla u) - \omega^2 \varepsilon u = f. \end{array} \right. \quad (1.1)$$

The connected bounded domain Ω is partitioned into two sub-domains Ω_+ , Ω_- such that $\overline{\Omega} = \overline{\Omega_+} \cup \overline{\Omega_-}$ and $\Omega_+ \cap \Omega_- = \emptyset$ (see Figure 1.2). We assume that Ω , $\Omega_{\pm}$ have Lipschitz boundaries and we denote by Σ the interface such that $\Sigma := \partial\Omega_+ \cap \partial\Omega_-$. Moreover in (1.1) f is a given source term, say in $L^2(\Omega)$, and $H_0^1(\Omega)$ stands for the Sobolev space of functions of $H^1(\Omega)$ that vanish on $\partial\Omega$. Since the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, to study the well-posedness of (1.1) in the

Fredholm sense, it suffices to investigate the properties of the problem

$$\left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ -\operatorname{div}(\sigma \nabla u) = f. \end{array} \right. \quad (1.2)$$

In (1.2), the function σ is such that $\sigma|_{\Omega_{\pm}} = \sigma_{\pm}$, where $\sigma_{\pm}$ are constants satisfying $\pm\sigma_{\pm} > 0$. Note that we assume that σ is piecewise constant to simplify the presentation. This can be weakened, which is an advantage of the variational tools presented in this section, compared for example to the methods based on surface integral equations [198, 267, 132, 81, 268], and we can work with functions $\sigma \in L^\infty(\Omega)$ such that $\sigma^{-1} \in L^\infty(\Omega)$. The variational formulation associated with (1.2) writes

$$\left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u, v) = \ell(v), \quad \forall v \in H_0^1(\Omega), \end{array} \right. \quad (1.3)$$

where

$$a(u, v) = \int_{\Omega} \sigma \nabla u \cdot \nabla v \, d\mathbf{x}, \quad \ell(v) = \int_{\Omega} f v \, d\mathbf{x}.$$

With the Riesz representation theorem, introduce the bounded operator $A : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

$$(Au, v)_{H_0^1(\Omega)} = a(u, v), \quad \forall u, v \in H_0^1(\Omega). \quad (1.4)$$

Here and below, we use the notation

$$(u, v)_{H_0^1(\Omega)} = \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x}, \quad \forall u, v \in H_0^1(\Omega).$$

It is easy to find functions $u \in H_0^1(\Omega)$ such that $a(u, u) = 0$. Therefore $a(\cdot, \cdot)$ is not coercive and we cannot apply directly the Lax-Milgram theorem to study (1.3). To cope with this difficulty, the method of the T-coercivity has been introduced during the PhD of C.-M. Zwölf in [79].

1.2.1 The T-coercivity

The idea of the T-coercivity is simple. First, one observes that if T is an isomorphism of $H_0^1(\Omega)$, then (1.3) is equivalent to the problem

$$\left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u, Tv) = \ell(Tv), \quad \forall v \in H_0^1(\Omega). \end{array} \right. \quad (1.5)$$

Then all the work consists in trying to construct isomorphisms such that $a(\cdot, T\cdot)$ is coercive. If this is possible, the Lax-Milgram theorem implies that (1.5) and so (1.3) are well-posed.

During my PhD, we revisited the T-coercivity approach by constructing well-suited T thanks to geometric considerations. This has allowed us to obtain sharper criteria than the ones in [79] ensuring that (1.3) is well-posed. Let us explain how we proceeded in [66].

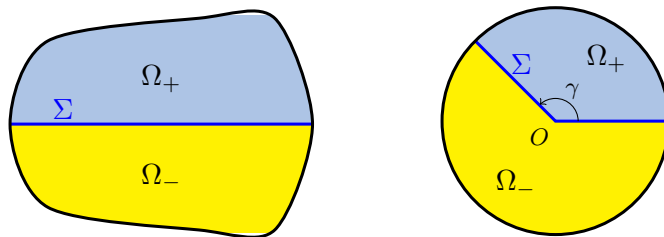


Figure 1.3: Left: symmetric setting. Right: interface with a corner.

Consider the symmetric setting of Figure 1.3 left where

$$\Omega \subset \mathbb{R}^2 \text{ is such that } \Omega_+ = \{(x, -y) \mid (x, y) \in \Omega_-\}. \quad (1.6)$$

Define the operator $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that for $u \in H_0^1(\Omega)$,

$$Tu = \begin{cases} u & \text{in } \Omega_+ \\ -u + 2Ru & \text{in } \Omega_- \end{cases} \quad (1.7)$$

where $Ru(x, y) = u(x, -y)$. Let us comment on this construction of T . First, we impose some “ $-u$ ” in Ω_- in order to compensate for the negative σ_- . But by doing so, we create some jump of trace at the interface Σ , which is not permitted in $H^1(\Omega)$. This is the reason why we add “ $2Ru$ ”, which guarantees that T is indeed valued in $H_0^1(\Omega)$. One can check that $T \circ T = \text{Id}$ which ensures that T is an involution and so an isomorphism of $H_0^1(\Omega)$. Besides, we find, for $u \in H_0^1(\Omega)$,

$$a(u, Tu) = \int_{\Omega} |\sigma| |\nabla u|^2 d\mathbf{x} + 2\sigma_- \int_{\Omega_-} \nabla u \cdot \nabla (Ru) d\mathbf{x}. \quad (1.8)$$

The first term in (1.8) is non-negative while the second one has no specific sign. Using Young’s inequality and the fact that the modulus of the Jacobian determinant is one, we can write, for all $\eta > 0$,

$$2 \left| \int_{\Omega_-} \nabla u \cdot \nabla (Ru) d\mathbf{x} \right| \leq \eta \int_{\Omega_-} |\nabla u|^2 d\mathbf{x} + \eta^{-1} \int_{\Omega_+} |\nabla u|^2 d\mathbf{x}. \quad (1.9)$$

Inserting (1.9) in (1.8), we get

$$a(u, Tu) \geq (\sigma_+ - \eta^{-1}|\sigma_-|) \int_{\Omega_+} |\nabla u|^2 d\mathbf{x} + (1 - \eta) |\sigma_-| \int_{\Omega_-} |\nabla u|^2 d\mathbf{x}. \quad (1.10)$$

Define the contrast

$$\kappa_{\sigma} := \frac{\sigma_-}{\sigma_+},$$

which is a negative constant. From (1.10), by taking $\eta \in (|\kappa_{\sigma}|; 1)$, we infer that if κ_{σ} satisfies $\kappa_{\sigma} > -1$, then $a(\cdot, T\cdot)$ is coercive. We deduce that A is an isomorphism of $H_0^1(\Omega)$ when $\kappa_{\sigma} > -1$. Working similarly with the operator $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that for $u \in H_0^1(\Omega)$,

$$Tu = \begin{cases} u - 2Ru & \text{in } \Omega_+ \\ -u & \text{in } \Omega_-, \end{cases} \quad (1.11)$$

R being the same as above, we prove that A is also an isomorphism when $\kappa_{\sigma} < -1$. On the other hand, one can prove that when $\kappa_{\sigma} = -1$, A has a kernel of infinite dimension. To proceed, consider some $\varphi \in \mathcal{C}_0^\infty(\Sigma)$, the space of infinitely differentiable functions compactly supported in Σ , and introduce $u_+ \in H^1(\Omega_+)$ the function satisfying

$$\Delta u_+ = 0 \text{ in } \Omega_+, \quad u_+ = \varphi \text{ on } \Sigma, \quad u_+ = 0 \text{ on } \partial\Omega_+ \setminus \bar{\Sigma}.$$

Then define u such that $u = u_{\pm}$ in $\Omega_{\pm}$ with $u_- := Ru_+$. We have $u \in H_0^1(\Omega)$ and

$$\begin{cases} \Delta u &= 0 & \text{in } \Omega_{\pm} \\ u_+ &= u_- & \text{on } \Sigma \\ \partial_y u_+ &= -\partial_y u_- & \text{on } \Sigma. \end{cases} \quad (1.12)$$

This is enough to guarantee that u belongs to $\ker A$ when $\sigma_+ = -\sigma_-$ (notice the third line of (1.12)), *i.e.* when $\kappa_{\sigma} = -1$. Since this is true for all $\varphi \in \mathcal{C}_0^\infty(\Sigma)$, this proves that $\dim \ker A = +\infty$ when $\kappa_{\sigma} = -1$. This allows us to write the following statement.

Theorem 1.2.1 *For the symmetric configuration defined in (1.6), A is an isomorphism of $H_0^1(\Omega)$ if and only if $\kappa_\sigma \neq -1$.*

Let us now consider the configuration of Figure 1.3 right. To describe it, introduce (r, θ) the usual polar coordinates such that $\mathbf{x} = (x, y) = (r \cos \theta, r \sin \theta)$ with $r \geq 0$, $\theta \in [0; 2\pi)$. For $\gamma \in (0; \pi)$, the aperture of the corner in the interface, set

$$\Omega := \{\mathbf{x} \mid r < 1\}, \quad \Omega_+ := \{\mathbf{x} \mid r < 1, 0 < \theta < \gamma\}, \quad \Omega_- := \{\mathbf{x} \mid r < 1, \gamma < \theta < 2\pi\}. \quad (1.13)$$

Define the operators $T_\pm : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

$$T_- u = \begin{cases} u & \text{in } \Omega_+ \\ -u + 2R_- u & \text{in } \Omega_- \end{cases}, \quad T_+ u = \begin{cases} u - 2R_+ u & \text{in } \Omega_+ \\ -u & \text{in } \Omega_- \end{cases}, \quad (1.14)$$

where

$$R_- u(r, \theta) = u\left(r, \frac{\gamma}{\gamma - 2\pi}(\theta - 2\pi)\right), \quad R_+ u(r, \theta) = u\left(r, \frac{\gamma - 2\pi}{\gamma}\theta + 2\pi\right). \quad (1.15)$$

By working as in the proof of Theorem 1.2.1 with the T defined in (1.14), we establish the following statement.

Theorem 1.2.2 *For the configuration (1.13) with a corner of aperture $\gamma \in (0; \pi)$ in the interface, A is an isomorphism of $H_0^1(\Omega)$ when $\kappa_\sigma \in (-\infty; 0) \setminus I_c$ where the critical interval I_c is given by*

$$I_c := \left[-\frac{2\pi - \gamma}{\gamma}; -\frac{\gamma}{2\pi - \gamma} \right]. \quad (1.16)$$

Let us make a few comments concerning this result.

- i) First, the case $\gamma \in (\pi; 2\pi)$ can be deduced from the above result by multiplying A by -1 and by exchanging the roles of Ω_+ and Ω_- .
- ii) Note that I_c always contains the value -1 . Moreover, when $\gamma = \pi$ (no corner in the interface), I_c reduces to $\{-1\}$, which is coherent with Theorem 1.2.1 because in that case we obtain the symmetric configuration described in (1.6).
- iii) For a right angle, i.e. $\gamma = \pi/2$, we have $I_c = [-3; -1/3]$.
- iv) When γ tends to zero, i.e. when the angle becomes very sharp, I_c tends to $(-\infty; 0)$.

Now we explain how to deal with more general interfaces. Let us first recall the definition of a Fredholm operator.

Definition 1.2.3 *Let X and Y be two Banach spaces, and let $L : X \rightarrow Y$ be a continuous linear map. The operator L is said to be a Fredholm operator if and only if the following two conditions are fulfilled*

- i) $\dim(\ker L) < +\infty$ and $\text{range } L$ is closed;
- ii) $\dim(\text{coker } L) < +\infty$ where $\text{coker } L := (Y/\text{range } L)$.

Besides, the index of a Fredholm operator L is defined by $\text{ind } L = \dim(\ker L) - \dim(\text{coker } L)$.

Assume that the interface Σ is of class $\mathcal{C}^1$ except at a finite number of points $S_1, \dots, S_N$ where it has corners (see Figure 1.4). For $j = 1, \dots, N$, we assume that in a neighborhood of S_j , Σ coincides with the union of two half-lines. Denote by $\gamma_j \in (0; 2\pi) \setminus \{\pi\}$ the aperture of the corner at S_j counted from Ω_- and set

$$\gamma = \min_{j=1, \dots, N} \{\gamma_j, 2\pi - \gamma_j\}, \quad (1.17)$$

as well as

$$I_c := \left[-\frac{2\pi - \gamma}{\gamma}; -\frac{\gamma}{2\pi - \gamma} \right] \cup \{-1\}. \quad (1.18)$$

Note that by convention, we simply have $I_c = \{-1\}$ when Σ has no corner.

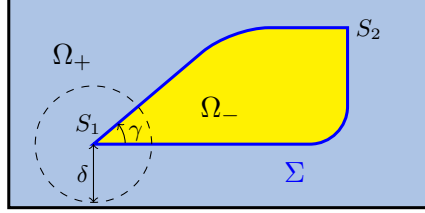


Figure 1.4: General configuration with a curvilinear polygonal interface.

To deal with Problem (1.3) in this case, we use techniques of localization. More precisely, by working with an adapted partition of unity together with the operators introduced in (1.7), (1.11), (1.14), when κ_σ satisfies $\kappa_\sigma \in (-\infty; 0) \setminus I_c$, we can construct a bounded linear map $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

$$A \circ T = \text{Id} + K,$$

where Id is the identity of $H_0^1(\Omega)$ and $K : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is compact. This shows that T is a right parametrix for A , *i.e.* an inverse modulo a compact operator. Since A is self-adjoint, because it is symmetric and bounded, this suffices to show that A is Fredholm of index zero. We summarize this result in the following statement.

Theorem 1.2.4 *For the general configuration described above (1.17), A is Fredholm of index zero when $\kappa_\sigma \in (-\infty; 0) \setminus I_c$ where I_c is defined in (1.18).*

Let us stress that the critical interval I_c reduces to $\{-1\}$ only when the interface Σ has no corner. Though the above statement has been obtained with a rather elementary method, it is actually optimal in the sense that the condition on κ_σ under which A is Fredholm cannot be improved. Let us explain why in the situation where Σ has a corner, say at O .

To understand what happens when κ_σ in I_c , it is necessary to compute the singularities of the problem. To set ideas, introduce $\delta > 0$ small enough so that

$$\Omega_- \cap B(O, \delta) = \{x \mid r < \delta, 0 < \theta < \gamma\}, \quad \Omega_+ \cap B(O, \delta) = \{x \mid r < \delta, \gamma < \theta < 2\pi\} \quad (1.19)$$

(see Figure 1.4). Here $B(O, \delta)$ stands for the open ball centered at O of radius δ . Singularities are defined as the functions of the form $\Psi(r)\phi(\theta)$, *i.e.* with separated variables in polar coordinates, which solve $\text{div}(\sigma \nabla(\Psi\phi)) = 0$ in $B(O, \delta)$. Since in this region σ depends only on θ , one finds that this is equivalent to have

$$\Psi(r) = A r^\lambda + B r^{-\lambda}$$

for some constants A, B , and (λ, ϕ) solution of the spectral problem

$$\begin{cases} \text{Find } \phi \in H_p^1(0; 2\pi) \setminus \{0\} \text{ such that} \\ -\partial_\theta(\sigma \partial_\theta \phi) = \lambda^2 \sigma \phi, \end{cases} \quad (1.20)$$

where $H_p^1(0; 2\pi) := \{\phi \in H^1(0; 2\pi) \mid \phi(0) = \phi(2\pi)\}$. Note that the spectral problem (1.20) is not standard because the sign-changing coefficient σ appears on both sides of the equation. In

particular, complex eigenvalues λ can exist. However, since this is a 1D problem, we can make explicit computations. The important point is that when

$$\kappa_\sigma \in \left(-\frac{2\pi - \gamma}{\gamma}; -\frac{\gamma}{2\pi - \gamma} \right) \setminus \{-1\}, \quad (1.21)$$

we find that (1.20) admits exactly two purely imaginary eigenvalues λ which are conjugated (see [206, 80]). This means that in this case, (1.3) has singularities of the form $r^{\pm i\eta}\phi(\theta)$ with $\eta > 0$ and $\phi \in H_p^1(0; 2\pi)$. Let us localize them by setting

$$s_\pm(\mathbf{x}) = \chi(r)r^{\pm i\eta}\phi(\theta) \quad (1.22)$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is a cut-off function such that $\chi = 1$ for $r \leq \delta/2$ and $\chi = 0$ for $r \geq \delta$. Note in particular that $s_\pm$ are just outside of $H^1(\Omega)$ in the sense that $s_\pm \notin H^1(\Omega)$ but $\mathbf{x} \mapsto r^\beta s_\pm \in H^1(\Omega)$ for all $\beta > 0$. To understand the first point, observe that

$$\int_\tau^1 |\partial_r(r^{i\eta})|^2 r dr = \eta^2 \int_\tau^1 r^{-1} dr = \eta^2 |\ln \tau| \xrightarrow{\tau \rightarrow 0^+} +\infty. \quad (1.23)$$

Now let us recall a classical lemma due to J. Peetre [415] (see also *e.g.* Theorem 12.12 in [474]).

Lemma 1.2.5 *Let X, Y, Z be three reflexive Banach spaces, such that X is compactly embedded into Z . Let $L : X \rightarrow Y$ be a continuous linear map. Then the assertions below are equivalent:*

- i) $\dim \ker L < +\infty$ and range L is closed in Y ;
- ii) there exists $C > 0$ such that $\|u\|_X \leq C(\|Lu\|_Y + \|u\|_Z)$, $\forall u \in X$.

From the $s_\pm$ in (1.22), we can construct a Weyl sequence and establish that there is no constant $C > 0$ such that

$$\|u\|_{H_0^1(\Omega)} \leq C(\|Au\|_{H_0^1(\Omega)} + \|u\|_{L^2(\Omega)}), \quad \forall u \in H_0^1(\Omega). \quad (1.24)$$

Combining this with Lemma 1.2.5, we deduce that A is not of Fredholm type when κ_σ satisfies (1.21). More precisely, by working with Sobolev spaces with weights at the origin, we can prove that A has a kernel of finite dimension (see after (1.31)). Therefore, for κ_σ as in (1.21), the operator A is not of Fredholm type because its range is not closed.

When $\kappa_\sigma = -(2\pi - \gamma)/\gamma$ or $\kappa_\sigma = -\gamma/(2\pi - \gamma)$, the reasoning and the result are similar but the singularities in (1.22) have a different form with some logarithmic/constant term in r . Finally, the case $\kappa_\sigma = -1$ is quite different. In this degenerated situation that we will not study more in the following, the loss of Fredholmness is not due to the existence of corner singularities but to what happens on the whole interface Σ . As we will see in (1.79), the well-posedness of (1.2) is directly related to the features of the difference of the Dirichlet-to-Neumann (DtN) maps associated with the operators $\sigma_\pm \Delta$ in $\Omega_\pm$. When $\sigma_+ = -\sigma_-$ and when Σ is flat (what we get by zooming at a curved interface), it is easy to see that these DtNs have the same symbol so that their difference is a regularizing operator which cannot be invertible. For references concerning this topic, we refer the reader to [407, 468, 40, 413, 116, 391, 394].

We summarize this discussion in the following statement.

Theorem 1.2.6 *For the general configuration described above (1.17), A is not of Fredholm type when $\kappa_\sigma \in I_c$ where I_c is defined in (1.18).*

In [66], simple 3D geometries like wedges and circular conical tips are also studied. In 3D however, there is a larger number of configurations and simple geometric arguments as above cannot always be used. For example, for a given conical tip which is not circular, it is not obvious with the above

approach how to obtain a criterion on the contrast ensuring that the problem is well-posed. On the other hand, for the so-called Fichera corner configuration represented in Figure 1.5, by working with symmetries, we have been able to prove that A is of Fredholm type when κ_σ does not belong to $[-7; -1/7]$ (note that 7 is the ratio blue volume/red volume). But other approaches, in particular one based on surface integral equations (see [269, 208]), suggest that this interval is not optimal. In other words, it seems that there are values of κ_σ inside $[-7; -1/7]$ such that A is of Fredholm type.

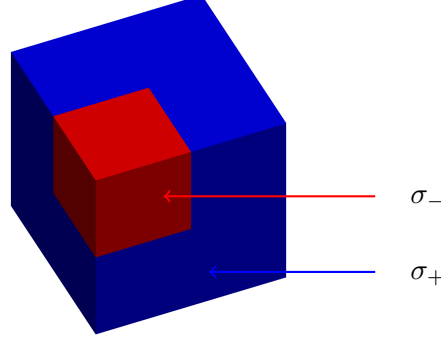


Figure 1.5: Fichera corner configuration.

Let us emphasize that when A is Fredholm of index zero, it may have a non zero kernel. This is a specificity of these problems with sign-changing coefficients which is used for certain applications. Indeed it allows one to construct small devices with small resonance frequencies, something which is impossible with standard (positive) materials. Let us mention that situations where A has a non zero kernel can be mathematically fully characterized. We will come back to this point in §1.6.2.

1.2.2 Numerical approximation

In [154], we show that the T-coercivity is equivalent to the *inf-sup* condition. In particular, we prove that A is an isomorphism if and only if there is some isomorphism T of $H_0^1(\Omega)$, not necessarily constructed via the geometric arguments above, such that $a(\cdot, T\cdot)$ is coercive. This is easy to establish. Indeed, we have already seen that if there is an isomorphism T of $H_0^1(\Omega)$ such that $a(\cdot, T\cdot)$ is coercive, then A is an isomorphism. Conversely, if A is an isomorphism, by observing that

$$a(u, Au) = (\nabla(Au), \nabla(Au))_\Omega = \|Au\|_{H_0^1(\Omega)}^2 \geq \|A^{-1}\|^{-2} \|u\|_{H_0^1(\Omega)}^2, \quad \forall u \in H_0^1(\Omega),$$

we infer that $a(\cdot, T\cdot)$ is coercive for $T = A$. Note that this is true for any continuous sesquilinear form $a(\cdot, \cdot)$ defined in some Hilbert space. For this reason, T-coercivity can be used to study a wide variety of problems, see *e.g.* [35, 181, 260].

In [154] however, our main goal is to study the classical discretization of (1.3) by finite element methods. This leads one to consider the problem

$$\left| \begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = \ell(v_h), \quad \forall v_h \in V_h, \end{array} \right. \quad (1.25)$$

where V_h is a classical Lagrange finite element space, of finite dimension, which approximates $H_0^1(\Omega)$. Here, as usual, h refers to the characteristic size of the mesh from which V_h is constructed. First, it is simple to prove that if there is some isomorphism T of $H_0^1(\Omega)$ such that $a(\cdot, T\cdot)$ is coercive and if V_h is such that $T(V_h) \subset V_h$, then u_h is uniquely defined and (u_h) converges to u as h tends to zero (basically, apply Céa's lemma to $a(\cdot, T\cdot)$). In [154], we explain how to relax this assumption on V_h (and so on the mesh). More precisely, we show that if the mesh satisfies certain “symmetry” properties, locally in a neighborhood of Σ , and related to the choice of the operator T in (1.7),

(1.11), (1.14), then, when A is an isomorphism, (1.25) has a unique solution for h small enough and the sequence $(u_h)_{h>0}$ converges to u , the solution of (1.3), as h tends to zero. As usual, the rate of convergence depends on the regularity of u and can become arbitrarily small for contrasts close to the critical interval. In [154], we also explain how to approximate u by u_h^τ , the solution of the problem

$$\left| \begin{array}{l} \text{Find } u_h^\tau \in V_h \text{ such that} \\ \int_{\Omega} (\sigma + i\tau) \nabla u_h^\tau \cdot \nabla \overline{v_h} d\mathbf{x} = \ell(v_h), \quad \forall v_h \in V_h, \end{array} \right. \quad (1.26)$$

where $\tau > 0$. Adding an imaginary part to σ in (1.26) can be seen as working with dissipative materials. For all $\tau > 0$, Problem (1.26) is well-posed because the corresponding sesquilinear form is coercive. We prove that by tuning τ with respect to h , we can ensure that (u_h^τ) converges to u as h goes to zero. Importantly, this works without constraint on the mesh. However the rate of convergence we obtain is only half of the optimal one we could hope for.

Let us mention that concerning the discretization of (1.3), several other approaches have been proposed since then, among them [397, 1, 63, 143, 258, 182, 2, 113, 259].

During the PhD of C. Carvalho [130], we considered spectral problems of the form

$$\left| \begin{array}{l} \text{Find } (\lambda, u) \in \mathbb{R} \times H_0^1(\Omega) \setminus \{0\} \text{ such that} \\ -\operatorname{div}(\sigma \nabla u) = \lambda \mu u \quad \text{in } \Omega. \end{array} \right. \quad (1.27)$$

When μ has a constant sign, for contrasts in σ outside of the critical interval I_c , the spectrum of (1.27) is discrete. With the *inf-sup* principle, one shows that it is made of two sequences of eigenvalues which converge to $\pm\infty$. In [131], we prove localization results of the corresponding eigenfunctions. More precisely, if μ is positive, the eigenfunctions localize more and more in Ω_+ (resp. Ω_-) as the index of the positive (resp. negative) eigenvalues tends to $+\infty$. This is illustrated with Figure 1.6. In [131], under some assumptions on the mesh similar to the ones of [154], we also explain how one can use classical finite element methods to approximate the spectrum of (1.27) as well as the eigenfunctions while avoiding spurious modes, including in situations with sign-changing σ and μ . To proceed, we reduce the analysis of (1.27) to the one of a compact operator and then apply the classical approach developed for instance in [30] by Babuška and Osborn.

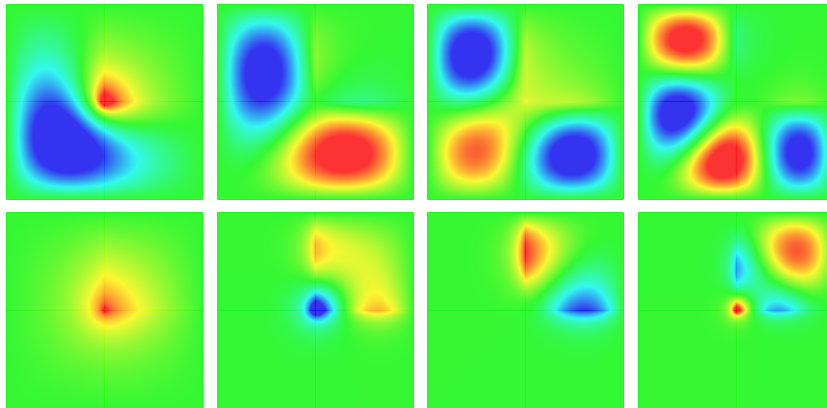


Figure 1.6: Top: eigenfunctions associated with the first four positive eigenvalues of (1.27). Bottom: eigenfunctions associated with the first four negative eigenvalues of (1.27). Here $\kappa_\sigma = -4$, $\mu \equiv 1$ in $\Omega = (-1; 1)^2$, $\Omega_- := (0; 1)^2$ and $\Omega_+ := \Omega \setminus \overline{\Omega_-}$.

1.3 New framework for the scalar problem

When the interface has a corner and for a contrast $\kappa_\sigma \in I_c \setminus \{-1\}$, we have seen that A is not of Fredholm type due to the existence of the non standard singularities $s_\pm$ introduced in (1.22). In this section, we explain how we can modify the functional framework to restore Fredholmness.

1.3.1 Analogy with a waveguide problem

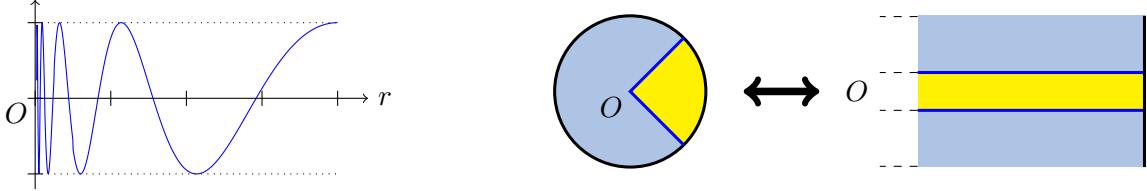


Figure 1.7: Left: real part of the radial behavior of the singularities $s_\pm$ introduced in (1.22) (note that $\Re e r^{\pm i\eta} = \cos(\eta \ln r)$). Right: schematic correspondence between the corner and the waveguide problem.

To help understanding, we highlight an analogy with a waveguide problem which appears naturally when considering the Euler change of variables

$$(r, \theta) \mapsto (\ln r, \theta) \quad (1.28)$$

in $B(O, \delta)$, where (r, θ) denote the polar coordinates centered at the corner and δ is introduced in (1.19). The map (1.28) is a smooth diffeomorphism from $(0; \delta) \times (0; 2\pi)$, the domain corresponding to $B(O, \delta) \setminus \{O\}$ in polar coordinates, to the semi-infinite strip $\mathcal{S} := (-\infty; \ln \delta) \times (0; 2\pi)$, the waveguide, which sends the corner O to infinity (see Figure 1.7 right). Its inverse is the map $(z, \theta) \mapsto (e^z, \theta)$. If u satisfies the equation $-\operatorname{div}(\sigma \nabla u) = f$ in $B(O, \delta) \setminus \{O\}$, i.e.

$$\begin{cases} -r^{-2}(\sigma(\theta)(r\partial_r)^2 + \partial_\theta(\sigma(\theta)\partial_\theta)) u(r, \theta) = f(r, \theta) & \text{in } (0; \delta) \times (0; 2\pi) \\ u(r, 0) = u(r, 2\pi) & \text{for } r < \delta \\ \partial_\theta u(r, 0) = \partial_\theta u(r, 2\pi) & \text{for } r < \delta, \end{cases} \quad (1.29)$$

then $\hat{u}$ such that $\hat{u}(z, \theta) = u(e^z, \theta)$ solves

$$\begin{cases} -(\sigma(\theta)\partial_z^2 + \partial_\theta(\sigma(\theta)\partial_\theta)) \hat{u}(z, \theta) = e^{2z} \hat{f}(z, \theta) & \text{in } \mathcal{S} \\ \hat{u}(z, 0) = \hat{u}(z, 2\pi) & \text{for } z < \ln \delta \\ \partial_\theta \hat{u}(z, 0) = \partial_\theta \hat{u}(z, 2\pi) & \text{for } z < \ln \delta. \end{cases} \quad (1.30)$$

Moreover, singularities of (1.29) of the form $r^{i\eta} \varphi(\theta)$ become propagating modes of (1.30) in the strip because we have

$$r^{\pm i\eta} \varphi(\theta) = e^{\pm i\eta \ln r} \varphi(\theta) = e^{\pm i\eta z} \varphi(\theta).$$

Thus for contrasts outside of the critical interval, from (1.20) we find that propagating modes do not exist for the waveguide problem (1.30). In this case H^1 is an adapted framework. On the contrary, for contrasts inside the critical interval, propagating modes exist and H^1 is not well-suited. However dealing with propagating waves is something which is well-known in waveguide theory. In this case, one looks for a solution which decomposes as a sum of evanescent modes plus a finite number of outgoing propagating modes. In particular, it is necessary to impose a radiation condition to select the outgoing behavior. This gives a natural framework to study (1.2) inside the critical interval: we have to look for a solution which combines one of the $s_\pm$ appearing in (1.22) and a remainder which decays at the corner. In the following, singularities (1.22) will be referred to as propagating singularities.

1.3.2 New framework

If from this formal analysis, the choice of the new functional framework is quite clear, the rigorous justification is a bit more complex. The main reason is that due to the change of sign of σ in (1.20), we are not able to show that the corresponding eigenfunctions form a Hilbert basis of $H_p^1(0; 2\pi)$ (see the definition in (1.20)), which prevents us from writing direct modal decompositions for the solutions of (1.29), something which is crucial in the study. During my PhD, with A.-S. Bonnet-Ben Dhia and X. Claeys, we established in [69] a similar result of modal decomposition by another approach, namely thanks to the Kondratiev theory (see [312] for the founding article or the monographs [382, 314, 352] for more recent presentations). Roughly speaking, instead of working with a basis in the transverse direction (θ variable here), we use the Mellin transform with respect to r which consists in making the Euler change of variables (1.28) and proceeding to a complex Fourier transform with respect to $z = \ln r$. Then with the residue theorem applied to the associated symbol, we are able to derive decompositions for the solutions of (1.22) as a singular part plus some remainder belonging to some weighted Sobolev space which allows us to control the decay at the corner.

Let us mention that this technique is more flexible than working with Hilbert bases. Indeed, it provides decompositions only for the solutions of the equation, which is sufficient for our needs, and not for all functions of the Hilbert space, which is much harder to get. This difficulty or impossibility to justify results of modal decompositions via the use of Hilbert bases appears in many other problems, like in elasticity for example. For this reason, the Kondratiev theory is a very interesting tool which is too little known, in particular in the physicists community.

In our study [69], we were more precisely inspired by the reviews [374, 371] concerning the theory of detached asymptotics in Kondratiev spaces as well as the papers [386, 378] where the authors consider problems of rotational cusps in water-waves and cuspidal geometries in elasticity for which singularities similar to ours appear. Note that after that, similar techniques have been used in [384] to study a Laplace operator with a degenerated Robin condition (see also [385]).

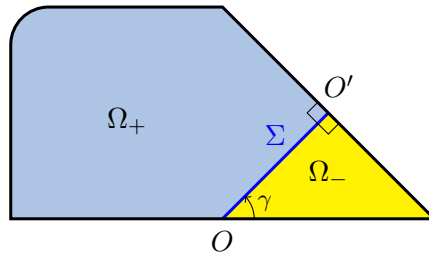


Figure 1.8: Half-corner geometry.

Let us describe the main ingredients of the method. To simplify the presentation, consider the 2D geometry of Figure 1.8 with a half-corner located at O . We call it a half-corner because if we proceed to a symmetry with respect to the line passing through O , we get a real corner. On the other hand, at O' no particular singularities appear because the interface Σ meets $\partial\Omega$ perpendicularly: if we make a local symmetry with respect to $\partial\Omega$, we obtain a flat interface. To set ideas, we assume that $\gamma \in (0; \pi/2)$. Again, (r, θ) denote the polar coordinates associated with O .

First, by using the variational tools of Section 1.2, we establish the following result.

Theorem 1.3.1 *The operator $A : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ defined in (1.4) is Fredholm of index zero if and only if $\kappa_\sigma \notin [-1; -\gamma/(\pi - \gamma)]$.*

Note that for this half-corner problem, we obtain a critical interval which is shorter than the one in (1.18) for the true corner. For the latter, ill-posedness for contrasts in $[-(\pi - \gamma)/\gamma; -1]$ is due to

the existence of propagating singularities for the problem in the geometry of Figure 1.8 where we impose a homogeneous Neumann boundary condition on the horizontal part of $\partial\Omega$.

Now we consider the case $\kappa_\sigma \in (-1; -\gamma/(\pi - \gamma))$. For $\beta \in \mathbb{R}$ and $m \in \mathbb{N}$, introduce the weighted Sobolev (Kondratiev) space $V_\beta^m(\Omega)$ defined as the closure of $\mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$ for the norm

$$\|\varphi\|_{V_\beta^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|r^{|\alpha|-m+\beta} \partial_x^\alpha \varphi\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Here $\mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$ is the space of infinitely differentiable functions which are supported in $\bar{\Omega} \setminus \{O\}$. We also denote by $\dot{V}_\beta^1(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ for the norm $\|\cdot\|_{V_\beta^1(\Omega)}$. We have the characterization

$$\dot{V}_\beta^1(\Omega) = \{\varphi \in V_\beta^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\}.$$

These weighted spaces will be useful to control the decaying part of the solution. In 2D, we can show the existence of some constant $C > 0$ such that

$$\|r^{-1}\varphi\|_{L^2(\Omega)} \leq C \|\nabla\varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega),$$

which proves that $\dot{V}_0^1(\Omega) = H_0^1(\Omega)$. Now set $\beta > 0$. Observe that we have

$$\dot{V}_{-\beta}^1(\Omega) \subset H_0^1(\Omega) \subset \dot{V}_\beta^1(\Omega),$$

which implies that their dual spaces satisfy $(\dot{V}_\beta^1(\Omega))^* \subset (H_0^1(\Omega))^* \subset (\dot{V}_{-\beta}^1(\Omega))^*$. Define the operators $A_{\pm\beta} : \dot{V}_{\pm\beta}^1(\Omega) \rightarrow (\dot{V}_{\mp\beta}^1(\Omega))^*$ such that

$$\langle A_{\pm\beta}\varphi, \varphi' \rangle = \int_\Omega \sigma \nabla\varphi \cdot \nabla\overline{\varphi'} d\mathbf{x}, \quad \forall \varphi \in \dot{V}_{\pm\beta}^1(\Omega), \varphi' \in \dot{V}_{\mp\beta}^1(\Omega), \quad (1.31)$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing $(\dot{V}_\beta^1(\Omega))^* \times \dot{V}_\beta^1(\Omega)$ or $(\dot{V}_{-\beta}^1(\Omega))^* \times \dot{V}_{-\beta}^1(\Omega)$ according to the case. By working with the Mellin transform with respect to the radial variable, as in (1.20), one is led to consider the spectral problem

$$\begin{cases} \text{Find } \phi \in H_0^1(0; \pi) \setminus \{0\} \text{ such that} \\ -\partial_\theta(\sigma \partial_\theta \phi) = \lambda^2 \sigma \phi. \end{cases} \quad (1.32)$$

One can show that there is $\beta_0 > 0$, depending only on κ_σ and γ , such that for all $\beta \in (0; \beta_0)$, A_β and $A_{-\beta}$ are Fredholm operators. Note that we have $\beta_0 = \min\{\Re \lambda \mid \lambda \text{ eigenvalue of (1.32) such that } \Re \lambda > 0\}$. Additionally, since $A_{-\beta}$ is the adjoint of A_β , we have

$$\text{ind} A_\beta = -\text{ind} A_{-\beta}. \quad (1.33)$$

We remind the reader that for a bounded linear operator between two Banach spaces $L : X \rightarrow Y$ whose range is closed, its index is defined as $\text{ind} L := \dim \ker L - \dim \text{coker} L$, with $\dim \text{coker} L = \dim(Y/\text{range}(L))$. On the other hand, by exploiting in particular that $s^\pm \in \dot{V}_\beta^1(\Omega) \setminus \dot{V}_{-\beta}^1(\Omega)$, one can prove that

$$\text{ind} A_\beta = \text{ind} A_{-\beta} + 2. \quad (1.34)$$

Gathering (1.33) and (1.34), we deduce that

$$\text{ind} A_\beta = 1 \quad \text{and} \quad \text{ind} A_{-\beta} = -1. \quad (1.35)$$

From (1.35), to obtain a Fredholm operator of index zero, we see that we have to construct an operator between $A_{-\beta}$ and A_β . To proceed, in accordance with the discussion at the beginning of

the section concerning the analogy with the waveguide problem, we add to the domain of $A_{-\beta}$ one singularity but not the other.

Thus, define the space

$$\mathring{V}^{\text{out}} := \text{span}(s^+) \oplus \mathring{V}_{-\beta}^1(\Omega) \quad (1.36)$$

where at this stage we choose arbitrarily to add s^+ instead of s^- (we come back to this point below). Endowed with the norm

$$\|\varphi\|_{V^{\text{out}}} = (|c|^2 + \|\tilde{\varphi}\|_{\mathring{V}_{-\beta}^1(\Omega)}^2)^{1/2}, \quad \forall \varphi = c s^+ + \tilde{\varphi} \in \mathring{V}^{\text{out}},$$

$\mathring{V}^{\text{out}}$ is a Banach space. Introduce the operator A^{out} such that for all $\varphi = c s^+ + \tilde{\varphi} \in \mathring{V}^{\text{out}}$ and $\varphi' \in \mathcal{C}_0^\infty(\Omega)$,

$$\langle A^{\text{out}} \varphi, \varphi' \rangle = \int_{\Omega} \sigma \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x} = -c \int_{\Omega} \text{div}(\sigma \nabla s^+) \overline{\varphi'} d\mathbf{x} + \int_{\Omega} \sigma \nabla \tilde{\varphi} \cdot \nabla \overline{\varphi'} d\mathbf{x}. \quad (1.37)$$

Note that due to the features of the cut-off function χ appearing in the definition of s^+ , we have $\text{div}(\sigma \nabla s^+) \in L^2(\Omega)$. And since $\text{div}(\sigma \nabla s^+) = 0$ in a neighborhood of O , we observe that there is a constant $C > 0$ such that $|\langle A^{\text{out}} \varphi, \varphi' \rangle| \leq C \|\varphi\|_{V^{\text{out}}} \|\varphi'\|_{V_{-\beta}^1(\Omega)}$. The density of $\mathcal{C}_0^\infty(\Omega)$ in $\mathring{V}_{-\beta}^1(\Omega)$ allows us to extend A^{out} as a continuous operator from $\mathring{V}^{\text{out}}$ to $(\mathring{V}_{-\beta}^1(\Omega))^*$. And we have

$$\langle A^{\text{out}} \varphi, \varphi' \rangle = \int_{\Omega} \sigma \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x}$$

where, for all $\varphi = c s^+ + \tilde{\varphi}$, $\varphi' \in \mathring{V}_{-\beta}^1(\Omega)$,

$$\int_{\Omega} \sigma \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x} = -c \int_{\Omega} \text{div}(\sigma \nabla s^+) \overline{\varphi'} d\mathbf{x} + \int_{\Omega} \sigma \nabla \tilde{\varphi} \cdot \nabla \overline{\varphi'} d\mathbf{x}. \quad (1.38)$$

Let us emphasize that this definition by density is necessary to give a sense to the first of the three integrals of (1.37) which is *a priori* not defined when $\varphi \in \mathring{V}^{\text{out}}$, $\varphi' \in \mathring{V}_{-\beta}^1(\Omega)$. Now we have everything to state the final result.

Theorem 1.3.2 *Assume that $\kappa_\sigma \in (-1; -\gamma/(\pi - \gamma))$. The operator $A^{\text{out}} : \mathring{V}^{\text{out}} \rightarrow (\mathring{V}_{-\beta}^1(\Omega))^*$ defined via (1.37) is Fredholm of index zero.*

Proof. Let us simply sketch the proof and assume to simplify that $A_{-\beta}$ is injective.

★ Let $\varphi = c s^+ + \tilde{\varphi}$, $c \in \mathbb{C}$, $\tilde{\varphi} \in \mathring{V}_{-\beta}^1(\Omega)$, be an element of $\ker A^{\text{out}}$. From (1.37), we find $\text{div}(\sigma \nabla \varphi) = 0$ in Ω in the distributional sense. Multiplying this equation by $\overline{\varphi}$ and integrating by parts in $\Omega \setminus \overline{B(O, \tau)}$ for $\tau > 0$ small, we find

$$0 = \int_{\Omega \setminus \overline{B(O, \tau)}} \sigma |\nabla \varphi|^2 d\mathbf{x} + \int_0^\pi \sigma \partial_r \varphi|_{r=\tau} \overline{\varphi}|_{r=\tau} \tau d\theta.$$

Taking the imaginary part above and the limit when $\tau \rightarrow 0$, exploiting also that $\tilde{\varphi} \in \mathring{V}_{-\beta}^1(\Omega)$, we obtain

$$0 = |c|^2 \eta \int_0^\pi \sigma(\theta) \phi^2(\theta) d\theta.$$

Besides, we can prove that there holds $\int_0^\pi \sigma(\theta) \phi^2(\theta) d\theta \neq 0$, which requires some work because of the presence of the sign-changing σ . Since $\eta > 0$ when $\kappa_\sigma \in (-1; -\gamma/(\pi - \gamma))$, we deduce that $c = 0$. Thus the above energy argument shows that φ belongs to $\ker A_{-\beta}$ and so we have $\varphi \equiv 0$.

★ Now consider some $f \in (\mathring{V}_{-\beta}^1(\Omega))^*$. Since A_β is the adjoint of $A_{-\beta}$, which is a Fredholm injective operator, we deduce that A_β is onto. As a consequence, there is $\varphi \in \mathring{V}_{-\beta}^1(\Omega)$ such that $A_\beta \varphi = f$.

Application of Kondratiev theory based in particular on the residue theorem (see [69, Thm. 5.2], [314, Thm. 5.4.2]) guarantees that if $\varphi \in \dot{V}_\beta^1(\Omega)$ is such that $A_\beta \varphi \in (\dot{V}_\beta^1(\Omega))^*$, the important point here being that $(\dot{V}_\beta^1(\Omega))^* \subset (\dot{V}_{-\beta}^1(\Omega))^*$, then there holds the following representation

$$\varphi = c^- s^- + c^+ s^+ + \tilde{\varphi} \quad \text{with } c^\pm \in \mathbb{C} \text{ and } \tilde{\varphi} \in \dot{V}_{-\beta}^1(\Omega).$$

On the other hand, since A_β has index one and is onto, we know that it admits a kernel of dimension one, generated by some

$$\varphi_\bullet = c_\bullet^- s^- + c_\bullet^+ s^+ + \tilde{\varphi}_\bullet \quad \text{with } c_\bullet^\pm \in \mathbb{C} \text{ and } \tilde{\varphi}_\bullet \in \dot{V}_{-\beta}^1(\Omega). \quad (1.39)$$

Working by contradiction and exploiting again the energy argument as above, one shows that $c_\bullet^- \neq 0$ and $c_\bullet^+ \neq 0$. Set

$$\varphi_{\text{sol}} = \varphi - (c^-/c_\bullet^-) \varphi_\bullet.$$

This function belongs to $\dot{V}^{\text{out}}$ and satisfies $A^{\text{out}} \varphi_{\text{sol}} = f$.

★ Finally A^{out} is a continuous operator which is both injective and onto, it is an isomorphism. ■

Let us comment this result.

i) To obtain a well-posed problem, we impose a radiation condition, incorporated in the space where we look for the solution, at the corner. It is important to understand that the corner here plays the role of infinity for classical scattering problems.

ii) We added in the space the singularity s^+ . We could have similarly worked with $\text{span}(s^-) \oplus \dot{V}_{-\beta}^1(\Omega)$ and would also have obtained an isomorphism. And more generally, we have an isomorphism when we work in $\text{span}(\alpha^- s^- + \alpha^+ s^+) \oplus \dot{V}_{-\beta}^1(\Omega)$ if and only if $(\alpha^-, \alpha^+) \in \mathbb{C}^2$, such that $|\alpha^-| + |\alpha^+| \neq 0$, is not proportional to the pair $(c_\bullet^-, c_\bullet^+)$ appearing in (1.39). Note that $\text{span}(\alpha^- s^- + \alpha^+ s^+)$ (of dimension one) is different from $\text{span}(s^-, s^+)$ (of dimension two).

iii) To decide which singularity we should add in the space, we use classically the limiting absorption principle. More precisely, we consider a slightly dissipative medium, which corresponds to adding a small uniformly positive imaginary part to the sign-changing coefficient as in (1.26). This allows us to obtain a well-posed problem in $H_0^1(\Omega)$. The physically relevant solution without dissipation is then defined as the limit of the solution with dissipation when the dissipation tends to zero. Note that due to the sign-changing coefficient, the selection of the outgoing behavior, $r^{i\eta}$ or $r^{-i\eta}$, cannot be deduced from the sign of η . Instead, one uses energy considerations: the outgoing propagating singularity is the one which carries energy to the corner. Going back to time-domain, for $\omega \neq 0$, the corresponding wave propagates to the corner but its velocity tends to zero so that it never reaches it. For this reason, it is sometimes called black-hole wave.

iv) For $\kappa_\sigma = -\gamma/(\pi - \gamma)$, the integral $\int_0^\pi \sigma(\theta) \phi^2(\theta) d\theta$ vanishes which prevents us from computing flux of energy at the corner point. In that case, the above approach should be adapted.

After this study, during the PhD of C. Carvalho, we extended the theory described above to a more realistic scattering problem by a bounded inclusion of negative material in free space at a frequency $\omega > 0$. In 2D, this led us to consider the system

$$\left| \begin{array}{l} \text{Find } u = u^{\text{inc}} + u^{\text{sca}} \text{ such that} \\ \text{div} \left(\varepsilon^{-1} \nabla u \right) + \omega^2 \mu u = 0 \quad \text{in } \mathbb{R}^2 \\ \lim_{\xi \rightarrow +\infty} \int_{|x|=\xi} \left| \frac{\partial u^{\text{sca}}}{\partial r} - i\omega u^{\text{sca}} \right|^2 ds = 0, \end{array} \right. \quad (1.40)$$

where u^{inc} is the incident field which satisfies $\Delta u^{\text{inc}} + \omega^2 u^{\text{inc}} = 0$ in $\mathbb{R}^2$ and the last line corresponds to the classical radiation condition at infinity. Adding adapted radiation conditions at the corners of the interface for contrasts $\kappa_\varepsilon \neq -1$ inside the critical interval, we showed in [62] that (1.40)

admits a unique solution. Note that uniqueness of the solution always holds here and can be proven thanks to Rellich's lemma (see *e.g.* [119, Thm. 3.5]) and Holmgren's theorem [277, Thm. 8.6.5] (see also [119, Lem. 5.23]).

1.3.3 An adapted numerical method

But the main goal of [62] was to consider the question of the approximation of the solution in this new functional framework. This is not trivial because the highly oscillating behavior cannot be captured by classical finite element methods: when refining the mesh, one observes some reflections of the propagating singularities and the numerical solution does not converge when the meshsize h tends to zero (see Figure 1.9).

Remark 1.3.3 *Note that this instability phenomenon is not so simple to exhibit in practice. The reason is that the propagating singularity is $2\pi/\eta$ -periodic in $\ln r$ scale. Therefore, roughly speaking, the mesh must be refined by a factor $e^{2\pi/\eta}$ to get a similar picture. Note for example that for $\eta = 1$, we have $e^{2\pi/\eta} \approx 535$ and so if the mesh is only slightly refined, the corresponding numerical solution does not change much (though it is incorrect). One way to enhance the instability is to work with contrasts close to -1 to have large values of η .*

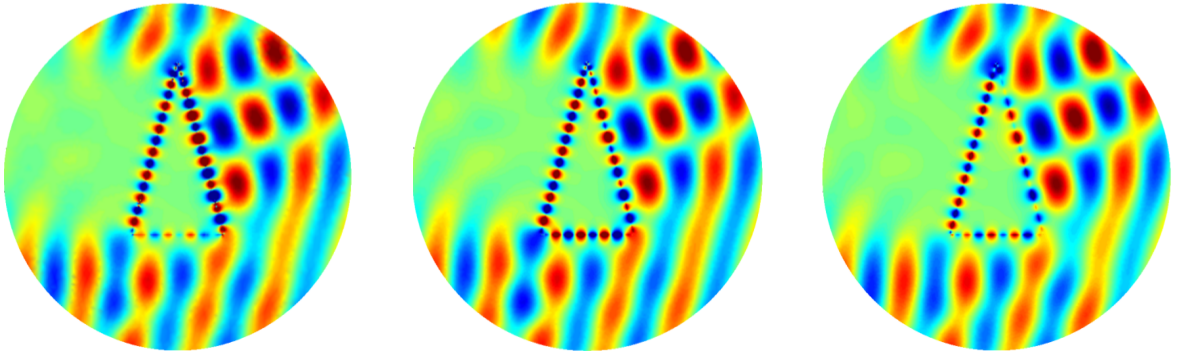


Figure 1.9: Solution obtained using a standard P2 finite element method for different meshes: from left to right, 13273 nodes, 56031 nodes and 100501 nodes. Here we consider the scattering of an incident plane wave coming from the right at non zero frequency by a triangular inclusion of negative material. Note that the computed field is not stable at the interface when the mesh is refined.

In [62] we presented an original method to approximate the solution of (1.40) when the contrast κ_ε belongs to the critical interval I_c . To proceed, we again exploited the analogy with the waveguide problem described in (1.28)–(1.30). The main difficulty is that when $\kappa_\varepsilon \in I_c$, there are propagating modes for Problem (1.30) in the semi-infinite strips obtained by zooming at the corners and considering the Euler change of variables (1.28). For this reason, for numerics, it is important to bound cleverly the domain to avoid spurious reflections on the artificial boundaries. But these are well-known problems in waveguide theory and different techniques exist. One of them consists in using a Perfectly Matched Layer (PML) [51, 38]. It consists in making a complex change of variable with respect to z in the strips. For the new problem one obtains, the solution is exponentially decaying in the strips, which correspond to small neighborhoods of the corners, while its restriction to the rest of the domain coincides with the initial solution. Then numerically when truncating the strips, one only gets exponentially small reflections, which yields a good approximation. In Figures 1.10, 1.11, we provide a few results of the method. Let us emphasize that it boils down to implementing PMLs at the corner points, which is something quite unusual. Let us mention also that we really mesh and solve the problem in the unfolded strip. To proceed, we match the fields on the dashed lines appearing in Figure 1.10. This way, we obtain a logarithmic mesh in the initial geometry (see Figure 1.12 right) which allows us to capture well the singularity.

In practice, this approach yields a robust technique to approximate the plasmonic wave propagating at the interface. Concerning the justification of the method, several questions remain open. First, one needs to control the error made when truncating the PMLs. This is not straightforward because of the change of sign of ε . However, using again Kondratiev spaces and working for example like in [291], one can reasonably hope to establish such a result. The problem of the justification of the convergence of the finite element methods seems more intricate. Without PMLs and outside of the critical interval, as we have seen above, the existing proofs require assumptions on meshes (see [154, 63]) and the question of knowing whether or not these assumptions are necessary is not solved. Here, due to the complex scaling $z \mapsto z/\alpha$ of the PMLs (typically $\alpha = e^{i\pi/4}$), it is not even clear how to work with the T-coercivity at the discrete level.

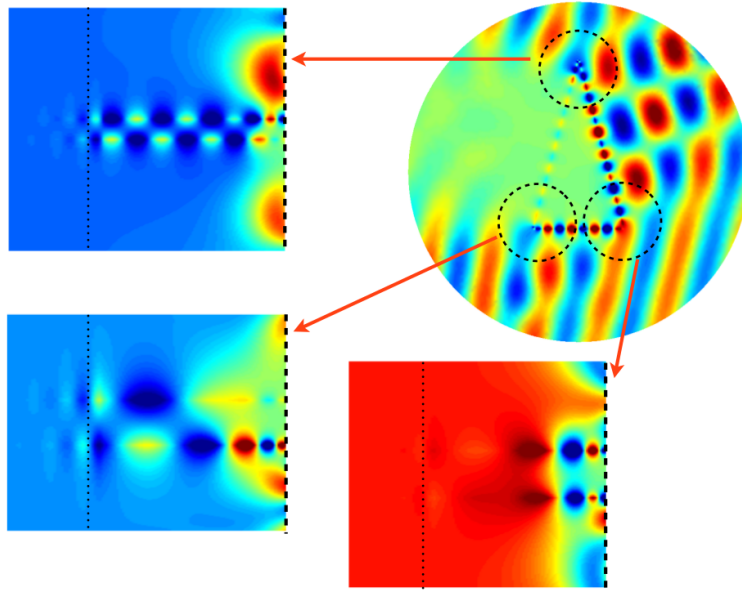


Figure 1.10: Numerical solution with PMLs at the corners. As in Figure 1.9, the incident wave is coming from the right. The bold dashed lines correspond to the interfaces where matching is made, the small dotted lines represent the beginning of the PML regions.

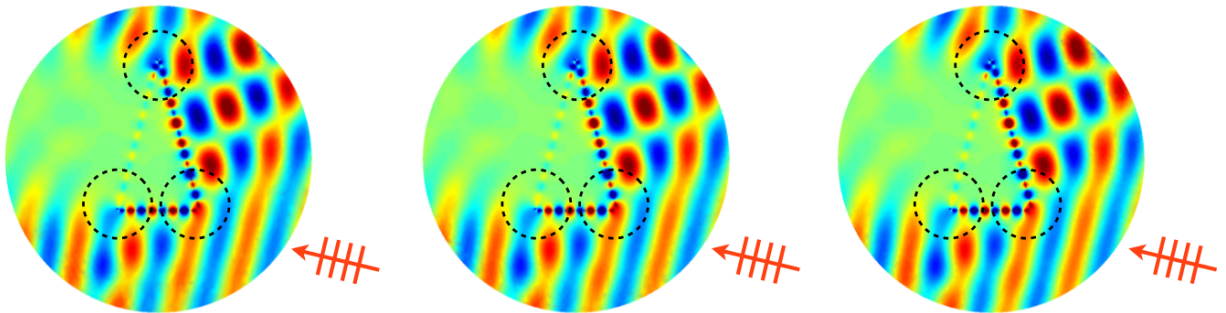


Figure 1.11: Same setting as in Figure 1.10 for three different meshes. When h tends to zero, from left to right, the numerical solution seems to converge. Note that we reconstitute the field inside the three disks using the transform $(z, \theta) \mapsto (e^z, \theta)$.

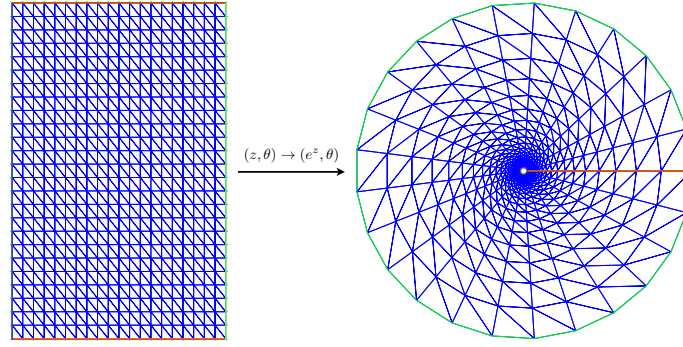


Figure 1.12: Left: mesh of the bounded strip. Right: resulting mesh after the change of variables $(z, \theta) \mapsto (e^z, \theta)$.

1.4 Maxwell's equations

1.4.1 Maxwell's equations in classical spaces

During my PhD, we also investigated Maxwell's equations with sign-changing coefficients. This leads one to study variational formulations such as (1.47) below set in some Maxwell's space $\mathbf{X}_N(\varepsilon)$. In the analysis, we face two obstacles. First, as in the scalar case, the principal part is not coercive when μ is sign-changing. However, for Maxwell's problem, it is not easy to work with symmetries around the interface to restore positivity due to the constraints for fields in $\mathbf{X}_N(\varepsilon)$. Second, we have to prove some compactness result for the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ in order to establish well-posedness in the Fredholm sense. Note that in the scalar case, this is straightforward because $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ when Ω is bounded. For Maxwell's problem, the space $\mathbf{X}_N(\varepsilon)$ depends on ε , which may be sign-changing.

To address these difficulties, first we proved a compactness result for the spaces appearing naturally in 2D in [153]. This analysis was based on a detailed study of the singularities again thanks to the Kondratiev theory. Then in [69], we used another approach and explained how to develop an efficient T-coercivity approach to study Maxwell's equations in 2D. There we showed their well-posedness when the contrasts for the scalar problems lie outside of the corresponding critical intervals (see also [195, 196] for another method based on volume integral equations). In [67], we studied the 3D problem. To state the main result, define the space

$$H_{\#}^1(\Omega) := \{\varphi \in H^1(\Omega) \mid \int_{\Omega} \varphi \, d\mathbf{x} = 0\}$$

and with the Riesz representation theorem, introduce the bounded operators $A^\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$, $A^\mu : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$ such that

$$(\nabla(A^\varepsilon \varphi), \nabla \varphi')_{\Omega} = \int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} \, d\mathbf{x}, \quad \forall \varphi, \varphi' \in H_0^1(\Omega) \quad (1.41)$$

$$(\nabla(A^\mu \varphi), \nabla \varphi')_{\Omega} = \int_{\Omega} \mu \nabla \varphi \cdot \nabla \overline{\varphi'} \, d\mathbf{x}, \quad \forall \varphi, \varphi' \in H_{\#}^1(\Omega). \quad (1.42)$$

In [67] we established that when A^ε , A^μ are of Fredholm type, then Maxwell's operators in the usual space are also of Fredholm type. In other words, all the analysis boils down to the one of the properties of A^ε , A^μ which can be done with the geometric T-coercivity as seen in Section 1.2. Let us explain how we showed this result.

Let Ω be a bounded domain of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. To simplify, we

assume that Ω is simply connected and that $\partial\Omega$ is connected. For a given frequency $\omega > 0$, the time-harmonic Maxwell's equations are given by

$$\mathbf{curl} \mathbf{E} - i\omega\mu\mathbf{H} = 0 \quad \text{and} \quad \mathbf{curl} \mathbf{H} + i\omega\varepsilon\mathbf{E} = \mathbf{J} \quad \text{in } \Omega. \quad (1.43)$$

Above $\mathbf{E}$ and $\mathbf{H}$ are respectively the electric and magnetic components of the electromagnetic field while $\mathbf{J}$ stands for the current density. We suppose that Ω is surrounded by a perfect conductor and we impose the boundary conditions

$$\mathbf{E} \times \nu = 0 \quad \text{and} \quad \mu\mathbf{H} \cdot \nu = 0 \quad \text{on } \partial\Omega. \quad (1.44)$$

Here ν denotes the unit outward normal vector to $\partial\Omega$. The dielectric permittivity ε and the magnetic permeability μ in (1.43)–(1.44) are assumed to be real valued functions such that $\varepsilon, \mu \in L^\infty(\Omega)$ and $\varepsilon^{-1}, \mu^{-1} \in L^\infty(\Omega)$. However their signs can change in Ω . As it is classical in the study of Maxwell's equations, we introduce the spaces

$$\begin{aligned} \mathbf{L}^2(\Omega) &:= (\mathbf{L}^2(\Omega))^3 \\ \mathbf{H}(\mathbf{curl}) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_N(\mathbf{curl}) &:= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}) \mid \mathbf{u} \times \nu = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

In (1.43), the current density belongs to $\mathbf{L}^2(\Omega)$ and is such that $\text{div } \mathbf{J} = 0$ in Ω . This leads us to introduce the spaces

$$\begin{aligned} \mathbf{X}_N(\xi) &:= \{\mathbf{u} \in \mathbf{H}_N(\mathbf{curl}) \mid \text{div}(\xi\mathbf{u}) = 0\} \\ \mathbf{X}_T(\xi) &:= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}) \mid \text{div}(\xi\mathbf{u}) = 0, \xi\mathbf{u} \cdot \nu = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

where ξ refers to a generic function of $L^\infty(\Omega)$ such that $\xi^{-1} \in L^\infty(\Omega)$. We endow $\mathbf{H}(\mathbf{curl})$, $\mathbf{H}_N(\mathbf{curl})$, $\mathbf{X}_N(\xi)$, $\mathbf{X}_T(\xi)$ with the inner product

$$(\cdot, \cdot)_{\mathbf{H}(\mathbf{curl})} := (\cdot, \cdot)_\Omega + (\mathbf{curl} \cdot, \mathbf{curl} \cdot)_\Omega.$$

Classically, one proves that if $(\mathbf{E}, \mathbf{H})$ satisfies (1.43)–(1.44), then $\mathbf{E}$ solves the second order problem

$$\left| \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that} \\ \mathbf{curl}(\mu^{-1}\mathbf{curl} \mathbf{E}) - \omega^2\varepsilon\mathbf{E} = i\omega\mathbf{J} \quad \text{in } \Omega. \end{array} \right. \quad (1.45)$$

Conversely, if $\mathbf{E}$ satisfies (1.45), then $(\mathbf{E}, \mathbf{H} = (i\omega\mu)^{-1}\mathbf{curl} \mathbf{E})$ solves (1.43)–(1.44). The variational formulation associated with (1.45) writes

$$\left| \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that} \\ \int_\Omega \mu^{-1}\mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \mathbf{E}' - \omega^2\varepsilon\mathbf{E} \cdot \mathbf{E}' \, dx = i\omega \int_\Omega \mathbf{J} \cdot \mathbf{E}' \, dx, \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}). \end{array} \right. \quad (1.46)$$

The analysis of (1.46) is not simple because the embedding of $\mathbf{H}_N(\mathbf{curl})$ in $\mathbf{L}^2(\Omega)$ is not compact (consider a sequence made of the gradients of an orthonormal basis of $\mathbf{H}_0^1(\Omega)$). As usual in the study of Maxwell's equations, we have to take into account the divergence free condition which is contained in (1.45). More precisely, one finds that if $\mathbf{E}$ satisfies (1.46) then $\mathbf{E}$ solves

$$\left| \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{X}_N(\varepsilon) \text{ such that} \\ \int_\Omega \mu^{-1}\mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \mathbf{E}' - \omega^2\varepsilon\mathbf{E} \cdot \mathbf{E}' \, dx = i\omega \int_\Omega \mathbf{J} \cdot \mathbf{E}' \, dx, \quad \forall \mathbf{E}' \in \mathbf{X}_N(\varepsilon). \end{array} \right. \quad (1.47)$$

Conversely, assume that $\mathbf{E}$ satisfies (1.47). If the operator $A^\varepsilon : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$ introduced in (1.41) is an isomorphism, for $\mathbf{E}' \in \mathbf{H}_N(\mathbf{curl})$, we can introduce $\varphi \in \mathbf{H}_0^1(\Omega)$ such that

$$\int_\Omega \varepsilon \nabla \varphi \cdot \nabla \varphi' \, dx = \int_\Omega \varepsilon \mathbf{E}' \cdot \nabla \varphi' \, dx, \quad \forall \varphi' \in \mathbf{H}_0^1(\Omega). \quad (1.48)$$

Then $\mathbf{E}' - \nabla\varphi$ is an element of $\mathbf{X}_N(\varepsilon)$ and so according to (1.47), there holds

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} (\mathbf{E}' - \nabla\varphi) - \omega^2 \varepsilon \mathbf{E} \cdot (\mathbf{E}' - \nabla\varphi) dx = i\omega \int_{\Omega} \mathbf{J} \cdot (\mathbf{E}' - \nabla\varphi) dx. \quad (1.49)$$

By exploiting that $\mathbf{E} \in \mathbf{X}_N(\varepsilon)$ and that $\mathbf{curl} \nabla\varphi = \operatorname{div} \mathbf{J} = 0$ in Ω , we deduce from (1.49) that

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \mathbf{E}' - \omega^2 \varepsilon \mathbf{E} \cdot \mathbf{E}' dx = i\omega \int_{\Omega} \mathbf{J} \cdot \mathbf{E}' dx.$$

Since this holds for all $\mathbf{E}' \in \mathbf{H}_N(\mathbf{curl})$, this is enough to conclude the equivalence of (1.46) and (1.47) when A^ε is an isomorphism.

Again, in (1.47), the principal part is not coercive when μ is sign-changing and we have to prove some compactness result for the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$. Both points will be treated thanks to the T-coercivity but with an approach different from the geometric one developed for the scalar problem (see (1.7)) which is inefficient to deal with Maxwell's equations. The following lemma plays a central role in the analysis.

Lemma 1.4.1 *Assume that Ω is simply connected with a connected boundary and that ε, μ are such that the operators A^ε, A^μ defined in (1.41), (1.42) are isomorphisms. Then there exists a bounded operator $\mathbb{T} : \mathbf{X}_N(\varepsilon) \rightarrow \mathbf{X}_N(\varepsilon)$ such that there holds*

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} (\mathbb{T} \mathbf{E}') dx = \int_{\Omega} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \overline{\mathbf{E}'} dx, \quad \forall \mathbf{E}, \mathbf{E}' \in \mathbf{X}_N(\varepsilon). \quad (1.50)$$

Proof. Consider some $\mathbf{E}' \in \mathbf{X}_N(\varepsilon)$. To annihilate the μ^{-1} in the left-hand side of (1.50), we would like to have $\mathbb{T}$ such that $\mathbf{curl} (\mathbb{T} \mathbf{E}') = \mu \mathbf{curl} \overline{\mathbf{E}'}$. By taking the divergence on both sides, we see that in general this is not possible. To circumvent this problem, let us work with gradients.

i) Since A^μ is an isomorphism, we can introduce $\zeta \in H_{\#}^1(\Omega)$ such that

$$\int_{\Omega} \mu \nabla \zeta \cdot \nabla \overline{\zeta'} dx = \int_{\Omega} \mu \mathbf{curl} \mathbf{E}' \cdot \nabla \overline{\zeta'} dx, \quad \forall \zeta' \in H_{\#}^1(\Omega).$$

ii) Then one observes that $\mu(\mathbf{curl} \mathbf{E}' - \nabla \zeta)$ is a divergence free element of $\mathbf{L}^2(\Omega)$ such that $\mu(\mathbf{curl} \mathbf{E}' - \nabla \zeta) \cdot \nu = 0$ on $\partial\Omega$. Since Ω is simply connected with a connected boundary, according to Theorem 3.17 in [11] (see also Theorem 3.6 in [238]), there exists a unique potential $\mathbf{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl} \mathbf{u} = \mu(\mathbf{curl} \mathbf{E}' - \nabla \zeta). \quad (1.51)$$

iii) Besides, since A^ε is an isomorphism, we can introduce $\varphi \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} dx = \int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \overline{\varphi'} dx, \quad \forall \varphi' \in H_0^1(\Omega).$$

Finally, we define the bounded operator $\mathbb{T} : \mathbf{X}_N(\varepsilon) \rightarrow \mathbf{X}_N(\varepsilon)$ such that, for $\mathbf{E}' \in \mathbf{X}_N(\varepsilon)$,

$$\mathbb{T} \mathbf{E}' = \mathbf{u} - \nabla \varphi.$$

For all $\mathbf{E}, \mathbf{E}' \in \mathbf{X}_N(\varepsilon)$, using in particular (1.51), we find

$$\begin{aligned} \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} (\mathbb{T} \mathbf{E}') dx &= \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} (\overline{\mathbf{u}} - \nabla \overline{\varphi}) dx \\ &= \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \overline{\mathbf{u}} dx \\ &= \int_{\Omega} \mathbf{curl} \mathbf{E} \cdot (\mathbf{curl} \overline{\mathbf{E}'} - \nabla \overline{\zeta}) dx = \int_{\Omega} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \overline{\mathbf{E}'} dx. \end{aligned}$$

This ends the proof. ■

Using Lemma 1.4.1 and the fact that the embedding of $\mathbf{X}_T(1)$ in $\mathbf{L}^2(\Omega)$ is compact, cf. [472, 11, 361] (note there is no sign-changing coefficient in the definition of $\mathbf{X}_T(1)$), we prove the following result.

Proposition 1.4.2 *Assume that Ω is simply connected with a connected boundary and that ε is such that the operator A^ε defined in (1.41) is an isomorphism. Then the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.*

With the Riesz representation theorem, for $\omega > 0$ define the bounded operator $\mathbb{A}_N(\omega) : \mathbf{X}_N(\varepsilon) \rightarrow \mathbf{X}_N(\varepsilon)$ such that,

$$(\mathbb{A}_N(\omega)\mathbf{E}, \mathbf{E}')_{\mathbf{H}(\mathbf{curl})} = \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \overline{\mathbf{E}'} - \omega^2 \varepsilon \mathbf{E} \cdot \overline{\mathbf{E}'} dx, \quad \forall \mathbf{E}, \mathbf{E}' \in \mathbf{X}_N(\varepsilon).$$

Combining Lemma 1.4.1 and Proposition 1.4.2, we establish the following theorem.

Theorem 1.4.3 *Assume that Ω is simply connected with a connected boundary and that ε, μ are such that the operators A^ε, A^μ defined in (1.41), (1.42) are isomorphisms. Then $\mathbb{A}_N(\omega)$ is an isomorphism for all $\omega > 0$ except possibly for a discrete set of values which accumulate only at $+\infty$.*

Let us give a concrete example of application. Consider the Fichera corner configuration represented in Figure 1.5 and assume that ε, μ are constant in $\Omega_{\pm}$. In [66], we proved that A^ε, A^μ are isomorphisms when the contrasts in ε and in μ do not belong to $[-7; -1/7]$. From Theorem 1.4.3 we infer that $\mathbb{A}_N(\omega)$ is an isomorphism for all $\omega > 0$ except possibly for a discrete set of values under the same constraints for ε, μ .

As mentioned above, we can slightly generalize the statement of Theorem 1.4.3 by considering the case where A^ε, A^μ are only Fredholm of index zero. Note however that this still excludes transmission problems with piecewise constant ε, μ such that the contrast in ε or in μ belongs to the corresponding critical interval. These configurations will be considered in §1.4.2.

Let us conclude this paragraph by showing that in 2D the operators A^ε and A^μ are closely related. To avoid any ambiguity, here we denote by $A_0^\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ (resp. $A_\#^\mu : H_\#^1(\Omega) \rightarrow H_\#^1(\Omega)$) instead of A^ε (resp. A^μ) the operator defined in (1.41) (resp. (1.42)). The following statement ([69, Thm. 4.6]) is useful to study the so-called Transverse Electric and Transverse Magnetic problems in 2D.

Proposition 1.4.4 *Assume that $\Omega \subset \mathbb{R}^2$ is simply connected and that $\varepsilon \in L^\infty(\Omega)$ is such that $\varepsilon^{-1} \in L^\infty(\Omega)$. Then A_0^ε is an isomorphism if and only if $A_\#^{1/\varepsilon}$ is an isomorphism.*

Proof. Suppose that A_0^ε is an isomorphism. For $\varphi, \varphi' \in H_\#^1(\Omega)$, remark that in 2D we have

$$(\varepsilon^{-1} \nabla \varphi, \nabla \varphi')_\Omega = (\varepsilon^{-1} \mathbf{curl} \varphi, \mathbf{curl} \varphi')_\Omega,$$

where classically for a function φ defined in a 2D domain, $\mathbf{curl} \varphi = (\partial_y \varphi, -\partial_x \varphi)^\top$. To show that $A_\#^{1/\varepsilon}$ is an isomorphism with the T-coercivity approach, mimicking the proof of Lemma 1.4.1, we would like to construct $\mathbf{T} : H_\#^1(\Omega) \rightarrow H_\#^1(\Omega)$ such that $\mathbf{curl}(\mathbf{T}\varphi) = \varepsilon \mathbf{curl} \varphi$ in Ω . By taking the divergence on both sides, we see that this is impossible in general. Again let us play with gradients. More precisely, for $\varphi \in H_\#^1(\Omega)$, since A_0^ε is an isomorphism, we can introduce $\zeta \in H_0^1(\Omega)$ such that

$$(\varepsilon \nabla \zeta, \nabla \zeta')_\Omega = (\varepsilon \mathbf{curl} \varphi, \nabla \zeta')_\Omega, \quad \forall \zeta' \in H_0^1(\Omega).$$

Observing that $\operatorname{div}(\varepsilon(\mathbf{curl} \varphi - \nabla \zeta)) = 0$ in Ω , from [238, Thm. 3.2] we know that there is a unique element of $H_\#^1(\Omega)$, that we denote by $\mathbf{T}\varphi$, such that

$$\mathbf{curl}(\mathbf{T}\varphi) = \varepsilon(\mathbf{curl} \varphi - \nabla \zeta) \quad \text{in } \Omega.$$

This defines a continuous operator $T : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$. Then for $\varphi \in H_{\#}^1(\Omega)$, we find

$$\begin{aligned} (\nabla(A_{\#}^{1/\varepsilon}\varphi), \nabla(T\varphi))_{\Omega} &= (\varepsilon^{-1}\nabla\varphi, \nabla(T\varphi))_{\Omega} = (\varepsilon^{-1}\mathbf{curl}\varphi, \mathbf{curl}(T\varphi))_{\Omega} \\ &= (\mathbf{curl}\varphi, \mathbf{curl}\varphi - \nabla\zeta)_{\Omega} = (\mathbf{curl}\varphi, \mathbf{curl}\varphi)_{\Omega} = \|\nabla\varphi\|_{\Omega}^2, \end{aligned}$$

which shows that $(\nabla(A_{\#}^{1/\varepsilon}\cdot), \nabla(T\cdot))_{\Omega}$ is coercive. From the Lax-Milgram theorem, this is enough to conclude that the selfadjoint operator $A_{\#}^{1/\varepsilon}$ is an isomorphism.

We establish in a very similar way that A_0^{ε} is an isomorphism when $A_{\#}^{1/\varepsilon}$ is an isomorphism. $\blacksquare$

Remark 1.4.5 *With this result, for the transmission problem (1.2) in the case where the interface has a corner located inside Ω , we are not surprised to obtain a critical interval of the form $[-(2\pi - \gamma)/\gamma; -\gamma/(2\pi - \gamma)]$ (see (1.16)). Indeed, by working with cut-off functions supported in Ω and equal to one in a neighborhood of the interface, it is easy to show that Fredholmness for A^{ε} is equivalent to Fredholmness for A^{μ} in that situation. This proposition also explains the discussion following Theorem 1.3.1 concerning singularities for problems with “half corners” in the interface.*

1.4.2 Maxwell's equations with contrasts inside the critical intervals

During the PhD of M. Rihani [434] that I supervised with A.-S. Bonnet-Ben Dhia, one of our goals was to study the 3D Maxwell's equations (1.43)–(1.44) when A^{ε} or A^{μ} is not of Fredholm type. Let us detail this.

Assumptions for ε, μ

Let $\Omega \subset \mathbb{R}^3$ be a simply connected domain with connected Lipschitz continuous boundary $\partial\Omega$. Consider a situation where Ω contains an inclusion Ω_- such that $\overline{\Omega_-} \subset \Omega$. We assume that $\partial\Omega_-$ is of class $\mathcal{C}^2$ except at the origin O where Ω_- coincides locally with a conical tip. More precisely, there are $\delta > 0$ and some smooth domain ϖ of the unit sphere $\mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = 1\}$ such that $B(O, \delta) \subset \Omega$ and

$$\Omega_- \cap B(O, \delta) = \mathcal{K} \cap B(O, \delta) \quad \text{with } \mathcal{K} := \{r\boldsymbol{\theta} \mid r > 0, \boldsymbol{\theta} \in \varpi\}.$$

Here $B(O, \delta)$ stands for the open ball of $\mathbb{R}^3$ centered at O and of radius δ . We assume that ε takes the constant value $\varepsilon_- < 0$ (resp. $\varepsilon_+ > 0$) in $\Omega_- \cap B(O, \delta)$ (resp. $\Omega_+ \cap B(O, \delta)$ where $\Omega_+ := \Omega \setminus \overline{\Omega_-}$). And we assume that the contrast $\kappa_{\varepsilon} := \varepsilon_-/\varepsilon_+ < 0$ and ϖ (which characterizes the geometry of the conical tip) are such that there exist singularities of the form

$$\mathbf{s}(\mathbf{x}) = r^{-1/2+i\eta}\Phi(\mathbf{x}/|\mathbf{x}|) \quad (1.52)$$

satisfying $\operatorname{div}(\varepsilon\nabla\mathbf{s}) = 0$ in $\mathcal{K}$ with $\eta \in \mathbb{R} \setminus \{0\}$. Here $r := |\mathbf{x}|$ while Φ is a function which is smooth in ϖ and in $\mathbb{S}^2 \setminus \overline{\varpi}$. These $\mathbf{s}$ are the 3D equivalent of the 2D singularities introduced in (1.22) (note that the power in r depends on the dimension of the space). Similarly, they are just outside of H^1 in the sense that $\mathbf{s} \notin H^1(\Omega)$ but $r^{\beta}\mathbf{s} \in H^1(\Omega)$ for all $\beta > 0$. To show the first point, observe that

$$\int_{\tau}^1 |\partial_r(r^{-1/2+i\eta})|^2 r^2 dr = |-1/2 + i\eta|^2 \int_{\tau}^1 r^{-1} dr = |-1/2 + i\eta|^2 |\ln \tau| \xrightarrow{\tau \rightarrow 0^+} +\infty.$$

With them, as in (1.24), we can show that the operator $A^{\varepsilon} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ defined in (1.41) is not Fredholm because its range is not closed.

For circular conical tips as in Figure 1.13, *i.e.* when

$$\varpi = \{(\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi) \mid -\pi \leq \theta \leq \pi, -\pi/2 \leq \phi < -\pi/2 + \gamma\} \text{ for some } \gamma \in (0; \pi), \quad (1.53)$$

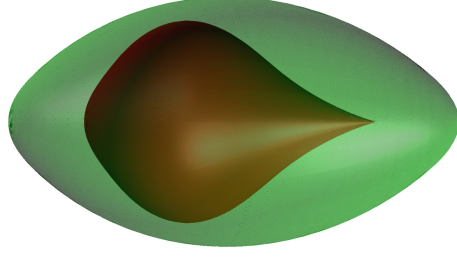


Figure 1.13: Configuration with a circular conical tip at the interface.

by adapting some results of [297, 270, 336], it can be shown that such $\mathfrak{s}$ exists for $\kappa_\varepsilon \in (-1; -I_\gamma)$ (resp. $\kappa_\varepsilon \in (-I_\gamma; -1)$) when $\gamma < \pi/2$ (resp. $\gamma > \pi/2$). Here I_γ is the constant defined by

$$I_\gamma := \frac{{}_2F_1(1/2, 1/2, 1, \cos^2(\gamma/2)) {}_2F_1(3/2, 3/2, 2, \sin^2(\gamma/2))}{{}_2F_1(1/2, 1/2, 1, \sin^2(\gamma/2)) {}_2F_1(3/2, 3/2, 2, \cos^2(\gamma/2))} > 0,$$

where ${}_2F_1$ stands for the Gauss's hypergeometric function. Note that we have $I_\gamma = 1/I_{\pi-\gamma}$ and $I_\gamma \in (0; 1)$ for $\gamma \in (0; \pi/2)$. Additionally, there holds for example $I_{\pi/4} \approx 0.218$ as well as $\lim_{\gamma \rightarrow \pi/2} I_\gamma = 1$, $\lim_{\gamma \rightarrow 0^+} I_\gamma = 0^+$, $\lim_{\gamma \rightarrow \pi^-} I_\gamma = +\infty$. Observe that contrary to what we got in 2D for a corner in an interface inside Ω , the interval is not of the form $(-c; -1/c)$ for some $c > 1$.

For a general smooth domain $\varpi \subset \mathbb{S}^2$ and a given contrast κ_ε , in order to know if such $\mathfrak{s}$ exist, one has to solve the spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda, \Phi) \in \mathbb{C} \times H^1(\mathbb{S}^2) \setminus \{0\} \text{ such that} \\ \int_{\mathbb{S}^2} \varepsilon \nabla_S \Phi \cdot \nabla_S \overline{\Phi'} ds = \lambda(\lambda + 1) \int_{\mathbb{S}^2} \varepsilon \Phi \overline{\Phi'} ds, \quad \forall \Phi' \in H^1(\mathbb{S}^2), \end{array} \right. \quad (1.54)$$

which is the 3D equivalent of (1.20), and see if among the eigenvalues some of them are of the form $\lambda = -1/2 + i\eta$ with $\eta \in \mathbb{R}, \eta \neq 0$. Above, ∇_S stands for the surface gradient. With a slight abuse, when ε is involved into integrals over $\mathbb{S}^2$, we write ε instead of $\varepsilon(\delta \cdot)$. Again in (1.54), the difficulty lies in the fact that ε appears on both sides of the equation. This time, the problem is set on the sphere. Note that since ε is real-valued, if $\lambda = -1/2 + i\eta$ is an eigenvalue, we have $\lambda(\lambda + 1) = -\eta^2 - 1/4$, so that $\lambda = -1/2 - i\eta$ is also an eigenvalue for the same eigenfunction. And since $\lambda(\lambda + 1) \in \mathbb{R}$, we can find a corresponding eigenfunction which is real-valued. From now on, we assume that $\lambda = -1/2 + i\eta$ is a simple eigenvalue of (1.54) (geometric multiplicity equal to one). Let us mention that this problem of existence of singularities of the form (1.52) is directly related to the problem of existence of essential spectrum for the so-called Neumann-Poincaré operator [299, 420, 89, 270]. A noteworthy difference with the 2D case of a corner in the interface is that several singularities of the form (1.52) with different values of $|\eta|$ can exist in 3D (this depends on ε and on ϖ). This is a result that we showed in [297] with some colleagues physicists of the Aalto University during my post-doc in Finland. To simplify the presentation here, we assume that for the case of interest, singularities of the form (1.52) exist for only one value of $|\eta|$. Moreover we assume that the quantity $\eta \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds$ does not vanish. In this case, exchanging η by $-\eta$ if necessary, we can set η so that

$$\eta \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds > 0.$$

Under this assumption, one can check that $\lambda = -1/2 + i\eta$ is of algebraic multiplicity equal to one. For the circular conical tip introduced in (1.53), say for $\gamma < \pi/2$, we find that the above assumptions are satisfied for contrasts $\kappa_\varepsilon \in (-I_\gamma^\dagger; -I_\gamma)$ with a certain $I_\gamma^\dagger \in (I_\gamma; 1)$. For $\kappa_\varepsilon \in (-1; -I_\gamma^\dagger)$, the number of propagating singularities is larger than two (counting $\pm$). Observe that once the geometry has been fixed, all the assumptions appearing above correspond to a condition on ε .

To fix notations, we set

$$s^\pm(\mathbf{x}) = \chi(r) r^{-1/2 \pm i\eta} \Phi(\mathbf{x}/|\mathbf{x}|),$$

where again $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is a cut-off function such that $\chi = 1$ for $r \leq \delta/2$ and $\chi = 0$ for $r \geq \delta$. In Figure 1.14, we give an illustration of $\Im m s^+|_{\Omega_-}$ in a case where Ω_- coincides in a neighborhood of O with a circular conical tip.

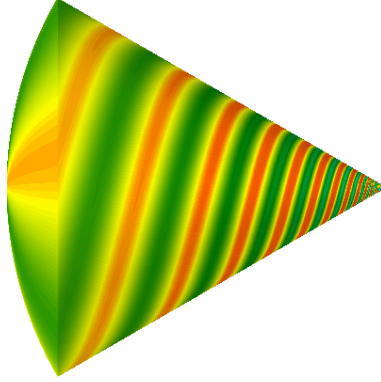


Figure 1.14: Example of singularity for a circular conical tip. For the sake of clarity, we simply represent its restriction to Ω_- .

Similarly to what has been shown in Section 1.3, we can modify the functional framework to recover a Fredholm operator. As in (1.36), set

$$\mathring{V}^{\text{out}} := \text{span}(s^+) \oplus \mathring{V}_{-\beta}^1(\Omega)$$

and define the operator $A^{\varepsilon, \text{out}} : \mathring{V}^{\text{out}} \rightarrow (\mathring{V}_{\beta}^1(\Omega))^*$ such that for all $\varphi = c s^+ + \tilde{\varphi} \in \mathring{V}^{\text{out}}$ and $\varphi' \in \mathring{V}_{\beta}^1(\Omega)$,

$$\langle A^{\varepsilon, \text{out}} \varphi, \varphi' \rangle = \oint_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x} \quad (1.55)$$

where, as in (1.38),

$$\oint_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x} := -c \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{\varphi'} d\mathbf{x} + \int_{\Omega} \varepsilon \nabla \tilde{\varphi} \cdot \nabla \overline{\varphi'} d\mathbf{x}.$$

Theorem 1.4.6 *The operator $A^{\varepsilon, \text{out}} : \mathring{V}^{\text{out}} \rightarrow (\mathring{V}_{\beta}^1(\Omega))^*$ defined in (1.55) is Fredholm of index zero.*

To simplify, additionally we assume that $A^{\varepsilon, \text{out}}$ is injective. Note that for a waveguide problem with propagating modes, this corresponds to assuming that so-called trapped modes do not exist.

On the other hand, we assume that μ is such that the scalar operator $A^{\mu} : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$ defined in (1.42) is an isomorphism. In other words, ε is critical while μ is not (situations with both critical ε and μ are discussed in the comments).

With these assumptions, we explain in [75] how one should change the functional framework to obtain a well-posed Maxwell's problem. To set ideas, we will work on the electric problem.

An adapted space for the electric field

For contrasts κ_{ε} outside of the critical interval, when we proved the equivalence between the formulations in $\mathbf{H}_N(\mathbf{curl})$ and in $\mathbf{X}_N(\varepsilon)$, we used a result of isomorphism of A^{ε} (see (1.48)). Here, instead, we will exploit that $A^{\varepsilon, \text{out}}$ is an isomorphism. This invites us to work with the following definition of the space of divergence free electric fields

$$\mathbf{X}_N^{\text{out}}(\varepsilon) := \{\mathbf{u} = c \nabla s^+ + \tilde{\mathbf{u}}, c \in \mathbb{C}, \tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega), \text{div}(\varepsilon \mathbf{u}) = 0 \text{ in } \Omega \setminus \{O\}, \mathbf{u} \times \nu = 0 \text{ on } \partial\Omega\}.$$

Note that in $\mathbf{X}_N^{\text{out}}(\varepsilon)$ the fields are singular but their $\mathbf{curl}$ are classically in $\mathbf{L}^2(\Omega)$. For $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}}$, the condition $\text{div}(\varepsilon \mathbf{u}) = 0$ in $\Omega \setminus \{O\}$ means that there holds

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \varphi \, d\mathbf{x} = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega \setminus \{O\}),$$

which after integration by parts and by density of $\mathcal{C}_0^\infty(\Omega \setminus \{O\})$ in $H_0^1(\Omega)$ is equivalent to

$$-c \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \varphi \, d\mathbf{x} + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega). \quad (1.56)$$

Note that we have $\mathbf{X}_N(\varepsilon) \subset \mathbf{X}_N^{\text{out}}(\varepsilon)$ and we can show that $\dim(\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)) = 1$. For $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}}$ with $c \in \mathbb{C}$ and $\tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega)$, we set

$$\|\mathbf{u}\|_{\mathbf{X}_N^{\text{out}}(\varepsilon)} = (|c|^2 + \|\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{curl} \, \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}.$$

Endowed with this norm, $\mathbf{X}_N^{\text{out}}(\varepsilon)$ is a Banach space. Moreover one can prove the following regularity result.

Lemma 1.4.7 *Assume that ε is as in §1.4.2. There is $\beta > 0$ such that for any $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$, we have $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$ and there is a constant $C > 0$ independent of $\mathbf{u}$ such that*

$$|c| + \|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl} \, \mathbf{u}\|_{\mathbf{L}^2(\Omega)}.$$

Up to the end of that section, we assume that β is fixed once and for all by this lemma.

Definition of the problem for the electric field

Our objective is to define the problem for the electric field as a variational formulation set in $\mathbf{X}_N^{\text{out}}(\varepsilon)$. For some $\tau > 0$, let $\mathbf{J}$ be an element of $\mathbf{V}_{-\tau}^0(\Omega)$ such that $\text{div} \, \mathbf{J} = 0$ in Ω . Consider the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \text{ such that} \\ \int_{\Omega} \mu^{-1} \mathbf{curl} \, \mathbf{u} \cdot \mathbf{curl} \, \bar{\mathbf{v}} \, d\mathbf{x} - \omega^2 \oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x} = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon), \end{array} \right. \quad (1.57)$$

where the term

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x} \quad (1.58)$$

has to be carefully defined. The difficulty comes from the fact that $\mathbf{X}_N^{\text{out}}(\varepsilon)$ is not a subspace of $\mathbf{L}^2(\Omega)$ so that this quantity cannot be considered as a classical integral.

Let $\mathbf{u} = c_u \nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. First, for $\tilde{\mathbf{v}} \in \mathbf{V}_{-\beta}^0(\Omega)$, it is natural to set

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x} := \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x}. \quad (1.59)$$

To complete the definition, we have to give a sense to (1.58) when $\mathbf{v} = \nabla s^+$. We start from the identity

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} \, d\mathbf{x} = -c_u \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \bar{\varphi} \, d\mathbf{x} + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi} \, d\mathbf{x}, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega \setminus \{O\}).$$

By density of $\mathcal{C}_0^\infty(\Omega \setminus \{O\})$ in $\dot{\mathbf{V}}_\beta^1(\Omega)$, this leads to set

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} \, d\mathbf{x} := -c_u \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \bar{\varphi} \, d\mathbf{x} + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi} \, d\mathbf{x}, \quad \forall \varphi \in \dot{\mathbf{V}}_\beta^1(\Omega).$$

With this definition, from (1.56) we obtain

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \varphi \, d\mathbf{x} = 0, \quad \forall \varphi \in \mathring{V}_{\beta}^1(\Omega).$$

In particular, since $s^+ \in \mathring{V}_{\beta}^1(\Omega)$, for all $\mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ we have

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \nabla s^+ \, d\mathbf{x} = 0 \quad \text{and so} \quad \oint_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla s^+ \, d\mathbf{x} = c_u \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) s^+ \, d\mathbf{x}. \quad (1.60)$$

Finally for all $\mathbf{u} = c_u \nabla s^+ + \tilde{\mathbf{u}}$ and $\mathbf{v} = c_v \nabla s^+ + \tilde{\mathbf{v}}$ in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, using (1.59) and (1.60), we find

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x} = c_u \int_{\Omega} \varepsilon \nabla s^+ \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x} + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x}.$$

But since $\mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$, we deduce from the second identity of (1.60) that

$$\int_{\Omega} \varepsilon \nabla s^+ \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x} = \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) s^+ \, d\mathbf{x}.$$

Summing up, we get

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x} = c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) s^+ \, d\mathbf{x} + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon). \quad (1.61)$$

Remark 1.4.8 Even if we use an integral symbol to keep the usual aspects of formulas and facilitate the reading, it is important to consider this new quantity as a sesquilinear form

$$(\mathbf{u}, \mathbf{v}) \mapsto \oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x}$$

on $\mathbf{X}_N^{\text{out}}(\varepsilon) \times \mathbf{X}_N^{\text{out}}(\varepsilon)$. In particular, we point out that this sesquilinear form is not hermitian in $\mathbf{X}_N^{\text{out}}(\varepsilon) \times \mathbf{X}_N^{\text{out}}(\varepsilon)$. Indeed, we have

$$\overline{\oint_{\Omega} \varepsilon \mathbf{v} \cdot \bar{\mathbf{u}} \, d\mathbf{x}} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} \, d\mathbf{x} + c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) s^+ \, d\mathbf{x}$$

so that

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x} - \overline{\oint_{\Omega} \varepsilon \mathbf{v} \cdot \bar{\mathbf{u}} \, d\mathbf{x}} = 2ic_u \bar{c}_v \Im \left(\int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) s^+ \, d\mathbf{x} \right).$$

On the other hand, we can show that

$$\Im \left(\int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) s^+ \, d\mathbf{x} \right) \neq 0.$$

In the sequel, we denote by $a_N(\cdot, \cdot)$ (resp. $\ell_N(\cdot)$) the sesquilinear form (resp. the antilinear form) appearing in the left-hand side (resp. right-hand side) of (1.57).

Equivalent formulation

Before considering the question of proving well-posedness in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, we have to make sure that a solution of (1.57) satisfies the initial problem. This leads us to introduce the following space

$$\mathbf{H}_N^{\text{out}}(\mathbf{curl}) := \operatorname{span}(\nabla s^+) \oplus \mathbf{H}_N(\mathbf{curl}) \supset \mathbf{X}_N^{\text{out}}(\varepsilon)$$

(without the divergence free condition) and to consider the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl}) \text{ such that} \\ a_N(\mathbf{u}, \mathbf{v}) = \ell_N(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl}), \end{array} \right. \quad (1.62)$$

where the definition of

$$\oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx$$

has to be extended to the space $\mathbf{H}_N^{\text{out}}(\mathbf{curl})$. Working exactly as in the beginning of the proof of Lemma 1.4.7, one can show that any $\mathbf{u} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl})$ admits the decomposition

$$\mathbf{u} = c_u \nabla s^+ + \nabla \varphi_u + \mathbf{curl} \psi_u,$$

with $c_u \in \mathbb{C}$, $\varphi_u \in H_0^1(\Omega)$ and $\psi_u \in \mathbf{X}_T(1)$, such that $\mathbf{curl} \psi_u \in \mathbf{V}_{-\beta}^0(\Omega)$. Then, for all $\mathbf{u} = c_u \nabla s^+ + \nabla \varphi_u + \mathbf{curl} \psi_u$ and $\mathbf{v} = c_v \nabla s^+ + \nabla \varphi_v + \mathbf{curl} \psi_v$ in $\mathbf{H}_N^{\text{out}}(\mathbf{curl})$, a natural extension of the previous definitions leads to set

$$\begin{aligned} \oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx &:= \int_{\Omega} \varepsilon (\nabla \varphi_u + \mathbf{curl} \psi_u) \cdot (\nabla \bar{\varphi}_v + \mathbf{curl} \bar{\psi}_v) \, dx \\ &+ \int_{\Omega} c_u \varepsilon \nabla s^+ \cdot \overline{\mathbf{curl} \psi_v} + \bar{c}_v \varepsilon \mathbf{curl} \psi_u \cdot \overline{\nabla s^+} \, dx \\ &- \int_{\Omega} c_u \bar{c}_v \operatorname{div} (\varepsilon \nabla s^+) \bar{s}^+ + c_u \operatorname{div} (\varepsilon \nabla s^+) \bar{\varphi}_v + \bar{c}_v \varphi_u \operatorname{div} (\varepsilon \nabla s^+) \, dx. \end{aligned} \quad (1.63)$$

One can verify that (1.63) is indeed an extension of (1.61). Moreover, we have the following result.

Proposition 1.4.9 *Assume that ε , μ are as in §1.4.2. The field $\mathbf{u}$ is a solution of (1.57) if and only if it solves Problem (1.62). As a consequence, if $\mathbf{u}$ satisfies (1.57), then $(\mathbf{E}, \mathbf{H}) := (\mathbf{u}, (i\omega\mu)^{-1} \mathbf{curl} \mathbf{u})$ is a solution of (1.43)–(1.44).*

Now let us explain how to deal with the formulation (1.57) set in $\mathbf{X}_N^{\text{out}}(\varepsilon)$.

Main analysis for the electric field

Define the continuous operators $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ and $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ such that for all $\mathbf{u}, \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$,

$$\langle \mathbb{A}_N^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx, \quad \langle \mathbb{K}_N^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \oint_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx.$$

With this notation, we have $\langle (\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}}) \mathbf{u}, \mathbf{v} \rangle = a_N(\mathbf{u}, \mathbf{v})$.

Proposition 1.4.10 *Assume that ε , μ are as in §1.4.2. The operator $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is an isomorphism.*

Proof. Adapting what has been done in the proof of Lemma 1.4.1, let us construct a continuous operator $\mathbb{T} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow \mathbf{X}_N^{\text{out}}(\varepsilon)$ such that for all $\mathbf{u}, \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$,

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} (\overline{\mathbb{T} \mathbf{v}}) \, dx = \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx.$$

Assume that $\mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ is given. We construct $\mathbb{T} \mathbf{v}$ in three steps.

i) Since $\mathbf{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)$ and $A^\mu : H_{\#}^1(\Omega) \rightarrow (H_{\#}^1(\Omega))^*$ is an isomorphism, there is a unique $\zeta \in H_{\#}^1(\Omega)$ such that

$$\int_{\Omega} \mu \nabla \zeta \cdot \nabla \zeta' \, dx = \int_{\Omega} \mu \mathbf{curl} \mathbf{v} \cdot \nabla \zeta' \, dx, \quad \forall \zeta' \in H_{\#}^1(\Omega).$$

Then the field $\mu(\mathbf{curl} \mathbf{v} - \nabla \zeta) \in \mathbf{L}^2(\Omega)$ is divergence free in Ω and satisfies $\mu(\mathbf{curl} \mathbf{v} - \nabla \zeta) \cdot \nu = 0$ on $\partial\Omega$.

ii) According to Theorem 3.17 in [11], we infer that there is $\psi \in \mathbf{X}_N(1)$ such that

$$\mu(\mathbf{curl} \mathbf{v} - \nabla \zeta) = \mathbf{curl} \psi.$$

Thanks to Lemma A.5 of [75], which requires a bit of work, we deduce that $\psi \in \mathbf{V}_{-\beta}^0(\Omega)$.

iii) Then we know from the previous step that $\operatorname{div}(\varepsilon \psi) \in (\dot{\mathbf{V}}_{\beta}^1(\Omega))^*$. On the other hand, since by assumption, $A^{\varepsilon, \text{out}} : \dot{\mathbf{V}}^{\text{out}} \rightarrow (\dot{\mathbf{V}}_{\beta}^1(\Omega))^*$ is an isomorphism, we can introduce $\varphi \in \dot{\mathbf{V}}^{\text{out}}$ such that $A^{\varepsilon, \text{out}} \varphi = -\operatorname{div}(\varepsilon \psi)$.

Finally, we set $\mathbb{T} \mathbf{v} = \psi - \nabla \varphi$. Clearly $\mathbb{T} \mathbf{v}$ is an element of $\mathbf{X}_N^{\text{out}}(\varepsilon)$. Moreover, for all $\mathbf{u}, \mathbf{v}$ in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, we have

$$\begin{aligned} \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \overline{\mathbb{T} \mathbf{v}} dx &= \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \overline{\psi} dx \\ &= \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \overline{\mathbf{v}} dx - \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \nabla \overline{\zeta} dx \\ &= \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \overline{\mathbf{v}} dx. \end{aligned}$$

With Lemma 1.4.7 and the Lax-Milgram theorem, we deduce that $\mathbb{T}^* \mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is an isomorphism. And by symmetry, permuting the roles of $\mathbf{u}$ and $\mathbf{v}$, it is obvious that $\mathbb{T}^* \mathbb{A}_N^{\text{out}} = \mathbb{A}_N^{\text{out}} \mathbb{T}$, which allows us to conclude that $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is an isomorphism. ■

To deal with the frequency dependent term in (1.57), we established the following compactness result.

Proposition 1.4.11 *Assume that ε, μ are as in §1.4.2. If $(\mathbf{u}_n = c_n \nabla s^+ + \tilde{\mathbf{u}}_n)$ is a sequence which is bounded in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, then we can extract a subsequence such that (c_n) and $(\tilde{\mathbf{u}}_n)$ converge respectively in $\mathbb{C}$ and in $\mathbf{V}_{-\beta}^0(\Omega)$. As a consequence, the operator $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is compact.*

We can now state the main theorem of the analysis of the problem for the electric field.

Theorem 1.4.12 *Assume that ε, μ are as in §1.4.2. For all $\omega \geq 0$ the operator $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is Fredholm of index zero.*

Proof. Since $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is compact (Proposition 1.4.11) and $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is an isomorphism (Proposition 1.4.10), $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$ is Fredholm of index zero. ■

The previous theorem guarantees that Problem (1.57) is well-posed if and only if uniqueness holds, that is, if and only if the only solution for $\mathbf{J} = 0$ is $\mathbf{u} = 0$. Since uniqueness holds for $\omega = 0$, the analytic Fredholm theorem ensures that (1.57) is well-posed except for at most a countable set of values of ω with no accumulation points (note that Theorem 1.4.12 remains true for $\omega \in \mathbb{C}$).

Comments

i) The material above was presented in the article [75]. In [75], we also propose a formulation for the magnetic field which is equivalent to the initial Maxwell's equations and which is well-posed. Quite logically from (1.43), it is set in some space $\mathbf{X}_T^{\text{out}}(\mu)$ of fields which are in $\mathbf{L}^2(\Omega)$ but which have singular $\mathbf{curl}$.

ii) In [75], we also considered, with ε, μ as in §1.4.2, the formulation for the electric field in the

classical space $\mathbf{X}_N(\varepsilon)$. We prove that it is well-posed in the Fredholm sense but we are not able to show that it is equivalent to Maxwell's equations (1.43)–(1.44). On the other hand, we established that the standard formulation for the magnetic field in $\mathbf{X}_T(\mu)$ is equivalent to (1.43)–(1.44). But the corresponding operator in $\mathbf{X}_T(\mu)$ is not Fredholm.

iii) To justify that our frameworks are physically relevant, we demonstrated some limiting absorption principle. More precisely, we proved that the solutions of Maxwell's equations with ε, μ having a small imaginary part $\delta > 0$ converge to our solutions as $\delta \rightarrow 0$ in some adapted norms.

iv) Finally, in a second article, namely [76], we studied the situation where propagating singularities exist both for the operators A^ε and A^μ . We obtained some well-posed formulation in the Fredholm sense, and equivalence with the initial Maxwell's equations, in a space of fields which are singular and which have singular **curl**. In [76], we also considered situations where several propagating singularities exist. Finally, the geometric setting that we considered in [76] was a bit more general than the one of [75]. In particular, we allowed the conical tip to touch the boundary of the domain. This might seem a minor technical point but it forces us to write more systematic proofs for the vector potential representations of singular fields, an important aspect which is hidden in the presentation above.

1.5 Asymptotic analysis

1.5.1 A curious instability phenomenon for rounded corners

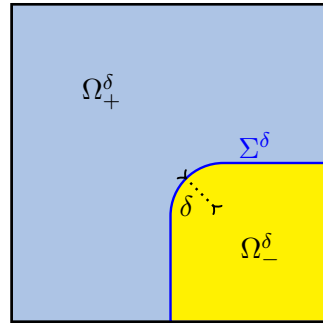


Figure 1.15: Interface with a rounded corner.

For certain applications, physicists specifically look to use metals with non-regular geometries to obtain high-intensity electromagnetic fields in plasmonic technologies (see, for example, [453, 57, 405]). In practice however, geometries are never perfectly singular and corners are always rounded. What does this imply?

Analysis of the source term problem

To study this question, with X. Claeys and S.A. Nazarov, we first considered the scalar Problem (1.2), with a given source term $f \in L^2(\Omega)$, in the geometry described by Figure 1.15 where the interface has a rounded corner of radius of curvature $\delta > 0$ small. Observe in particular that we get a true corner at the limit $\delta \rightarrow 0$. We denote by $(\mathcal{P}^\delta)$ this problem (set in $H_0^1(\Omega)$) and by σ^δ the function such that $\sigma^\delta = \sigma_\pm$ in $\Omega_\pm^\delta$ where $\sigma_\pm$ are constants satisfying $\pm\sigma_\pm > 0$.

For positive materials, it is known that for $\delta > 0$, $(\mathcal{P}^\delta)$ admits a unique solution u_δ and that the sequence (u_δ) converges in $H^1(\Omega)$ to u_0 , the solution of $(\mathcal{P}^0)$, the problem with an interface having a true corner. Moreover, the speed of convergence depends on the regularity of u_0 .

With a sign-changing σ^δ , the situation is less clear. In particular when $\sigma_-/\sigma_+ \neq -1$ belongs

to the critical interval, one may expect some unconventional phenomena to happen. Indeed for $\delta > 0$, the interface is smooth and so A^δ , the operator associated to $(\mathcal{P}^\delta)$, is Fredholm of index zero. However for $\delta = 0$, A^0 is not Fredholm. In the article [156], we decided to investigate precisely this configuration.

First, by assuming that the limit problem for $\delta = 0$ is injective in $H_0^1(\Omega)$, so that the operator A^{out} defined in (1.37) is an isomorphism, we showed that A^δ is injective for $\delta \in (0; \delta_0] \cap I(\alpha)$ where

$$I(\alpha) := \bigcup_{k=-\infty}^{\infty} [\delta_\star e^{(k+\alpha)\pi/\eta}, \delta_\star e^{(k+1-\alpha)\pi/\eta}] \quad (1.64)$$

(see the representation of Figure 1.16). Here α can be chosen as desired in $(0; 1/2)$, $\delta_0 > 0$ depends on α , $\delta_\star > 0$ is a particular constant that we can characterize and $\eta \in \mathbb{R} \setminus \{0\}$ is the singular exponent appearing in (1.22).

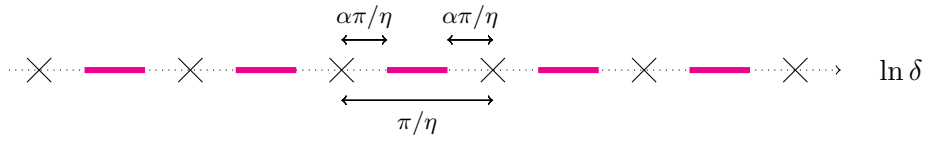


Figure 1.16: The crosses represent the set $\{\ln(\delta_\star e^{k\pi/\eta}) = \ln \delta_\star + k\pi/\eta, k \in \mathbb{Z}\}$. The thick segments represent the set $\{\ln \delta, \delta \in I(\alpha)\}$.

We deduce that $(\mathcal{P}^\delta)$ admits a unique solution $u_\delta \in H_0^1(\Omega)$ for all $\delta \in (0; \delta_0] \cap I(\alpha)$. Additionally, we proved the following important *a priori* estimate: there exists a constant $C > 0$ such that

$$\|u_\delta\|_{H_0^1(\Omega)} \leq C |\ln \delta| \|f\|_{L^2(\Omega)}, \quad \forall \delta \in (0; \delta_0] \cap I(\alpha). \quad (1.65)$$

Let us comment this result. First, obtaining (1.65), which is the basic brick to establish error estimates, has been very technical. With coercive or coercive plus compact problems, one usually works by contradiction. Due to the sign-changing coefficient, we cannot proceed so simply here. On the other hand, because the interface changes when δ tends to zero, we were unable to work with some T-coercivity argument. Instead, we exploited the very powerful technique of overlapping cut-off functions introduced in [352, Chap. 2], [370]. Roughly speaking, it allows one to establish these types of *a priori* estimates as soon as the far field operator, appearing when taking the limit $\delta \rightarrow 0$, and the inner field operator, obtained by rescaling the problem by a factor δ^{-1} and considering the limit $\delta \rightarrow 0$, are isomorphisms in *ad hoc* weighted Sobolev spaces. Second, in (1.65), we are not surprised to see the appearance of a $|\ln \delta|$ term. Indeed, we know that at the limit $\delta = 0$, Problem (1.2) is ill-posed in $H_0^1(\Omega)$ due to the existence of the two propagating singularities (see in particular the calculus (1.23)). Finally in (1.65), we emphasize that we exclude a neighborhood of a sequence of points which accumulates at zero. But we will see below that there is no hope to get better, *i.e.* to derive an estimate for all $\delta \in (0; \delta_0]$.

Then we constructed an asymptotic expansion $\hat{u}_\delta$ of u_δ as δ tends to zero and with (1.65) proved the error estimate: there exist $\tau > 0$ and $C > 0$ such that

$$\|u_\delta - \hat{u}_\delta\|_{H_0^1(\Omega)} \leq C \delta^\tau \|f\|_{L^2(\Omega)}, \quad \forall \delta \in (0; \delta_0] \cap I(\alpha). \quad (1.66)$$

Finally, by using the rather explicit form of $\hat{u}_\delta$, which depends on δ , we have been able to prove that $(\hat{u}_\delta)$ does not converge as δ tends to zero, even in $L^2(\Omega)$. With (1.66), this guarantees that (u_δ) does not converge, even in $L^2(\Omega)$. Indeed, if there were some $u_0 \in L^2(\Omega)$ such that $\lim_{\delta \rightarrow 0} \|u_\delta - u_0\|_{L^2(\Omega)} = 0$, then we could write

$$\begin{aligned} \|\hat{u}_\delta - u_0\|_{L^2(\Omega)} &\leq \|\hat{u}_\delta - u_\delta\|_{L^2(\Omega)} + \|u_\delta - u_0\|_{L^2(\Omega)} \\ &\leq C \|\hat{u}_\delta - u_\delta\|_{H_0^1(\Omega)} + \|u_\delta - u_0\|_{L^2(\Omega)} \xrightarrow{\delta \rightarrow 0} 0, \end{aligned}$$

which is false. In other words, (1.66) proves that (u_δ) gets closer and closer to a term which does not converge as δ tends to zero. This shows that Problem (1.2) is unstable with respect to a small corner rounding: two very similar geometries with almost identical (but different) rounded corners can lead to solutions that have nothing to do with each other. From a physical point of view, this result is highly unusual and difficult to accept. As such, it is of great interest. Indeed, it demonstrates the limitations of the model we work on. There is no doubt that in practice something happens at the rounded corner that the equations we use do not describe. To get more realistic phenomena, refined models with non local effects have been considered since then, see *e.g.* [441, 179, 53]. They are still being studied.

Let us mention that when $\sigma_-/\sigma_+ \neq -1$ does not belong to the critical interval, the results are different and look like the case of positive materials. More precisely, using again the technique of overlapping cut-off functions, one can show that if A^0 is injective, then there is δ_0 such that A^δ is injective for all $\delta \in (0; \delta_0]$. Moreover (u_δ) converges to u_0 in $H_0^1(\Omega)$.

As an illustration, let us present some numerics in a canonical geometry. For $\delta \in (0; 1)$, set

$$\begin{aligned}\Omega_+^\delta &:= \{(r \cos \theta, r \sin \theta) \mid \delta < r < 1, \pi/4 < \theta < \pi\}; \\ \Omega_-^\delta &:= \{(r \cos \theta, r \sin \theta) \mid \delta < r < 1, 0 < \theta < \pi/4\}; \\ \Omega^\delta &:= \{(r \cos \theta, r \sin \theta) \mid \delta < r < 1, 0 < \theta < \pi\}\end{aligned}\tag{1.67}$$

(see Figure 1.18). Admittedly this framework is not exactly the one described above. In particular, we do not really have a rounded corner. However the problem is completely similar because for all $\delta \in (0; 1)$, the interface touches the boundary perpendicularly so that A^δ is Fredholm as soon as the contrast is different from -1 . And for $\delta = 0$, A^0 is Fredholm if and only if $\sigma_-/\sigma_+ \notin [-1; -1/3]$. The advantage of this setting is that we can work with separation of variables, which allows us for example to identify if A^δ is injective or not.

• OUTSIDE THE CRITICAL INTERVAL $[-1; -1/3]$

In Figure 1.17, we display the variation of $\|u_\delta\|_{H_0^1(\Omega^\delta)}$, computed with a finite element method, with respect to $1 - \delta$ for $\sigma_-/\sigma_+ = -1 - 10^{-4}$. The source term f is equal to one hundred for $x \leq -1/2$, zero otherwise. Figure 1.18 represents u_δ for this contrast and for eight values of δ . As expected, u_δ is stable when $\delta \rightarrow 0$.

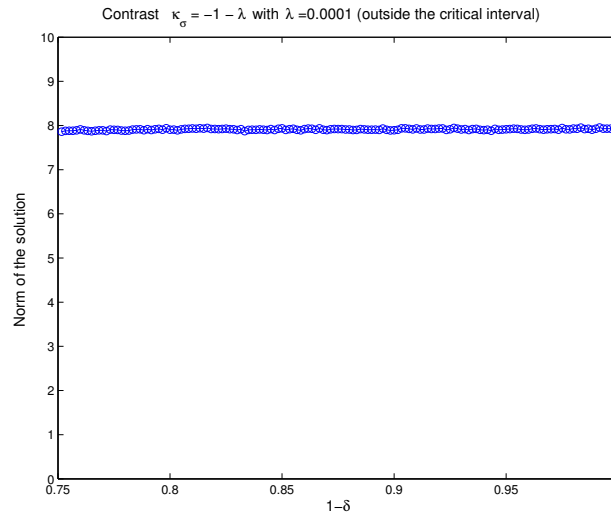


Figure 1.17: Variation of $\|u_\delta\|_{H_0^1(\Omega^\delta)}$ with respect to $1 - \delta$.

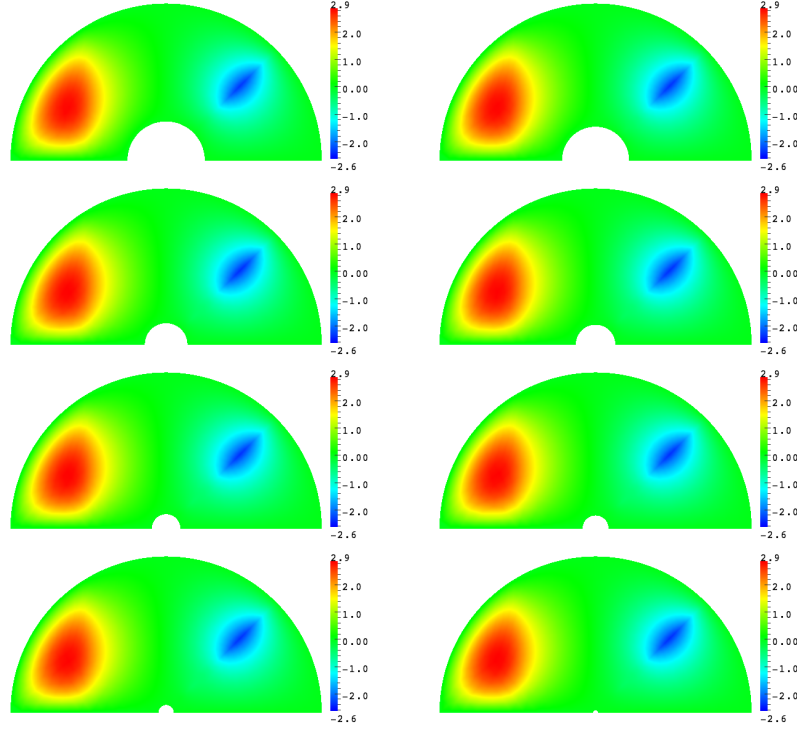


Figure 1.18: Solution u_δ for eight values of δ where here δ stands for the radius of the inner half disk. The contrast σ_-/σ_+ is chosen equal to $-1 - 10^{-4}$ (outside of the critical interval). The sequence (u_δ) converges when $\delta \rightarrow 0$.

• INSIDE THE CRITICAL INTERVAL $(-1; -1/3)$

In Figure 1.19, we display the variation of $\|u_\delta\|_{H_0^1(\Omega^\delta)}$ with respect to $1 - \delta$ for $\sigma_-/\sigma_+ = -1 + 10^{-4}$. In accordance with the analytical computations, we observe peaks which correspond to the values $\delta = \delta^n$ for which A^δ fails to be injective. More generally, the numerical experiments confirm the following important idea: when the contrast lies inside the critical interval, the solution in the geometry with a rounded corner, when it is well-defined, critically depends on the parameter δ . This appears clearly in Figure 1.20 where we see that the solution is not stable with respect to δ not only in a neighborhood of the corner point O but everywhere in Ω . Concerning the numerical analysis point of view, let us make the following comments. First, notice that for small values of δ , it is very expensive to use a mesh adapted to the geometry. Therefore, the mesh size is chosen small but independent of δ . This explains why peaks do not appear for $\delta \leq 0.02$. Remark also that we work here with a contrast very close to -1 . This may seem surprising because for $\sigma_-/\sigma_+ = -1$, the operators A^δ are not of Fredholm type. However, this allows us to obtain several peaks without being obliged to use a very refined mesh. Indeed, from (1.64) we find that the period of peaks, in $\ln \delta$ scale, is equal to π/η . On the other hand, there holds $\lim_{\sigma_-/\sigma_+ \rightarrow -1+} \eta = +\infty$. From a numerical point of view, it only requires using meshes which are locally symmetric with respect to the interface to avoid instability phenomena (see again [154]).

Remark 1.5.1 By proceeding to the Euler change of variables (1.28) in the particular geometry (1.67), we clearly understand why the stability estimate (1.65), which implies injectivity, cannot be valid uniformly with respect to δ . Indeed, with (1.28), Ω^δ is transformed into the rectangle $(\ln \delta; 0) \times (0; \pi)$. In this rectangle on the boundary of which we impose homogeneous Dirichlet boundary conditions, with the propagating modes existing for contrasts inside the critical interval, it is easy to create resonant modes for periodic values of δ in $\ln \delta$ scale. Let us emphasize that this phenomenon is not specific to the canonical configuration chosen here for the illustration, it is generic.

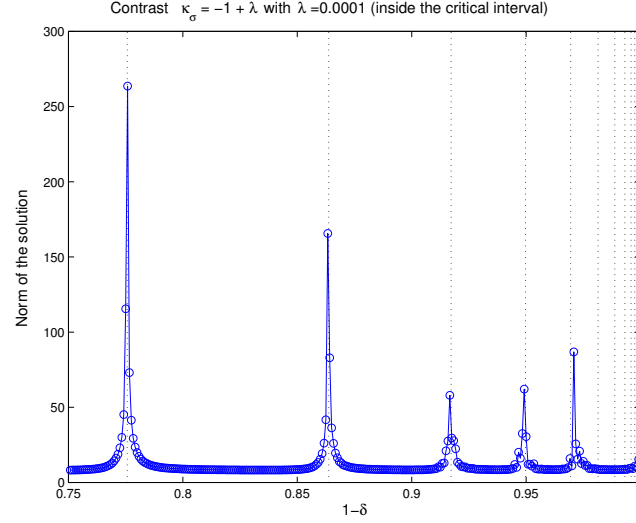


Figure 1.19: Variation of $\|u_\delta\|_{H_0^1(\Omega^\delta)}$ with respect to $1 - \delta$. The dotted lines correspond to the expected values of $\delta = \delta^n$ for which A^δ fails to be injective.

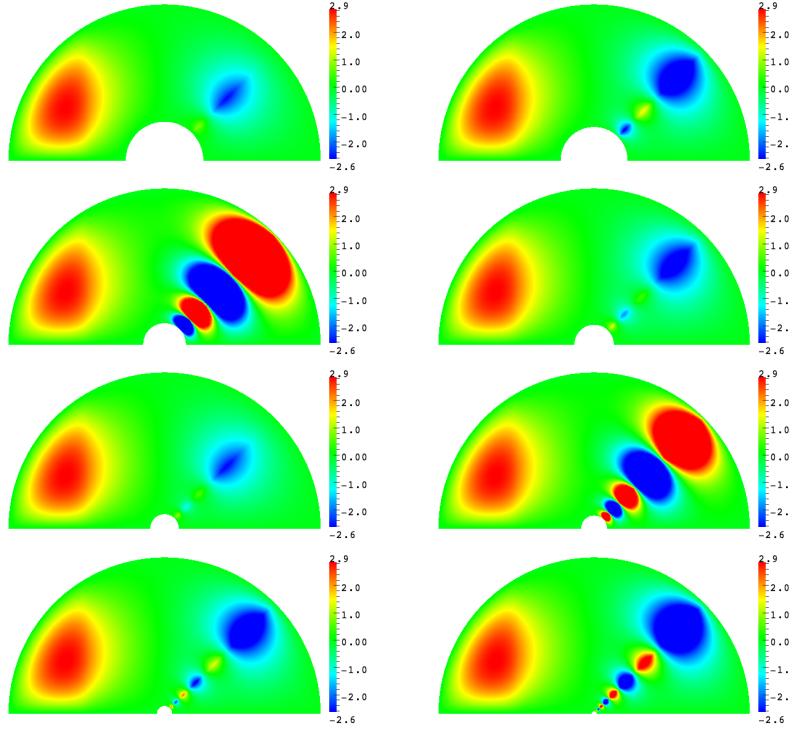


Figure 1.20: Same setting as in Figure 1.18 but with $\sigma_-/\sigma_+ = -1 + 10^{-4}$ (inside the critical interval). The sequence (u_δ) does not converge when $\delta \rightarrow 0$.

Analysis of the spectral problem

To complete our understanding of the instability phenomenon, in the article [158], we turned our attention to the asymptotic analysis of the spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda_\delta, u_\delta) \in \mathbb{R} \times (H_0^1(\Omega) \setminus \{0\}) \text{ such that} \\ -\operatorname{div}(\sigma_\delta \nabla u_\delta) = \lambda_\delta u_\delta \quad \text{in } \Omega \end{array} \right. \quad (1.68)$$

as the rounding parameter of the corner δ tends to zero. Again, we considered the most interesting case where $\sigma_-/\sigma_+ \neq -1$ is located inside the critical interval. Define the unbounded operator

$A^\delta : D(A^\delta) \rightarrow L^2(\Omega)$ such that

$$\begin{cases} A^\delta v = -\operatorname{div}(\sigma^\delta \nabla v) \\ D(A^\delta) := \{v \in H_0^1(\Omega) \mid \operatorname{div}(\sigma^\delta \nabla v) \in L^2(\Omega)\}. \end{cases} \quad (1.69)$$

For $\delta > 0$, since Σ^δ is smooth and intersects $\partial\Omega$ with right angles, the operator A^δ is densely defined, closed, self-adjoint and admits compact resolvent. Besides, we have the following proposition.

Proposition 1.5.2 *The spectrum of A^δ , denoted $\mathfrak{S}(A^\delta)$, consists of two sequences, one nonnegative and one negative, of real eigenvalues of finite multiplicity:*

$$\dots \lambda_{-m}^\delta \leq \dots \leq \lambda_{-1}^\delta < 0 \leq \lambda_0^\delta \leq \lambda_1^\delta \leq \dots \leq \lambda_m^\delta \dots$$

Moreover, there hold $\inf \mathfrak{S}(A^\delta) = -\infty$ and $\sup \mathfrak{S}(A^\delta) = +\infty$.

We are interested in finding an operator that would be the “limit” of A^δ as $\delta \rightarrow 0$. Let us first introduce the unbounded operator $\mathfrak{A} : D(\mathfrak{A}) \rightarrow L^2(\Omega)$ defined by

$$\begin{cases} \mathfrak{A}v := -\operatorname{div}(\sigma^0 \nabla v) \\ D(\mathfrak{A}) := \{v \in H_0^1(\Omega) \mid \operatorname{div}(\sigma^0 \nabla v) \in L^2(\Omega)\}. \end{cases}$$

The operator $\mathfrak{A}$ is symmetric. From the point of view of spectral analysis, it is desirable to determine whether it is self-adjoint. The following result is established in [432, Chap. 7] (see also [80]).

Proposition 1.5.3 *The domain of the operator $\mathfrak{A}^*$ is given by $D(\mathfrak{A}^*) = \operatorname{span}\{s^+, s^-\} \oplus D(\mathfrak{A})$.*

Since $D(\mathfrak{A}^*) \neq D(\mathfrak{A})$, this shows that $\mathfrak{A}$ is not self-adjoint. Actually, it is possible to completely describe all the self-adjoint extensions of $\mathfrak{A}$ (for the general theory, see e.g. [433, Sect. X.1]).

Proposition 1.5.4 *The self-adjoint extensions of $\mathfrak{A}$ are the unbounded operators $\mathfrak{A}(\tau)$, $\tau \in \mathbb{R}$, such that $\mathfrak{A}(\tau) : D(\mathfrak{A}(\tau)) \rightarrow L^2(\Omega)$ is defined by*

$$\begin{cases} \mathfrak{A}(\tau)v = -\operatorname{div}(\sigma^0 \nabla v) \\ D(\mathfrak{A}(\tau)) = \operatorname{span}\{s^+ + e^{i\tau}s^-\} \oplus D(\mathfrak{A}). \end{cases} \quad (1.70)$$

For any given $\tau \in \mathbb{R}$, we can show that the spectrum of $\mathfrak{A}(\tau)$ admits the same qualitative properties as the one of A^δ for $\delta > 0$.

Proposition 1.5.5 *Pick $\tau \in \mathbb{R}$. The operator $\mathfrak{A}(\tau)$ is closed, densely defined, self-adjoint and admits compact resolvent. Its spectrum consists of two sequences, one nonnegative and one negative, of real eigenvalues of finite multiplicity:*

$$\dots \Lambda_{-m}(\tau) \leq \dots \leq \Lambda_{-1}(\tau) < 0 \leq \Lambda_0(\tau) \leq \Lambda_1(\tau) \leq \dots \leq \Lambda_m(\tau) \dots$$

Moreover, there hold $\inf \mathfrak{S}(\mathfrak{A}(\tau)) = -\infty$ and $\sup \mathfrak{S}(\mathfrak{A}(\tau)) = +\infty$.

Then we computed an asymptotic expansion of the eigenpairs of A^δ . We showed that they get closer and closer to the eigenpairs of the model operator $\mathfrak{A}^\delta \subset \mathfrak{A}^*$ such that

$$\begin{cases} \mathfrak{A}^\delta v = -\operatorname{div}(\sigma^0 \nabla v) \\ D(\mathfrak{A}^\delta) = \operatorname{span}\{s^+ + (\delta/\delta_\bullet)^{2i\eta}s^-\} \oplus D(\mathfrak{A}). \end{cases} \quad (1.71)$$

Here $\delta_\bullet > 0$ is some quantity that we can characterize. Since $|(\delta/\delta_\bullet)^{2i\eta}| = 1$, according to Proposition 1.5.4, we know that $\mathfrak{A}^\delta$ is self-adjoint for $\delta > 0$ (observe that with the definition (1.70), we have $\mathfrak{A}^\delta = \mathfrak{A}(2\eta \ln(\delta/\delta_\bullet))$). Denote $\{\Lambda_j^\delta\}_{j \in \mathbb{Z}}$ the set of ordered eigenvalues, counted with their multiplicities, of $\mathfrak{A}^\delta$. For $\delta_0 > 0$, one can show that for each fixed $j \in \mathbb{Z}$, the function $\delta \mapsto \Lambda_j^\delta$ is bounded over $(0; \delta_0]$.

Remark 1.5.6 When $\delta_1, \delta_2 > 0$ are such that $(\delta_1/\delta_\bullet)^{2i\eta} = (\delta_2/\delta_\bullet)^{2i\eta} \Leftrightarrow \ln \delta_1 = \ln \delta_2 + k\pi/\eta$ for some $k \in \mathbb{Z}$, we have $D(\mathfrak{A}^{\delta_1}) = D(\mathfrak{A}^{\delta_2})$, so that $\mathfrak{A}^{\delta_1} = \mathfrak{A}^{\delta_2}$ and $\mathfrak{S}(\mathfrak{A}^{\delta_1}) = \mathfrak{S}(\mathfrak{A}^{\delta_2})$. Therefore, for each fixed $j \in \mathbb{Z}$, the map $\delta \mapsto \Lambda_j^\delta$ is π/η -periodic in $\ln \delta$ -scale.

Remark 1.5.7 As already used in (1.39), there is an element $\zeta \in D(\mathfrak{A}^*) \setminus D(\mathfrak{A})$ admitting the decomposition $\zeta = s^- + c_\zeta s^+ + \tilde{\zeta}$ with $c_\zeta \in \mathbb{C}$ such that $|c_\zeta| = 1$ and $\tilde{\zeta} \in D(\mathfrak{A})$. When $\delta > 0$ is such that $(\delta/\delta_\bullet)^{2i\eta} = c_\zeta$, the function ζ belongs to $D(\mathfrak{A}^\delta)$. In this case, we have $0 \in \mathfrak{S}(\mathfrak{A}^\delta)$. Together with Remark 1.5.6, this shows that the set of values of δ such that $\mathfrak{A}^\delta$ is not injective accumulates at zero. Figure 1.21 illustrates the phenomenon.

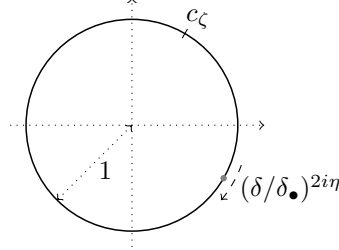


Figure 1.21: As δ goes to zero, $(\delta/\delta_\bullet)^{2i\eta}$ runs on the unit circle and hits c_ζ infinitely many times.

The main theorem proved in [158] is as follows.

Theorem 1.5.8 Assume that $\sigma_-/\sigma_+ \neq -1$ belongs to the critical interval and that the only $u \in H_0^1(\Omega)$ such that $\operatorname{div}(\sigma^0 \nabla u) = 0$ is zero.

Let $\{\lambda_j^\delta\}_{j \in \mathbb{Z}}$ refer to the set of ordered eigenvalues of the original operator A^δ defined in (1.69).

Let $\{\Lambda_j^\delta\}_{j \in \mathbb{Z}}$ refer to the set of ordered eigenvalues of the model operator $\mathfrak{A}^\delta$ defined in (1.71).

Then for any $j \in \mathbb{Z}$, $\beta > 0$, there is a constant $C_{j,\beta} > 0$ independent of $\delta \in (0; \delta_0]$ such that

$$\inf_{\lambda \in \mathfrak{S}(A^\delta)} |\Lambda_j^\delta - \lambda| + \inf_{\Lambda \in \mathfrak{S}(\mathfrak{A}^\delta)} |\lambda_j^\delta - \Lambda| \leq C_{j,\beta} \delta^{1-\beta}.$$

Theorem 1.5.8 guarantees that all the eigenvalues of the original operator A^δ behave as the eigenvalues of the model operator $\mathfrak{A}^\delta$ as δ goes to zero. Since the eigenvalues of $\mathfrak{A}^\delta$ are periodic in $\ln \delta$ -scale, this demonstrates that, asymptotically, all the eigenvalues of A^δ are periodic in $\ln \delta$ -scale as $\delta \rightarrow 0$.

Let us conclude this paragraph by presenting some numerical illustrations of the above results. We work again in the configuration described in (1.67) and solve (1.68) by a finite element method.

• OUTSIDE THE CRITICAL INTERVAL $[-1; -1/3]$

In Figure 1.22, we display the ten eigenvalues of smallest modulus of (1.68) with respect to $-\ln \delta$ for a contrast $\sigma_-/\sigma_+ = -1 - 10^{-4} \notin [-1; -1/3]$. In this case, the limit problem for $\delta = 0$ is well-posed in the Fredholm sense in $H_0^1(\Omega)$. The operator $A^0 : D(A^0) \rightarrow L^2(\Omega^0)$ is self-adjoint and has compact resolvent. The dashed lines in Figure 1.22 represent the approximation of the ten eigenvalues of smallest modulus of the limit operator A^0 . The numerical experiments suggest that the spectrum of A^δ converges to the spectrum of A^0 as $\delta \rightarrow 0$, which is something that we can prove rigorously.

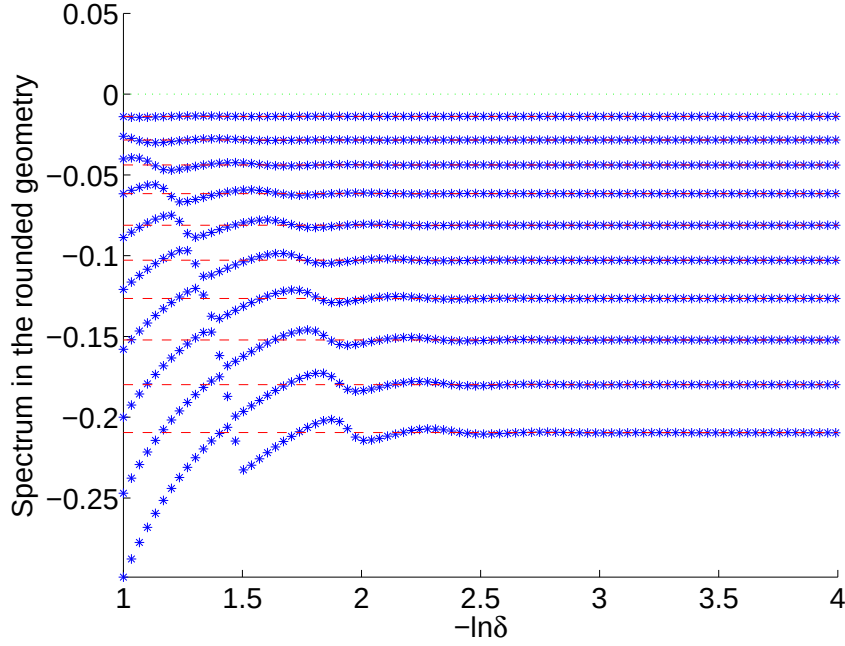


Figure 1.22: For a given $\delta \in (0; \delta_0]$, we approximate the ten eigenvalues of smallest modulus of the operator A^δ . Then, we make δ tend to zero. The figure represents the approximation of the spectrum of A^δ with respect to $-\ln \delta$. The horizontal dashed lines correspond to the approximation of the ten eigenvalues of smallest modulus of the limit operator A^0 .

- INSIDE THE CRITICAL INTERVAL $(-1; -1/3)$

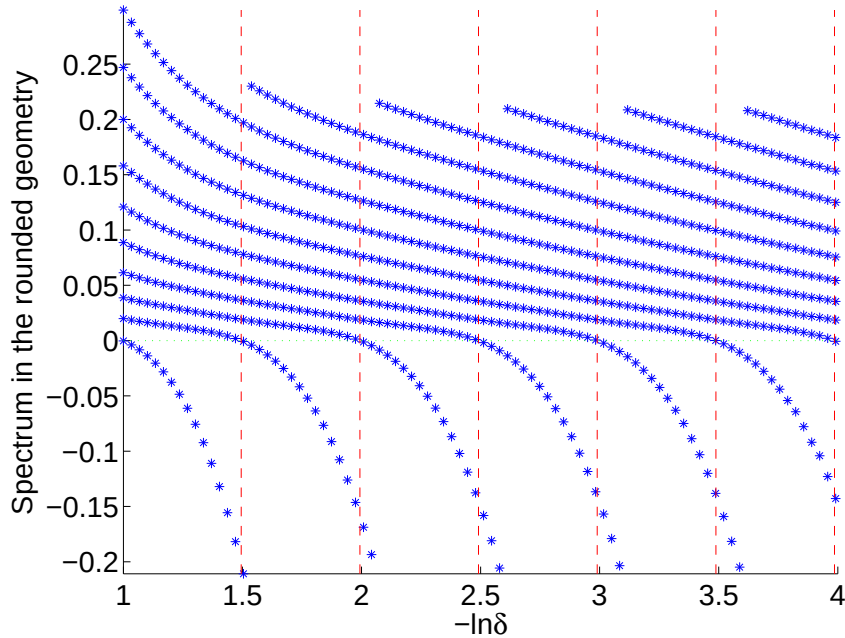


Figure 1.23: Same quantities as in Figure 1.22 but this time for $\sigma_-/\sigma_+ = -1 + 10^{-4}$ (inside the critical interval). The vertical dashed lines correspond to the expected values of $\delta = \delta^n$ for which A^δ fails to be injective.

In Figure 1.23, we display the ten eigenvalues of smallest modulus of (1.68) with respect to $-\ln \delta$ for a contrast $\kappa_\sigma = \sigma_-/\sigma_+ = -1 + 10^{-4} \in (-1; -1/3)$. We observe that the spectrum of A^δ depends on δ even for small δ . In other words, it does not converge to the spectrum of some

operator independent of δ . The dashed lines correspond to the expected values of $\delta = \delta^n$, computed explicitly using separation of variables, for which A^δ fails to be injective, or equivalently, for which zero belongs to the spectrum of A^δ . Notice that the spectrum computed numerically indeed passes through zero for these values of δ . Figures 1.24 and 1.25 represent respectively the approximation of the first positive and the first negative eigenvalue of A^δ with respect to $-\ln \delta$. Note the periodic behavior. This is consistent with what we proved in Theorem 1.5.8.

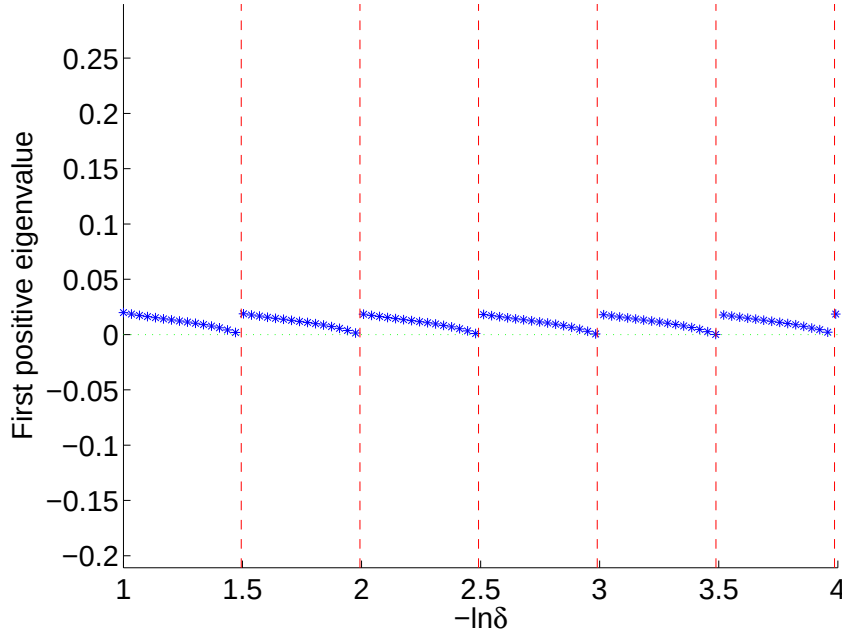


Figure 1.24: First positive eigenvalue of A^δ with respect to $-\ln \delta$ (same setting as in Figure 1.23). The vertical dashed lines correspond to the expected values of $\delta = \delta^n$ for which A^δ is not injective.

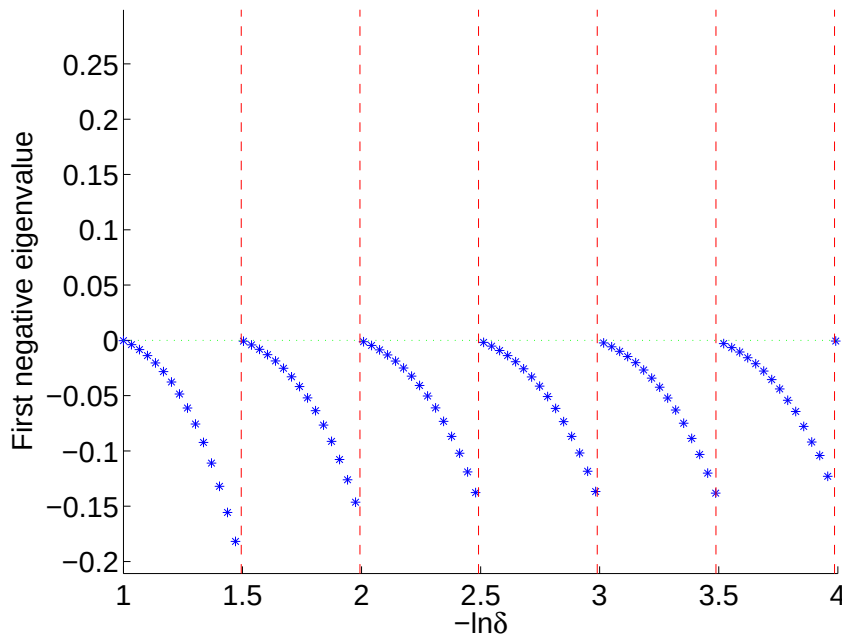


Figure 1.25: First negative eigenvalue of A^δ with respect to $-\ln \delta$ (same setting as in Figure 1.23). The vertical dashed lines correspond to the expected values of $\delta = \delta^n$ for which A^δ is not injective.

Remark 1.5.9 For an interface with a corner and a contrast inside the critical interval, adding

dissipation as in (1.26) and rounding the corner as above can be seen as two ways of regularizing Problem (1.2) which is ill-posed in $H_0^1(\Omega)$. It is interesting to note however that they lead to different limit behaviors when the regularization parameter tends to zero. In the first case, the solution converges to the one in the framework $\text{span}(s^+) \oplus \mathring{V}_{-\beta}^1(\Omega)$. This is the limiting absorption principle mentioned in item iii) after (1.39) which selects a limit operator which is not self-adjoint. In the second case, the regularized operator gets closer and closer to a self-adjoint operator which depends on the regularization parameter δ and does not converge when $\delta \rightarrow 0$.

1.5.2 Spectrum for a small inclusion of negative material

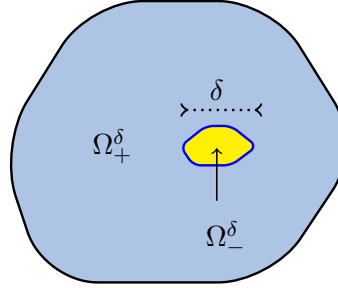


Figure 1.26: Small inclusion of negative material.

With X. Claeys and S.A. Nazarov, in [157], we were also interested in the asymptotic analysis of the spectral problem (1.68) in presence of a small smooth inclusion of negative material of diameter proportional to δ (see Figure 1.26) when $\kappa_\sigma \neq -1$. The question that motivated us was the following. When δ is positive, as in Proposition 1.5.2, the spectrum of (1.68) is made of two sequences of eigenvalues tending respectively to $\pm\infty$. At the limit $\delta = 0$, the inclusion disappears, $\sigma_0 = \sigma_+$ is positive everywhere, and therefore the spectrum of (1.68) is a sequence of eigenvalues tending to $+\infty$. What happens to the negative eigenvalues when $\delta \rightarrow 0$?

In [157], we showed that the positive spectrum of (1.68) converges to the spectrum of the limit problem without the inclusion of negative material. On the other hand, we proved that negative eigenvalues behave as $\delta^{-2}\Lambda_n$, where the $\Lambda_n < 0$ are the eigenvalues of some inner field problem set in the whole space $\mathbb{R}^2$ with a normalized negative inclusion. We further established that the eigenfunctions associated with negative eigenvalues are localized in the vicinity of the small inclusion (see the illustration of Figure 1.27).

The main difficulty in the analysis comes from the justification of the formal asymptotic analysis. To proceed, we establish uniform boundedness of the inverse of A_δ in terms of weighted norms involving the small parameter δ . The proof of this important result (Proposition 3.1 in [157]) is made difficult by the change of sign of σ^δ which prevents the use of variational arguments. Again we use the efficient technique of overlapping cut-off functions introduced in [352, Chap. 2], [370]. Finally, let us mention that for other spectral problems involving sign-changing coefficient, one can look at [351] where the authors present a study of bifurcations for a nonlinear problem and [133] where scattering resonances in free space are considered.

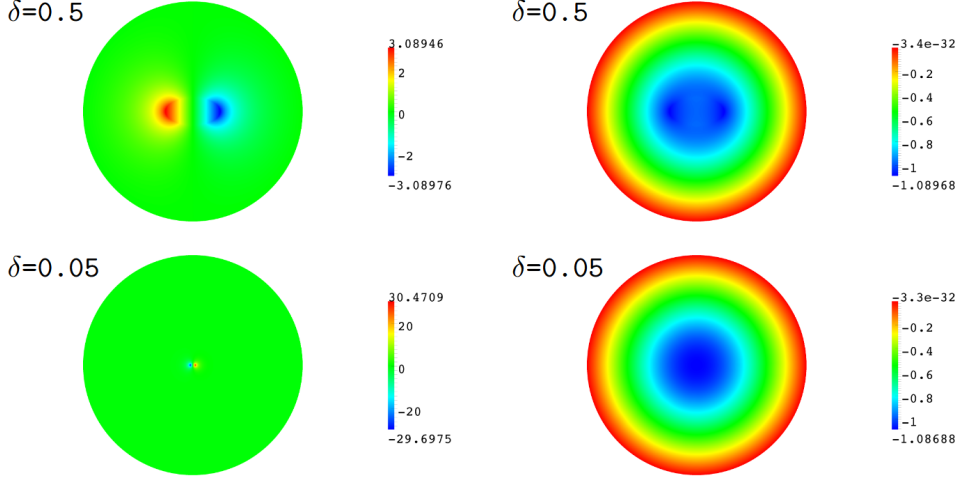


Figure 1.27: Eigenfunctions associated with the first negative (left) and positive (right) eigenvalues of (1.68). First line: $\delta = 0.5$, second line $\delta = 0.05$. The inclusion of negative material is an ellipse of diameter proportional to δ and $\kappa_\sigma = -2.5$. Eigenfunctions associated with the positive (resp. negative) eigenvalues are localized in the positive (resp. negative) material.

1.6 Homogenization of Maxwell's equations

During the PhD of M. Rihani, with R. Bunoïu and K. Ramdani, we studied in [108] the homogenization process for time-harmonic Maxwell's equations in the presence of δ -periodically distributed inclusions of negative material embedded into a dielectric material (see Figure 1.28 for a typical configuration). The main objective was to clarify if the homogenization process is doable in this context and if so, to determine what are the properties of the corresponding homogenized material. For scalar problems, the first homogenization results have been obtained in [109] using the T-coercivity approach of [79]. More precisely, it is shown therein that for negative contrasts close to zero, the scalar problem with Dirichlet boundary conditions can be homogenized. In particular, it is proved that under this assumption on the contrast, the solution of the problem in the composite material is well-defined for δ small enough and that it two-scale converges (see [389, 9]) to the solution of a well-posed problem set in a homogeneous material. These results have been extended in [85], through the analysis of the spectrum of the Neumann-Poincaré operator. The authors demonstrate that the homogenization process is possible provided that the contrast belongs to $(-\infty; -1/\alpha)$ or $(-\alpha; 0)$ for some $\alpha \in (0; 1)$. However, the value of α is not explicit. The outcome of [108] is double. First, we characterize these bounds appearing in the assumptions on the contrast under which homogenization can be performed. Second, we explain how to deal with Maxwell's equations. Let us describe this in more detail.

1.6.1 Main analysis

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain as in §1.4.1. We consider a situation where Ω is filled with a composite electromagnetic material constituted of periodically distributed inhomogeneous cells of small size $\delta > 0$. More precisely, let $Y = (0; 1)^3$ denote the reference cell and assume that Y contains two materials:

- a metamaterial with negative dielectric permittivity $\varepsilon_- < 0$ and magnetic permeability $\mu_- < 0$ located inside a connected domain $Y_- \subset Y$ with Lipschitz boundary ∂Y_- such that $\overline{Y_-} \subset Y$;
- a dielectric material with positive dielectric permittivity $\varepsilon_+ > 0$ and magnetic permeability $\mu_+ > 0$ filling the region $Y_+ := Y \setminus \overline{Y_-}$.

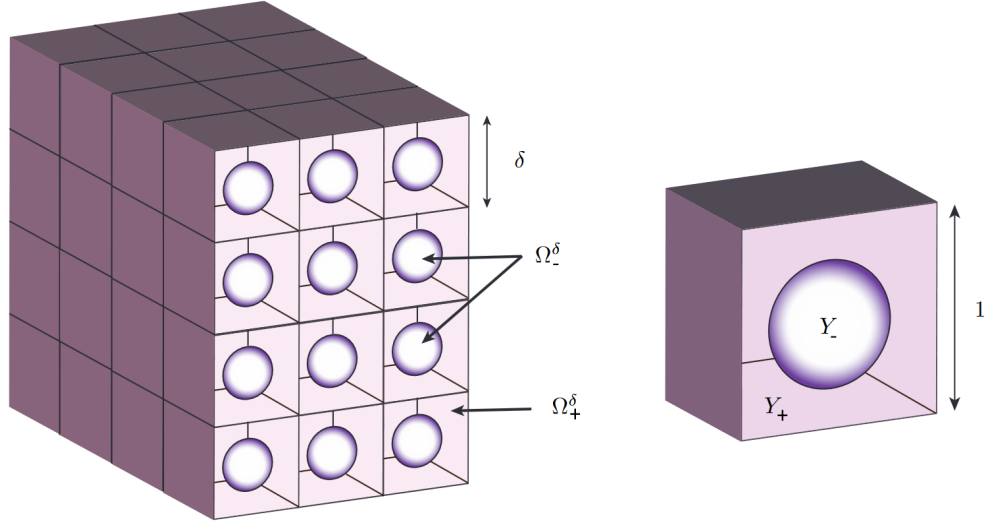


Figure 1.28: Example of a periodic material (left) and corresponding reference cell Y (right).

The assumption $\overline{Y_-} \subset Y$ is important. When the inclusion Y_- meets the boundary of the cell ∂Y , for example in laminar materials, phenomena different from the ones described below can appear. We refer the reader to [85, Appendix A] for more details concerning the scalar problem in this case. To simplify the presentation, we assume that ε_- , ε_+ , μ_- and μ_+ are constant. Set

$$\kappa_\varepsilon := \frac{\varepsilon_-}{\varepsilon_+} < 0, \quad \kappa_\mu := \frac{\mu_-}{\mu_+} < 0,$$

and define in the reference cell the two real-valued functions $\varepsilon, \mu \in L^\infty(Y)$ such that

$$\varepsilon(y) = \varepsilon_+ \mathbb{1}_{Y_+}(y) + \varepsilon_- \mathbb{1}_{Y_-}(y), \quad \mu(y) = \mu_+ \mathbb{1}_{Y_+}(y) + \mu_- \mathbb{1}_{Y_-}(y),$$

where for a set S , $\mathbb{1}_S(\cdot)$ stands for the indicator function of S . For any $\delta > 0$ and any integer vector $k \in \mathbb{Z}^3$, we define the shifted and scaled sets Y_{-k}^δ , Y_{+k}^δ , Y_k^δ such that

$$\begin{aligned} Y_{-k}^\delta &:= \{\mathbf{x} \in \mathbb{R}^3 \mid (\mathbf{x} - k)/\delta \in Y_-\} \\ Y_{+k}^\delta &:= \{\mathbf{x} \in \mathbb{R}^3 \mid (\mathbf{x} - k)/\delta \in Y_+\} \\ Y_k^\delta &:= \{\mathbf{x} \in \mathbb{R}^3 \mid (\mathbf{x} - k)/\delta \in Y\}. \end{aligned}$$

We denote by K^δ the set of $k \in \mathbb{Z}^3$ such that $Y_k^\delta \subset \Omega$. We assume that the metamaterial fills the region

$$\Omega_-^\delta := \bigcup_{k \in K^\delta} Y_{-k}^\delta,$$

while the complementary set in Ω

$$\Omega_+^\delta = \Omega \setminus \overline{\Omega_-^\delta}$$

is occupied by the dielectric. Finally we define the macroscopic dielectric permittivity ε^δ and the magnetic permeability μ^δ in Ω such that

$$\varepsilon^\delta(x) = \varepsilon_+ \mathbb{1}_{\Omega_+^\delta}(x) + \varepsilon_- \mathbb{1}_{\Omega_-^\delta}(x), \quad \mu^\delta(x) = \mu_+ \mathbb{1}_{\Omega_+^\delta}(x) + \mu_- \mathbb{1}_{\Omega_-^\delta}(x).$$

As in Section 1.4, we can reduce the analysis of Maxwell's equations to the study of the following problem for the electric field

$$(\mathcal{P}^\delta) \left| \begin{array}{l} \text{Find } \mathbf{E}^\delta \in \mathbf{H}_N(\mathbf{curl}) \text{ such that} \\ \int_{\Omega} (\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}^\delta \cdot \mathbf{curl} \mathbf{E}' - \omega^2 \varepsilon^\delta \mathbf{E}^\delta \cdot \mathbf{E}' d\mathbf{x} = i\omega \int_{\Omega} \mathbf{J} \cdot \mathbf{E}' d\mathbf{x}, \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}). \end{array} \right. \quad (1.72)$$

As we have seen in Section 1.4, the well-posedness of (1.72) relies on the invertibility of the two scalar operators $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$, $A_\mu^\delta : H_\#^1(\Omega) \rightarrow H_\#^1(\Omega)$ such that

$$\begin{aligned} (\nabla(A_\varepsilon^\delta \varphi), \nabla \varphi')_\Omega &= \int_\Omega \varepsilon^\delta \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x}, & \forall \varphi, \varphi' \in H_0^1(\Omega) \\ (\nabla(A_\mu^\delta \varphi), \nabla \varphi')_\Omega &= \int_\Omega \mu^\delta \nabla \varphi \cdot \nabla \overline{\varphi'} d\mathbf{x}, & \forall \varphi, \varphi' \in H_\#^1(\Omega). \end{aligned}$$

More precisely, in order to consider the limit of (1.72) as δ tends to zero, we need a result of uniform invertibility of A_ε^δ , A_μ^δ , *i.e.* we have to show the existence of some $\delta_0 > 0$ such that A_ε^δ , A_μ^δ are invertible for all $\delta \in (0; \delta_0]$ together with the estimate

$$\|(A_\varepsilon^\delta)^{-1}\| + \|(A_\mu^\delta)^{-1}\| \leq C,$$

where $C > 0$ is independent of $\delta \in (0; \delta_0]$. Introduce the Hilbert spaces of functions defined in the reference cell Y

$$\begin{aligned} \mathcal{H}_0 &:= \{\varphi \in H_0^1(Y) \mid \Delta \varphi = 0 \text{ in } Y_+ \cup Y_-\} \\ \mathcal{H}_\diamond &:= \{\varphi \in H_\diamond^1(Y) \mid \Delta \varphi = 0 \text{ in } Y_+ \cup Y_-\} \end{aligned}$$

where $H_\diamond^1(Y) := \{\varphi \in H^1(Y) \mid \int_{\partial Y_-} \varphi d\sigma = 0\}$. Define the function $\varphi_D \in \mathcal{H}_0$ such that $\varphi_D = 1$ in Y_- and set

$$\begin{aligned} \hat{\mathcal{H}}_0 &:= \{\varphi \in \mathcal{H}_0 \mid (\nabla \varphi, \nabla \varphi_D)_Y = 0\} \\ \hat{\mathcal{H}}_\diamond &:= \{\varphi \in \mathcal{H}_\diamond \mid \partial_n \varphi = 0 \text{ on } \partial Y\}. \end{aligned} \tag{1.73}$$

Then define the (positive) constants

$$m := \inf_{\varphi \in \hat{\mathcal{H}}_0 \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_-}^2}{\|\nabla \varphi\|_{Y_+}^2}, \quad M := \sup_{\varphi \in \hat{\mathcal{H}}_\diamond \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_-}^2}{\|\nabla \varphi\|_{Y_+}^2}. \tag{1.74}$$

We emphasize that m and M are independent of δ . Our criteria of uniform invertibility of A_ε^δ , A_μ^δ are as follows ([108, Thm. 3.14], see also §1.6.2 below for some intuition).

Theorem 1.6.1 *Let m, M be the constants defined in (1.74).*

When $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$, $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is uniformly invertible as $\delta \rightarrow 0$.

When $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$, $A_\mu^\delta : H_\#^1(\Omega) \rightarrow H_\#^1(\Omega)$ is uniformly invertible as $\delta \rightarrow 0$.

We then turn to the definition of the homogenized problem associated with (1.72). Denote by $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$ the subset of functions of $\mathcal{C}^\infty(\bar{Y})$ satisfying periodic boundary conditions on ∂Y . Let $H_{\text{per}}^1(Y)$ be the closure of $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$ for the norm of $H^1(Y)$. Then set

$$H_{\text{per},\diamond}^1(Y) := \{\varphi \in H_{\text{per}}^1(Y) \mid \int_{\partial Y_-} \varphi d\sigma = 0\}.$$

We endow this space with the inner product $(\nabla \cdot, \nabla \cdot)_Y$. Assume that the contrasts κ_ε and κ_μ are located in $(-\infty; -1/m) \cup (-1/M; 0)$. For $\eta = \varepsilon$ or μ and $j = 1, 2, 3$, we can show ([108, Thm. 4.1]) that there are unique functions $\chi_j^\eta \in H_{\text{per},\diamond}^1(Y)$ such that

$$(\eta \nabla \chi_j^\eta, \nabla \xi)_Y = (\eta \nabla y_j, \nabla \xi)_Y, \quad \forall \xi \in H_{\text{per},\diamond}^1(Y).$$

The homogenized tensor associated with η is classically defined as the 3×3 symmetric matrix $\mathcal{H}(\eta) = (\mathcal{H}_{jk}(\eta))_{1 \leq j,k \leq 3}$ given by (see, for instance, [183])

$$\mathcal{H}_{jk}(\eta) = \frac{1}{|Y|} \int_Y \eta \nabla(y_j - \chi_j^\eta) \cdot \nabla(y_k - \chi_k^\eta) d\mathbf{y}.$$

Though $\mathcal{H}(\varepsilon)$, $\mathcal{H}(\mu)$ involve sign-changing coefficients, we proved the following non-intuitive result ([108, Prop. 4.2]).

Proposition 1.6.2 *Assume that κ_ε (resp. κ_μ) $\in (-\infty; -1/m) \cup (-1/M; 0)$ where m, M are defined in (1.74). Then the matrix $\mathcal{H}(\varepsilon)$ (resp. $\mathcal{H}(\mu)$) is positive definite.*

Then we can define the effective problem

$$(\mathcal{P}^{\text{eff}}) \quad \left| \begin{array}{l} \text{Find } \mathbf{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that for all } \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}) \\ \int_{\Omega} \mathcal{H}(\mu)^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}} \cdot \mathbf{curl} \mathbf{E}' - \omega^2 \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \mathbf{E}' dx = i\omega \int_{\Omega} \mathbf{J} \cdot \mathbf{E}' dx. \end{array} \right. \quad (1.75)$$

By exploiting Proposition 1.6.2, when $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$, one classically proves that $(\mathcal{P}^{\text{eff}})$ admits a unique solution for all

$$\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}} \quad (1.76)$$

where Λ^{eff} is a discrete subset of $[0; +\infty)$. Then by working with the T-coercivity for Maxwell's equations as in Section 1.4 (a result of uniform compactness is also needed), we establish the following stability result.

Proposition 1.6.3 *Assume that $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ where m, M are defined in (1.74). Assume that $\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}}$ where Λ^{eff} appears in (1.76). Then, there exists $\delta_0 > 0$ such that for all $\delta \in (0; \delta_0]$, problem $(\mathcal{P}^\delta)$ admits a unique solution $\mathbf{E}^\delta$. Moreover we have the estimate*

$$\|\mathbf{E}^\delta\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{curl} \mathbf{E}^\delta\|_{\mathbf{L}^2(\Omega)} \leq C \|\mathbf{J}\|_{\mathbf{L}^2(\Omega)} \quad (1.77)$$

where $C > 0$ is independent of $\delta \in (0; \delta_0]$.

Finally, by applying classical tools of homogenization which are insensitive to the change of sign of ε, μ , we obtained in [108] the following final result.

Theorem 1.6.4 *Assume that $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ where m, M are defined in (1.74). Assume that $\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}}$ where Λ^{eff} appears in (1.76). Then, there exists $\delta_0 > 0$ such that for $\delta \in (0; \delta_0]$, the solution $\mathbf{E}^\delta$ of problem $(\mathcal{P}^\delta)$, which is well-defined according to Proposition 1.6.3, satisfies*

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{and} \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{weakly in } \mathbf{L}^2(\Omega)$$

where $\mathbf{E}^{\text{eff}}$ is the unique solution of problem $(\mathcal{P}^{\text{eff}})$ given by (1.75).

Let us mention that in [108], we only prove weak convergence results. Strong convergence results (using correctors) for Maxwell's equations with positive materials have been obtained in [454, 455]. It would be interesting to understand if we can adapt the approach proposed in these two articles to our setting. On the other hand, it is not completely clear that doing homogenization of negative materials is some goal for physicists. In practice, it is already difficult to get negative coefficients without too much dissipation. And at the end, we see that we lose the negative material effect. The interest here lies more in the mathematical justification of the homogenization process, and more precisely in the derivation of bounds for the fields which are uniform with respect to δ , the main difficulty being that the geometry changes as δ tends to zero so that non zero kernel that we should avoid may appear.

1.6.2 Optimality of the criterion and connection to the Neumann-Poincaré operator

Let us discuss the criterion of invertibility of the scalar operators we have obtained above. We focus our attention on the analysis of A_ε^δ and to simplify, assume that Ω is a bounded domain of $\mathbb{R}^2$ which contains only one connected inclusion Ω_- such that $\overline{\Omega_-} \subset \Omega$ (see Figure 1.2). Therefore below we remove the index δ . We also assume that $\partial\Omega_-$ is of class $\mathcal{C}^2$. This is important to ensure

that the spectrum of Problem (1.79) below is discrete. In this case A_ε is Fredholm of index zero when $\kappa_\varepsilon \neq -1$. Therefore when $\kappa_\varepsilon \neq -1$, the operator A_ε is an isomorphism if and only if it is injective. As it has been observed in different works, see in particular [85], and as we recall below, the question of the injectivity of A_ε is directly linked to the spectrum of the so-called Neumann-Poincaré operator. The latter has been widely studied when Ω is the whole space $\mathbb{R}^d$. For this problem, among the references, let us cite [424, 5, 440, 442, 299, 246, 245, 419, 87, 88, 244]. Below, we use a symmetrization argument similar to the one used in [299]. We work with Dirichlet-to-Neumann maps following the approach of [244].

Spectrum of the Neumann-Poincaré operator

Set $\Sigma := \partial\Omega_-$ and introduce the two Dirichlet-to-Neumann operators $\Lambda_+ : H^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$, $\Lambda_- : H^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$ such that for $\varphi \in H^{1/2}(\Sigma)$, we have $\Lambda_+\varphi = \partial_{n_+}u_+$, $\Lambda_-\varphi = \partial_{n_-}u_-$ where u_+ , u_- solve respectively the problems

$$\left\{ \begin{array}{ll} \Delta u_+ = 0 & \text{in } \Omega_+ \\ u_+ = 0 & \text{on } \partial\Omega \\ u_+ = \varphi & \text{on } \Sigma \end{array} \right. \quad \left\{ \begin{array}{ll} \Delta u_- = 0 & \text{in } \Omega_- \\ u_- = \varphi & \text{on } \Sigma. \end{array} \right. \quad (1.78)$$

Here $n_\pm$ denotes the outward unit normal vector to $\partial\Omega_\pm$. Define also the lifting operator $\mathcal{R} : H^{1/2}(\Sigma) \rightarrow H_0^1(\Omega)$ such that $\mathcal{R}\varphi = u_\pm$ in $\Omega_\pm$, where u_+ , u_- are the solutions to (1.78).

If u belongs to $\ker A_\varepsilon \setminus \{0\}$ then we have $\operatorname{div}(\varepsilon \nabla u) = 0$ in Ω and so in particular

$$\varepsilon_+ \partial_{n_+} u_+ = \varepsilon_- \partial_{n_-} u_- \quad \Leftrightarrow \quad \partial_{n_+} u_+ = -\kappa_\varepsilon \partial_{n_-} u_- \quad \text{on } \Sigma.$$

Therefore $\varphi := u|_\Sigma \in H^{1/2}(\Sigma) \setminus \{0\}$ satisfies $\Lambda_+\varphi = -\kappa_\varepsilon \Lambda_-\varphi$. This implies

$$(\Lambda_+ - \Lambda_-)\varphi = -(\kappa_\varepsilon + 1)\Lambda_-\varphi, \quad (\Lambda_+ + \Lambda_-)\varphi = (1 - \kappa_\varepsilon)\Lambda_-\varphi,$$

and so we find that the pair (α, φ) , with $\alpha := (\kappa_\varepsilon + 1)/(\kappa_\varepsilon - 1) \in (-1; 1)$, is a solution to the generalized eigenvalue problem

$$\left\{ \begin{array}{l} \text{Find } (\alpha, \varphi) \in \mathbb{R} \times (H^{1/2}(\Sigma) \setminus \{0\}) \text{ such that} \\ \Lambda_d \varphi = \alpha \Lambda_s \varphi \end{array} \right. \quad (1.79)$$

with $\Lambda_d := \Lambda_+ - \Lambda_-$ (d for difference), $\Lambda_s := \Lambda_+ + \Lambda_-$ (s for sum). Reciprocally, assume that (α, φ) is a solution to (1.79) with $\alpha \in (-1; 1)$. Then, $\mathcal{R}\varphi \in H_0^1(\Omega)$ is an element of $\ker A_\varepsilon \setminus \{0\}$ for $\kappa_\varepsilon = (\alpha + 1)/(\alpha - 1) \in (-\infty; 0)$. This shows that it is sufficient to determine the eigenvalues of Problem (1.79) to study the injectivity of A_ε . Note that the spectrum of (1.79) coincides with the spectrum of the Neumann-Poincaré operator.

Theorem 1.6.5 *The spectrum of the generalized eigenvalue problem (1.79) is discrete and coincides with two sequences of real numbers $(\alpha_n^-)_{n \in \mathbb{N}^*}$, $(\alpha_n^+)_{n \in \mathbb{N}}$ such that*

$$-1 < \alpha_1^- \leq \alpha_2^- \leq \dots \leq 0 \quad \text{and} \quad 1 = \alpha_0^+ > \alpha_1^+ \geq \dots \geq 0$$

and $\lim_{n \rightarrow +\infty} \alpha_n^\pm = 0$.

Proof. First, we show that $\Lambda_s : H^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$ is an isomorphism. Consider some $\psi \in H^{-1/2}(\Sigma)$. If $\varphi \in H^{1/2}(\Sigma)$ satisfies $\Lambda_s \varphi = \psi$, then $\mathcal{R}\varphi$ is a solution to

$$\left\{ \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ (\nabla u, \nabla v) = \langle \psi, v \rangle_\Sigma, \quad \forall v \in H_0^1(\Omega). \end{array} \right. \quad (1.80)$$

Reciprocally, assume that u is a solution to (1.80). Then the function $\varphi := u|_{\Sigma}$ satisfies $\Lambda_s \varphi = \psi$. According to the Lax-Milgram theorem, Problem (1.80) admits a unique solution for all $\psi \in H^{-1/2}(\Sigma)$. We infer that $\Lambda_s : H^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$ is indeed an isomorphism.

Now, remarking that Λ_+ , Λ_- have the same principal symbol and using standard arguments of pseudo-differential operators theory (work as in the proof of [245, Thm. 1]), we can show that $\Lambda_d = \Lambda_+ - \Lambda_- : H^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$ is compact. We emphasize that the assumption of smoothness of Σ here is important.

Using the Riesz representation theorem, define the operator $K : H^{1/2}(\Sigma) \rightarrow H^{1/2}(\Sigma)$ such that

$$(K\varphi, \varphi')_{\Sigma} = \langle \Lambda_d \varphi, \varphi' \rangle_{\Sigma} \quad \text{for all } \varphi, \varphi' \in H^{1/2}(\Sigma). \quad (1.81)$$

Here, we use the notation $(\cdot, \cdot)_{\Sigma} := \langle \Lambda_s \cdot, \cdot \rangle_{\Sigma}$. Note that according to the features of Λ_s , the latter form is an inner product in $H^{1/2}(\Sigma)$ equivalent to the usual one. Remark that (α, φ) is an eigenpair for (1.79) if and only if we have $K\varphi = \alpha\varphi$. But due to the properties of Λ_d , K is a selfadjoint and compact operator. Therefore, the spectrum of (1.79) coincides with a sequence of real eigenvalues which accumulate at zero. We can use the *min-max* principle (see [473, Chap. 3]) to characterize these eigenvalues. We have

$$\alpha_1^+ = \sup_{\varphi \in H^{1/2}(\Sigma) \setminus \{0\}} \frac{\langle \Lambda_d \varphi, \varphi \rangle_{\Sigma}}{\langle \Lambda_s \varphi, \varphi \rangle_{\Sigma}}.$$

By the *min-max* principle, we know that this sup is attained for some φ_1^+ . By induction, for $n \geq 2$, we define

$$\alpha_n^+ = \sup_{\substack{\varphi \in H^{1/2}(\Sigma) \setminus \{0\}, \\ \varphi \perp \{\varphi_1^+, \dots, \varphi_{n-1}^+\}}} \frac{\langle \Lambda_d \varphi, \varphi \rangle_{\Sigma}}{\langle \Lambda_s \varphi, \varphi \rangle_{\Sigma}}. \quad (1.82)$$

Here, if φ, φ' are two elements of $H^{1/2}(\Sigma)$, we write $\varphi \perp \varphi'$ when $(\varphi, \varphi')_{\Sigma} = \langle \Lambda_s \varphi, \varphi' \rangle_{\Sigma} = (\nabla(\mathcal{R}\varphi), \nabla(\mathcal{R}\varphi'))_{\Omega} = 0$. Similarly, we define

$$\alpha_1^- = \inf_{\varphi \in H^{1/2}(\Sigma) \setminus \{0\}} \frac{\langle \Lambda_d \varphi, \varphi \rangle_{\Sigma}}{\langle \Lambda_s \varphi, \varphi \rangle_{\Sigma}}, \quad (1.83)$$

and, by induction, for $n \geq 2$,

$$\alpha_n^- = \inf_{\substack{\varphi \in H^{1/2}(\Sigma) \setminus \{0\}, \\ \varphi \perp \{\varphi_1^-, \dots, \varphi_{n-1}^-\}}} \frac{\langle \Lambda_d \varphi, \varphi \rangle_{\Sigma}}{\langle \Lambda_s \varphi, \varphi \rangle_{\Sigma}}. \quad (1.84)$$

Observing that for all $\varphi \in H^{1/2}(\Sigma) \setminus \{0\}$ we have

$$\frac{\langle \Lambda_d \varphi, \varphi \rangle_{\Sigma}}{\langle \Lambda_s \varphi, \varphi \rangle_{\Sigma}} = \frac{1-a}{1+a}, \quad \text{with } a = \langle \Lambda_- \varphi, \varphi \rangle_{\Sigma} / \langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma} \geq 0,$$

we deduce that there holds $\alpha_n^{\pm} \in [-1; 1]$ for all $n \in \mathbb{N}^*$. Taking $\varphi = \varphi_D|_{\Sigma}$ where

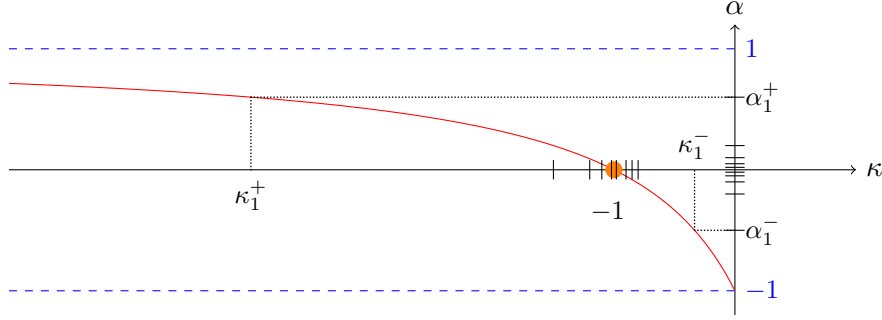
$$\varphi_D \in H_0^1(\Omega) \text{ is the function such that } \Delta \varphi_D = 0 \text{ in } \Omega_+ \cup \Omega_-, \varphi_D = 1 \text{ on } \Sigma, \quad (1.85)$$

we find $a = 0$ and so $\langle \Lambda_d \varphi, \varphi \rangle_{\Sigma} / \langle \Lambda_s \varphi, \varphi \rangle_{\Sigma} = 1$. We deduce that $\alpha_0^+ = 1$. Now, if $\alpha_1^+ = 1$, then there is $\varphi \in H^{1/2}(\Sigma) \setminus \{0\}$ such that $\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma} = 0$ and $\mathcal{R}\varphi \in \hat{\mathcal{H}}_0 \setminus \{0\}$ where as in (1.73),

$$\begin{aligned} \mathcal{H}_0 &:= \{\varphi \in H_0^1(\Omega) \mid \Delta \varphi = 0 \text{ in } \Omega_+ \cup \Omega_-\} \\ \hat{\mathcal{H}}_0 &:= \{\varphi \in \mathcal{H}_0 \mid (\nabla \varphi, \nabla \varphi_D)_{\Omega} = 0\}. \end{aligned}$$

This is impossible and therefore there holds $\alpha_1^+ < 1$. Similarly, if $\alpha_1^- = -1$, then there exists $\varphi \in H^{1/2}(\Sigma) \setminus \{0\}$ such that $\langle \Lambda_s \varphi, \varphi \rangle_{\Sigma} = 0$. This cannot happen, which implies that $\alpha_1^- > -1$. ■

Optimal invertibility conditions

Figure 1.29: Correspondence between the $\kappa_n^\pm$ and the $\alpha_n^\pm$.

From the discussion preceding the statement of Theorem 1.6.5, we obtain the following result.

Theorem 1.6.6 *For $\kappa_\varepsilon \in (-\infty; 0) \setminus \{-1\}$, the operator $A_\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is an isomorphism if and only if*

$$\kappa_\varepsilon \notin \{\kappa_n^+, n \in \mathbb{N}^*\} \cup \{\kappa_n^-, n \in \mathbb{N}^*\} \quad \text{where} \quad \kappa_n^\pm := \frac{\alpha_n^\pm + 1}{\alpha_n^\pm - 1}, \quad (1.86)$$

the $\alpha_n^\pm$ being defined in (1.82)–(1.84).

Observing that the map $\alpha \mapsto (\alpha + 1)/(\alpha - 1)$ is decreasing on $(-1; 1)$ (see Figure 1.29), we deduce in particular from Theorem 1.6.6 that A_ε is an isomorphism for

$$\kappa_\varepsilon \in (-\infty; \kappa_1^+) \cup (\kappa_1^-; 0). \quad (1.87)$$

Comparison with existing literature

In certain articles, see *e.g.* [79, 109], authors considered the T-coercivity approach with some operators T constructed globally and not via local geometric transformations. In particular, they worked with $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

$$T\varphi = \begin{cases} \varphi & \text{in } \Omega_+ \\ -\varphi + 2\mathcal{R}(\varphi|_\Sigma) & \text{in } \Omega_- \end{cases} \quad (1.88)$$

where the harmonic lifting $\mathcal{R}$ is defined in (1.78). We have $T \circ T = \text{Id}$ which shows that T is an isomorphism of $H_0^1(\Omega)$. On the other hand, for all $\varphi \in H_0^1(\Omega)$, we find

$$(\nabla(A_\varepsilon\varphi), \nabla(T\varphi))_\Omega = \varepsilon_+ \|\nabla\varphi\|_{L^2(\Omega_+)}^2 + |\varepsilon_-| \|\nabla\varphi\|_{L^2(\Omega_-)}^2 + 2\varepsilon_- (\nabla\varphi, \nabla\mathcal{R}(\varphi|_\Sigma))_{\Omega_-}. \quad (1.89)$$

Set

$$\tilde{M} := \sup_{\varphi \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla\mathcal{R}(\varphi|_\Sigma)\|_{\Omega_-}^2}{\|\nabla\varphi\|_{\Omega_+}^2}.$$

Using Young's inequality, from (1.89) we infer that for all $\tau > 0$, there holds

$$(\nabla(A_\varepsilon\varphi), \nabla(T\varphi))_\Omega \geq (\varepsilon_+ - \tau^{-1}|\varepsilon_-|\tilde{M}) \|\nabla\varphi\|_{L^2(\Omega_+)}^2 + |\varepsilon_-|(1 - \tau) \|\nabla\varphi\|_{L^2(\Omega_-)}^2.$$

We deduce that when $\varepsilon_+ > |\varepsilon_-|\tilde{M} \Leftrightarrow \kappa_\varepsilon = \varepsilon_-/\varepsilon_+ > -1/\tilde{M}$, the operator A_ε is an isomorphism of $H_0^1(\Omega)$. But using the characterization of α_1^- in (1.83), we can prove that the bound $-1/\tilde{M}$ coincides with the κ_1^- appearing in (1.87). Therefore the T-coercivity approach with T constructed as in (1.88) shows that A_ε is an isomorphism for $\kappa_\varepsilon \in (\kappa_1^-; 0)$. Thus, though this T is not optimal to establish that A_ε is a Fredholm operator, it is very efficient to demonstrate that it is an isomorphism.

T-coercivity operator in the general case

Finally, we explain how to construct operators of T-coercivity for contrasts κ_ε as in the statement of Theorem 1.6.6, in particular for contrasts in

$$(\kappa_1^+; \kappa_1^-).$$

First, we reindex the eigenvalues $\{\alpha_n^-\}_{n \geq 1}$, $\{\alpha_n^+\}_{n \geq 1}$ and denote them $\{\alpha_n\}_{n \geq 1}$. Let (φ_n) be a family of eigenfunctions of the operator K introduced in (1.81) associated with the α_n . We choose them so that the functions $\mathcal{R}\varphi_n$, $n \geq 1$, form an orthonormal basis of $\hat{\mathcal{H}}_0$ (see (1.73)). Now we define the operator $\mathbf{T}_D : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that for $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$ with $\hat{\varphi}_h = \sum_{n \in \mathbb{N}^*} \gamma_n \mathcal{R}\varphi_n \in \hat{\mathcal{H}}_0$, $\Phi_h \in \text{span}\{\varphi_D\}$ (see (1.85) for the definition of φ_D) and $\tilde{\varphi} \in H_0^1(\Omega_+ \cup \Omega_-)$, there holds

$$\mathbf{T}_D \varphi = \begin{cases} \sum_{n \in \mathbb{N}^*} t_n \gamma_n \mathcal{R}\varphi_n + \Phi_h + \tilde{\varphi} & \text{in } \Omega_+ \\ \sum_{n \in \mathbb{N}^*} t_n \gamma_n \mathcal{R}\varphi_n + \Phi_h - \tilde{\varphi} & \text{in } \Omega_- \end{cases} \quad (1.90)$$

Here we take $t_n = 1$ for n such that $\kappa_\varepsilon > \kappa_n := (\alpha_n + 1)/(\alpha_n - 1)$ and $t_n = -1$ otherwise. The operator $\mathbf{T}_D$ is valued in $H_0^1(\Omega)$ and we have $\mathbf{T}_D \circ \mathbf{T}_D = \text{Id}$ which guarantees that $\mathbf{T}_D$ is an isomorphism of $H_0^1(\Omega)$.

Proposition 1.6.7 *Assume that $\kappa_\varepsilon \neq -1$ is such that for all $n \in \mathbb{N}^*$, we have $\kappa_\varepsilon \neq \kappa_n$ with*

$$\kappa_n = \frac{\alpha_n + 1}{\alpha_n - 1}.$$

Let $\mathbf{T}_D : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ denote the isomorphism defined in (1.90). Then $(\nabla(A_\varepsilon \cdot), \nabla(\mathbf{T}_D \cdot))_\Omega$ is coercive in $H_0^1(\Omega)$. As a consequence, $A_\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is an isomorphism.

Proof. For all $\varphi \in H_0^1(\Omega)$, we find

$$(\varepsilon \nabla \varphi, \nabla(\mathbf{T}_D \varphi))_\Omega = \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_\Omega + (\varepsilon_+ \nabla \Phi_h, \nabla \Phi_h)_{\Omega_+} + (|\varepsilon| \nabla \tilde{\varphi}, \nabla \tilde{\varphi})_\Omega. \quad (1.91)$$

But by the definition of the κ_n , we have, for all $n \in \mathbb{N}^*$,

$$(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_+} = -\kappa_n (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_-}.$$

This allows us to write

$$\begin{aligned} \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_\Omega &= \varepsilon_+ \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\kappa_\varepsilon - \kappa_n) (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_-} \\ &= \varepsilon_+ \sum_{n \in \mathbb{N}^*} |\gamma_n|^2 |\kappa_\varepsilon - \kappa_n| (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_-}. \end{aligned} \quad (1.92)$$

Observing that we have $\|\nabla(\mathcal{R}\varphi_n)\|_{L^2(\Omega_-)}^2 \geq \inf_{m \in \mathbb{N}^*} |\kappa_m|^{-1} \|\nabla(\mathcal{R}\varphi_n)\|_{L^2(\Omega_+)}^2$ (note that the sequence $(|\kappa_m|)$ is bounded), from (1.92) we obtain

$$\sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_\Omega \geq C \inf_{n \in \mathbb{N}^*} |\kappa_\varepsilon - \kappa_n| \sum_{n \in \mathbb{N}^*} |\gamma_n|^2 \|\nabla(\mathcal{R}\varphi_n)\|_{L^2(\Omega)}^2. \quad (1.93)$$

Finally, using (1.93) into (1.91), we get $(\varepsilon \nabla \varphi, \nabla(\mathbf{T}_D \varphi))_\Omega \geq C \inf_{n \in \mathbb{N}^*} |\kappa_\varepsilon - \kappa_n| \|\nabla \varphi\|_{L^2(\Omega)}^2$ for all $\varphi \in H_0^1(\Omega)$ which proves that $(\nabla(A_\varepsilon \cdot), \nabla(\mathbf{T}_D \cdot))_\Omega$ is coercive in $H_0^1(\Omega)$. $\blacksquare$

Remark 1.6.8 For the problem of homogenization (1.72), for each value of $\delta > 0$, from Theorem 1.6.6, we deduce that well-posedness is ensured when $\kappa_\varepsilon \neq -1$, $\kappa_\mu \neq -1$ are such that

$$\kappa_\varepsilon \notin \{\kappa_{\varepsilon,n}^{\delta+}, n \in \mathbb{N}^*\} \cup \{\kappa_{\varepsilon,n}^{\delta-}, n \in \mathbb{N}^*\}, \quad \kappa_\mu \notin \{\kappa_{\mu,n}^{\delta+}, n \in \mathbb{N}^*\} \cup \{\kappa_{\mu,n}^{\delta-}, n \in \mathbb{N}^*\},$$

where the $\kappa_{\varepsilon,n}^{\delta\pm}$, $\kappa_{\mu,n}^{\delta\pm}$ are defined as the $\kappa_n^\pm$ in (1.86) for the operators A_δ^ε , A_δ^μ . To obtain the uniform estimate (1.77), we need κ_ε , κ_μ to avoid the sets

$$\bigcup_{\delta \in (0; \delta_0]} \{\kappa_{\varepsilon,n}^{\delta+}, n \in \mathbb{N}^*\} \cup \{\kappa_{\varepsilon,n}^{\delta-}, n \in \mathbb{N}^*\}, \quad \bigcup_{\delta \in (0; \delta_0]} \{\kappa_{\mu,n}^{\delta+}, n \in \mathbb{N}^*\} \cup \{\kappa_{\mu,n}^{\delta-}, n \in \mathbb{N}^*\}. \quad (1.94)$$

We proved in Theorem 1.6.1 that this is true when κ_ε , κ_μ belong to $(-\infty; -1/m) \cup (-1/M; 0)$, where m , M are defined in (1.74). We do not know if these bounds are optimal or not. On the other hand, we could imagine that in certain situations the sets (1.94) are not intervals and that they have gaps. For contrasts inside these gaps, one could think to prove homogenization results but this has to be studied.

Historical digression

Following the very interesting introduction of [299], let us explain how the name of Poincaré appears in this story, though plasmonic technologies and negative metamaterials were unknown to him. In 1897, in the article [424], he considered the following question:

Let Σ be a smooth closed surface enclosing some domain $\Omega_- \subset \mathbb{R}^3$. Let u_φ be the potential associated to an electrostatic charge φ distributed on Σ . If we normalize the total energy in $\mathbb{R}^3$, what is the minimum of energy in Ω_- ?

With our notation, this writes: what is

$$\inf_{\varphi \neq 0} \frac{\int_{\Omega_-} |\nabla u_\varphi|^2 d\mathbf{x}}{\int_{\mathbb{R}^3} |\nabla u_\varphi|^2 d\mathbf{x}} \quad \text{where } u_\varphi \text{ satisfies } \begin{cases} \Delta u_\varphi = 0 & \text{in } \mathbb{R}^3 \setminus \Sigma \\ u_\varphi = \varphi & \text{on } \Sigma \end{cases} ?$$

The answer is simple, this infimum is zero and it is attained for $\varphi \equiv 1$. But then Poincaré continued.

What is the infimum of the same quantity under the additional condition u_φ orthogonal to u_1 ?

In our language: what is

$$\inf_{\varphi \neq 0, u_\varphi \perp u_1} \frac{\int_{\Omega_-} |\nabla u_\varphi|^2 d\mathbf{x}}{\int_{\mathbb{R}^3} |\nabla u_\varphi|^2 d\mathbf{x}} \quad \text{where } u_\varphi \perp u_1 \text{ means } \int_{\mathbb{R}^3} \nabla u_\varphi \cdot \nabla u_1 d\mathbf{x} = 0?$$

With our modern tools, we can show that this infimum, equal to $\lambda \in (0; 1)$, is attained, for some $\varphi = \varphi_2$. Classically, it is characterized by the variational equality

$$\int_{\Omega_-} \nabla u_2 \cdot \nabla v d\mathbf{x} = \lambda \int_{\mathbb{R}^3} \nabla u_2 \cdot \nabla v d\mathbf{x}, \quad \forall v \in \mathcal{C}_0^\infty(\mathbb{R}^3), \quad (1.95)$$

where we write u_2 instead of u_{φ_2} . Identity (1.95) can be equivalently expressed as

$$\int_{\Omega_+} \nabla u_2 \cdot \nabla v d\mathbf{x} + (1 - \lambda^{-1}) \int_{\Omega_-} \nabla u_2 \cdot \nabla v d\mathbf{x} = 0,$$

where $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega_-}$. Therefore we see that u_2 satisfies

$$\operatorname{div}(\varepsilon \nabla u_2) = 0 \text{ in } \mathbb{R}^3 \quad \text{with} \quad \varepsilon = \begin{cases} 1 & \text{in } \Omega_+ \\ 1 - \lambda^{-1} & \text{in } \Omega_- \end{cases} \quad (1.96)$$

Thus solving the Poincaré problem gives a ratio of energy such that the parameter ε defined via (1.96), which changes sign in Ω , is such that the operator $\operatorname{div}(\varepsilon \nabla \cdot) : H^1(\mathbb{R}^3) \rightarrow (H^1(\mathbb{R}^3))'$ has a non zero kernel. Then we can iterate the process by looking at the successive infima with the constraint $u_\varphi \perp \{u_1, \dots, u_n\}$, where we write u_n instead of u_{φ_n} , φ_n being a charge for which the infimum is attained at step n .

1.7 Future directions and open questions

Let us present here additional questions or directions that it would be interesting to consider.

i) Concerning the approximation of the solutions of (1.3) by classical finite element methods (see Problem (1.25)), without mesh assumptions, we still do not know how to justify the convergence when h tends to zero. Numerically, in general we do not observe much problems when the contrast is not too close to the critical interval and as soon as the interface is included in the skeleton of the mesh but this may appear, as pointed out in the recent work [261]. For alternative methods whose convergence can be justified, we refer the reader to [182, 2, 113, 259].

ii) In Section 1.3, we explained how to define a new functional framework to recover Fredholmness when the interface has a corner in 2D and for a contrast inside the critical interval. During the PhD of M. Rihani, we extended the analysis in 3D for conical tips. In 3D, one can also have edges or intersections of edges in the interface between the two materials. In that case, computing the singularities and playing with them to define a new setting where one has well-posedness is an open question. Let us mention that this study shares similarities with certain works concerning time-harmonic Maxwell's equations in magnetized plasma. Indeed, some cold plasma models lead to consider smooth ε which change sign. Due to the smoothness of ε , there are lines or surfaces where ε vanishes, something that we exclude in our case. This is responsible for the phenomenon of hybrid resonance near the interface (see *e.g.* [210, 398]) which obliges one to modify the usual functional framework to get well-posed problems.

iii) In 3D for the conical tip, we mentioned that one may have several propagating singularities. We explained that we define, in accordance with the limiting absorption principle, the outgoing singularities as the ones which bring energy to the corner point. In certain situations, this leads us to select outgoing singularities with η_n (in the power $r^{-1/2+i\eta_n}$) of different signs. This is like having so called inverse modes in waveguides. In that case, the numerical method proposed in §1.3.3 based on the use of the PMLs at the tips fails to capture the physical solution and must be adapted (see [64] for a repair).

iv) In the introduction, we mentioned that negative metamaterials are actually assemblies of small resonators and that the negative effect is obtained after a homogenization process. In his PhD [48] supervised by X. Claeys and S. Fliss, C. Beneteau proposed enriched homogenized models in presence of boundaries. The goal was to derive models describing more precisely what happens at the interface between a metamaterial and a usual dielectric. He considered planar interfaces. It would be interesting to look at interfaces with corners and see if in the refined model there is a trace of the black-hole waves.

v) The numerical analysis of Maxwell's equations with sign-changing coefficients still has to be studied. For contrasts outside of the critical intervals, one can hope to show that classical methods based on edge finite elements converge. However, this has to be proved. On the other hand, the question of the approximation of the solution in the new framework proposed in §1.4.2 is widely open. For the moment it is not clear if the method with PMLs presented in §1.3.3 to deal with the

scalar problem can be adapted to Maxwell's problem.

vi) For the numerical aspects, methods based on surface integral equations seem to be more appropriate. The convergence should be simple to establish. I am not sure that this is written properly somewhere. On the other hand, it would be interesting to adapt the new functional framework with the propagating singularities proposed in §1.4.2 for the surface integral equations. Is it just a transposition or should one prove new results? For a closely related question, namely the computation of eigenvalues embedded in the continuous spectrum for the Neumann-Poincaré operator, we refer the reader to the nice article [229].

vii) In practice, the coefficients ε and μ depend on the frequency ω . For example for metals at optical frequencies, one sometimes works with the non dissipative Drude model

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right),$$

where ω_p is the plasma frequency. It would be relevant to incorporate this aspect in the analysis as it has been done for the scalar problems in [264]. For the frameworks with singular behaviors, both for the scalar and Maxwell problems, this seems far from being obvious because the singular exponent, and even sometimes the number of black-hole waves, depend on ω .

viii) Another interesting direction is to study what happens in time-domain regime. Can we show some limiting amplitude principle? This question has been studied in detail by M. Cassier during his PhD [134] supervised by C. Hazard and P. Joly and after (see the corresponding articles [137, 138]). Planar interfaces have been considered. Interfaces with corners still wait for investigations.

PROPAGATION OF WAVES IN WAVEGUIDES

2.1 Introduction

In this chapter, we are interested in the propagation of waves for scalar equations in guides which are unbounded in one direction. Such problems arise in many fields of physics. For example air ducts and horns in acoustics carry sound waves in musical instruments as well as in loudspeakers. Conductive metal pipes are exploited to propagate high frequency radio waves while optical fibers serve as waveguides for light in electromagnetics. Waveguide problems also appear in water-waves theory, in classical mechanics or in quantum mechanics. In general, the diffraction of an incident wave in such structures in presence of an obstacle generates a reflection and a transmission characterized by some scattering coefficients. Broadly speaking, our goal is to play with the geometry, the frequency and/or the index material to control these scattering coefficients.

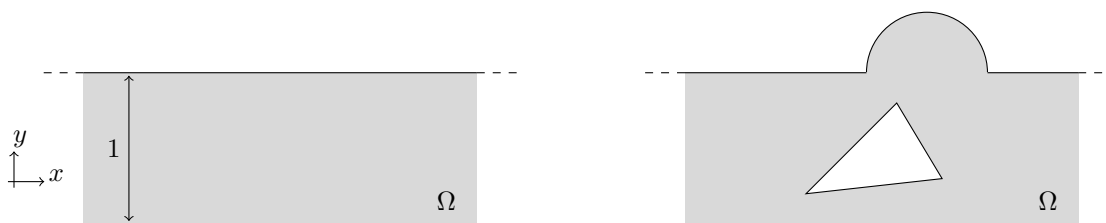


Figure 2.1: Left: reference strip. Right: perturbed waveguide.

To be more concrete, let us introduce the simplest problem of propagation of waves that we will consider. Let $\Omega \subset \mathbb{R}^2$ be a waveguide which coincides with the reference strip $\mathcal{S} := \{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside of a compact region (see Figure 2.1). The domain Ω is connected with Lipschitz boundary. Propagation of acoustic waves in Ω is governed by the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Here u stands for the acoustic pressure in Ω , $k > 0$ is the wavenumber and ∂_ν refers to the outward normal derivative on $\partial\Omega$. We mention acoustics here but the study of this problem is also relevant in optics, in microwaves or in water-waves theory. Now we introduce what we usually call the modes of the waveguide. These are defined as the functions with separated variables which solve (2.1) in the strip $\mathcal{S}$. After a classical calculation, one finds that they coincide with the family $\{w_n^\pm\}_{n \in \mathbb{N}}$ with

$$w_n^\pm(x, y) = e^{\pm i\beta_n x} \varphi_n(y), \quad \beta_n := \sqrt{k^2 - n^2 \pi^2}, \quad \varphi_n(y) = \begin{cases} 1 & \text{if } n = 0 \\ \sqrt{2} \cos(n\pi y) & \text{for } n > 0. \end{cases} \quad (2.2)$$

For a complex number $z = \rho e^{i\eta}$ with $\rho \geq 0$ and $\eta \in [0; 2\pi)$, its square root is defined as $\sqrt{z} = \sqrt{\rho} e^{i\eta/2}$. As a consequence, for any $z \in \mathbb{C}$, there holds $\Im m \sqrt{z} \geq 0$. Let us emphasize that for a fixed $k > 0$, there is a finite number of propagating modes in (2.2) which do not decay at infinity and an infinite number of modes which are exponentially growing/decaying in the two branches of Ω . One can show that any solution of (2.1) decomposes, outside of the geometric perturbation, on these modes. This allows one to get a semi-explicit representation of the field which is interesting for several reasons, for example to impose radiation condition at infinity in (2.1) which is crucial to have a well-posed problem.

To stick to the simplest setting, we assume that k belongs to $(0; \pi)$ so that only the plane waves $w^\pm$, with

$$w^\pm(x, y) = w_0^\pm(x, y) = e^{\pm i k x}, \quad (2.3)$$

can propagate in Ω . The scattering of the right-going plane wave in Ω leads to consider the solution of (2.1) admitting the expansion

$$u = \begin{cases} w^+ + R w^- + \tilde{u} & \text{for } x \leq -d \\ T w^+ + \tilde{u} & \text{for } x \geq d, \end{cases} \quad (2.4)$$

where $R, T \in \mathbb{C}$ are constants and $\tilde{u}$ is a remainder which decays as $O(e^{-\beta_1|x|})$ as $x \rightarrow \pm\infty$. Moreover here $d > 0$ is such that $\Omega = \mathcal{S}$ for $|x| \geq d$. Due to conservation of energy, classically one shows that the reflection and transmission coefficients satisfy the relation

$$|R|^2 + |T|^2 = 1. \quad (2.5)$$

The general goal of the first works presented in this chapter (up to Section 2.9 included), is to find situations, by playing with the geometry, the wavenumber k , ... where we have some sort of invisibility. The weakest invisibility that one can look for is $R = 0$. In the sequel, we shall say that one has zero reflection or that the defect is non reflecting. In such a situation, an observer generating an incident plane wave in the left branch, located a bit far from the defect in the geometry and measuring the resulting backscattering field will only measure the evanescent component. Due to noise, her/his measurements will not differ much from that in the reference strip and therefore she/he will be unable to detect the presence of the obstacle. Note that due to conservation of energy (2.5), the fact that $R = 0$ implies $|T| = 1$ and so $T = e^{i\theta}$ for a certain $\theta \in \mathbb{R}$. As a consequence in general there is a phase shift in the transmitted field which indicates the presence of the defect.

A more demanding definition of invisibility is to have $T = 1$. In that situation, we shall say that the defect is perfectly invisible or that we have perfect transmission without phase shift.

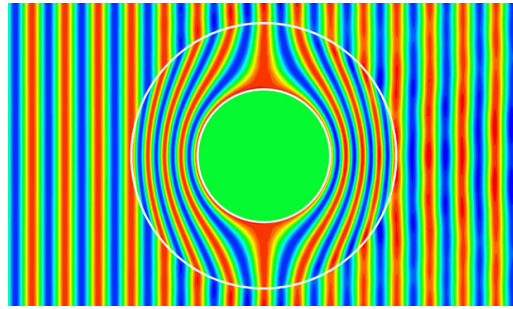


Figure 2.2: Cloaking by transformation optics.

Talking about invisibility, it is important to mention that our strategies are quite different from some ideas existing in the literature. The problem of cloaking an object has a large number of

applications and has been the subject of intense studies over the last decade in the theory of wave propagation. Thus, the methods based on anomalous resonances [356, 93, 311, 392, 349] and on transformation optics [417, 335, 401, 390, 146, 242] have encountered a large success which is related to the development of metamaterials. Our approach is different from these techniques, in particular from the cloaking via transformation optics pictured in Figure 2.2. Let us describe this latter device. Imagine that one wishes to hide a given obstacle. One way to proceed consists in surrounding it by a well chosen penetrable material, localized in the annulus around the green region on the picture, so that an incoming wave leaves the whole device as if there was nothing. Said differently, on the picture, an observer probing the field outside of the white thin circle obtains the same measurements as in free space and so cannot detect the presence of the obstacle. Mathematically, this idea is quite simple to implement, it boils down to a change of variable. However this change of variable is singular and for this reason the physical parameters of the *ad hoc* material in the cloaking device should take infinite values. Therefore, for the moment designing such structures, even by working with metamaterials, is still unreachable, in particular due to the presence of important losses.

What we propose is less ambitious because we only wish to control the scattering coefficients and not the inner field. In short, we simply wish to cloak defects at infinity. For this reason, it is more easily doable. Actually, we will not even need to work with penetrable materials. We will see that working with a homogeneous material and playing only with the geometry suffices. On the other hand, though our setting is less ambitious, we think it is still relevant in numerous applications. Indeed, the evanescent part of the field that we neglect is exponentially decaying at infinity and therefore is really difficult to distinguish from noise a few wavelengths far from the obstacle.

Let us emphasize that the main difficulty of the problem lies in the fact that the dependence of the scattering coefficients with respect to the parameters (geometry, k , ...) is not explicit and not linear. In order to address it, techniques of optimization have been considered. We refer the reader in particular to [6, 328, 329, 327]. However, due to the features of the Helmholtz equation, the functionals involved in the analysis are non convex and unsatisfying local minima exist. Moreover, these methods do not allow the user to control the main features of the shape compared to the approaches we propose.

In what follows, we present several ideas to reach invisibility and different variations in that direction. Let us mention that this shares connections with different topics of wave physics such as extraordinary optical transmission [215], topological states immune to backscattering [471, 345], perfect transmission resonances [479], transmission eigenchannels through disordered media [300, 443], reflectionless metamaterials [430] or metasurfaces [476, 17].

2.2 Non reflecting smooth perturbations of the geometry

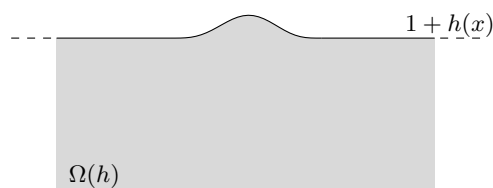


Figure 2.3: Perturbed waveguide $\Omega(h)$.

The first work that led to the emergence of this whole theme was the article [82]. There the

authors develop perturbative techniques to build non reflecting geometries. Let us describe the method which in this specific case is similar to the proof of the implicit function theorem. Let us mention that before [82], the approach was developed in [376, 377] to construct examples of non symmetric waveguides for which eigenvalues exist in the continuous spectrum, a question which has remained open for a while.

2.2.1 General scheme

Consider some real valued profile function $h \in \mathcal{C}_0^\infty(\mathbb{R})$, the space of infinitely differentiable functions with compact support, and let $\Omega(h)$ be the waveguide whose upper boundary coincides with the graph of the function $1 + h$ (see Figure 2.3). Denote by $R(h)$ the reflection coefficient of the scattering solution appearing in (2.4) in the geometry $\Omega(h)$. Importantly, we have $R(0) = 0$ because when $h \equiv 0$, $\Omega(h)$ is simply the reference strip $\mathcal{S} = \mathbb{R} \times (0; 1)$. With this notation, the problem we consider writes

$$\left| \begin{array}{l} \text{Find } h \neq 0 \text{ such that} \\ R(h) = 0. \end{array} \right.$$

Let us look for non reflecting geometries which are small perturbations of $\mathcal{S}$. To proceed, let us look for $h = \varepsilon\mu$ with $\varepsilon > 0$ small and $\mu \in \mathcal{C}_0^\infty(\mathbb{R})$ to be determined. Since ε is small, we can write an asymptotic expansion of R with respect to ε . We obtain

$$\begin{aligned} R(\varepsilon\mu) &= R(0) + \varepsilon dR(0)(\mu) + \varepsilon^2 \tilde{R}(\varepsilon\mu) \\ &= \varepsilon dR(0)(\mu) + \varepsilon^2 \tilde{R}(\varepsilon\mu) \end{aligned} \quad (2.6)$$

where $dR(0)(\mu)$ stands for the differential of R at zero in the direction μ and $\tilde{R}(\varepsilon\mu)$ is an abstract remainder. Since $dR(0) : \mathcal{C}_0^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ is a linear map from a space of infinite dimension to a space of dimension two (remember that we work with real valued functions), we have $\dim \ker dR(0) = +\infty$ and we can pick $\mu_0 \neq 0$ such that

$$dR(0)(\mu_0) = 0. \quad (2.7)$$

By choosing $h = \varepsilon\mu = \varepsilon\mu_0$, we obtain a perturbation of order ε which produces a reflection in $O(\varepsilon^2)$. This is interesting because this is almost zero reflection but not completely satisfactory yet. To compensate for the remainder, we need to work a bit more. Below we will explain how to compute $dR(0)$ and prove that $dR(0) : \mathcal{C}_0^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ is onto. This allows us to introduce $\mu_1 \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\mu_2 \in \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$dR(0)(\mu_1) = 1 \quad \text{and} \quad dR(0)(\mu_2) = i. \quad (2.8)$$

Finally, we look for μ of the form

$$\mu = \mu_0 + \tau_1\mu_1 + \tau_2\mu_2$$

where $\tau_1, \tau_2 \in \mathbb{R}$ are parameters to tune. Inserting this μ in the expansion (2.6), to get $R(\varepsilon\mu) = 0$, we see that we must have

$$0 = \varepsilon dR(0)(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2) + \varepsilon^2 \tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)).$$

By exploiting (2.8), this yields

$$0 = \tau_1 + i\tau_2 + \varepsilon \tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)).$$

In other words, we find that the vector $\vec{\tau} := (\tau_1, \tau_2) \in \mathbb{R}^2$ must satisfy the fixed point equation

$$\vec{\tau} = G^\varepsilon(\vec{\tau}) \quad \text{with} \quad G^\varepsilon(\vec{\tau}) = -\varepsilon(\Re \tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)), \Im \tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2))). \quad (2.9)$$

Now by proving uniform (with respect to $\vec{\tau}$) error estimates in the asymptotic expansions as ε tends to zero, one can show that for any $r > 0$, there is $\varepsilon_0 > 0$ such that the map $\vec{\tau} \mapsto G^\varepsilon(\vec{\tau})$ is a contraction from $\overline{B(O, r)}$ to $\overline{B(O, r)}$ for all $\varepsilon \in (0; \varepsilon_0]$. Here $B(O, r)$ denotes the open ball of $\mathbb{R}^2$ centered at O of radius r . Therefore the Banach fixed point theorem guarantees that (2.9) admits a unique solution $\vec{\tau}^{\text{sol}} \in \overline{B(O, r)}$. Then for $h^{\text{sol}} = \varepsilon(\mu_0 + \tau_1^{\text{sol}}\mu_1 + \tau_2^{\text{sol}}\mu_2)$, we have $R(h^{\text{sol}}) = 0$. This proves the existence of non reflecting geometries.

2.2.2 Comments

Let us give a few comments concerning this method.

i) Let us prove that the constructed h^{sol} is indeed not trivial. To proceed, let us work by contradiction and assume that $h^{\text{sol}} \equiv 0$. Then we get

$$0 = dR(0)(h^{\text{sol}}) = dR(0)(\varepsilon(\mu_0 + \tau_1^{\text{sol}}\mu_1 + \tau_2^{\text{sol}}\mu_2)) = \varepsilon(\tau_1^{\text{sol}} + i\tau_2^{\text{sol}}),$$

and so $\tau_1^{\text{sol}} = \tau_2^{\text{sol}} = 0$. This implies $h^{\text{sol}} = \varepsilon\mu_0 \equiv 0$ which is not true due to our choice for μ_0 .

ii) Observe that this technique guarantees the existence of a space of infinite dimension of non reflecting perturbations. First, h^{sol} depends on $\varepsilon \in (0; \varepsilon_0]$ (remark that it is not only a scaling because the nonlinear terms are also involved). Additionally, we have some freedom for the choice of μ_0 because $\dim \ker dR(0) = +\infty$.

iii) We can establish that there is a constant $C > 0$ independent of ε such that we have $|G^\varepsilon(\vec{\tau})| \leq C\varepsilon$ in $B(O, r)$. As a consequence, as ε tends to zero, we have $|\vec{\tau}^{\text{sol}}| = O(\varepsilon)$ and the shape of the perturbation is mainly characterized by μ_0 .

iv) Let us clarify the connection with the implicit function theorem. Introduce the functional

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (\tau_0, \vec{\tau}) &\mapsto (\Re R(\tau_0\mu_0 + \tau_1\mu_1 + \tau_2\mu_2), \Im R(\tau_0\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)) \end{aligned}$$

which is of class $\mathcal{C}^1$ in a neighborhood of $0_{\mathbb{R}^3}$. From what will be shown below, we will be able to deduce that $\partial_{\vec{\tau}} F(0, 0_{\mathbb{R}^2}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is well-defined and bijective. On the other hand, we remark that $F(0, 0_{\mathbb{R}^2}) = 0$. The implicit function theorem applies: there are some neighborhoods $\mathcal{U} \subset \mathbb{R}$, $\mathcal{V} \subset \mathbb{R}^2$ of $0, 0_{\mathbb{R}^2}$ and a unique function $\varphi : \mathcal{U} \rightarrow \mathcal{V}$ of class $\mathcal{C}^1$ such that

$$[(\tau_0, \vec{\tau}) \in \mathcal{U} \times \mathcal{V} \text{ and } F(\tau_0, \vec{\tau}) = 0] \quad \Leftrightarrow \quad \vec{\tau} = \varphi(\tau_0).$$

2.2.3 Application to our problem

To apply the generic strategy described above, we need to compute $dR(0)$. To proceed, we use classical techniques of asymptotic analysis. For $\varepsilon > 0$ and a given $\mu \in \mathcal{C}_0^\infty(\mathbb{R})$, denote by Ω^ε the domain $\Omega(h)$ with $h = \varepsilon\mu$. Let u^ε stand for the solution introduced in (2.1) in the geometry Ω^ε . It satisfies

$$\begin{cases} \Delta u^\varepsilon + k^2 u^\varepsilon = 0 & \text{in } \Omega^\varepsilon \\ \partial_{\nu^\varepsilon} u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases} \quad (2.10)$$

Note that ν^ε , the outward unit normal vector to $\partial\Omega^\varepsilon$, depends on ε . For u^ε , as ε tends to zero, we consider the ansatz

$$u^\varepsilon = u_0 + \varepsilon u_1 + \dots \quad (2.11)$$

where u_0, u_1 are functions to be determined and the dots correspond to higher order terms. On $\partial\Omega^\varepsilon$, we have the expansions

$$\nu^\varepsilon = \frac{1}{\sqrt{1 + \varepsilon^2(\mu'(x))^2}} \begin{pmatrix} -\varepsilon\mu'(x) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} -\mu'(x) \\ 0 \end{pmatrix} + \dots \quad (2.12)$$

$$\nabla u^\varepsilon(x, 1 + \varepsilon\mu(x)) = \nabla u^\varepsilon(x, 1) + \varepsilon\mu(x) \begin{pmatrix} \partial_{xy}^2 u^\varepsilon(x, 1) \\ \partial_{yy}^2 u^\varepsilon(x, 1) \end{pmatrix} + \dots \quad (2.13)$$

Now we insert (2.11) in (2.10) and exploit (2.12), (2.13). Collecting the terms of orders $\varepsilon^0, \varepsilon^1$, we find that u_0, u_1 satisfy respectively the problems

$$\begin{aligned} \left| \begin{array}{l} \Delta u_0 + k^2 u_0 = 0 \quad \text{in } \mathcal{S} \\ \partial_{\nu} u_0 = 0 \quad \text{on } \partial\mathcal{S} \end{array} \right| \quad \begin{array}{ll} \Delta u_1 + k^2 u_1 = & 0 \quad \text{in } \mathcal{S} \\ \partial_{\nu} u_1 = & 0 \quad \text{on } \mathbb{R} \times \{0\} \\ \partial_{\nu} u_1 = & \mu'(x)\partial_x u_0 - \mu(x)\partial_{yy}^2 u_0 \quad \text{on } \mathbb{R} \times \{1\}. \end{array} \quad (2.14) \end{aligned}$$

Using additionally that $u^\varepsilon - w^+$ is outgoing, we get first

$$u_0(x, y) = w^+(x, y) = e^{ikx}.$$

Since w^+ is independent of y , we deduce that u_1 satisfies the condition

$$\partial_\nu u_1 = \mu'(x) \partial_x w^+ \quad \text{on } \mathbb{R} \times \{1\}.$$

Denote by R^ε , T^ε the scattering coefficients of u^ε and set $\Sigma_{\pm L} := \{\pm L\} \times (0; 1)$. Assuming that μ is supported in $(-L; L)$, using the decomposition (2.4) of u^ε , we can show the formula

$$2ikR^\varepsilon = \int_{\Sigma_{-L} \cup \Sigma_L} \partial_\nu u^\varepsilon w^+ - u^\varepsilon \partial_\nu w^+ dy, \quad 2ikT^\varepsilon = \int_{\Sigma_{-L} \cup \Sigma_L} \partial_\nu u^\varepsilon w^- - u^\varepsilon \partial_\nu w^- dy, \quad (2.15)$$

where $\partial_\nu = \pm \partial_x$ on $\Sigma_{\pm L}$. Inserting the expansion (2.11) of u^ε in (2.15), integrating by parts and exploiting the boundary conditions of (2.14), we obtain

$$R^\varepsilon = 0 - \frac{\varepsilon}{2ik} \int_{\mathbb{R}} \mu'(x) \partial_x w^+ w^+ dx, \quad T^\varepsilon = 1 - \frac{\varepsilon}{2ik} \int_{\mathbb{R}} \mu'(x) \partial_x w^+ w^- dx.$$

This gives the formulas

$$\begin{aligned} dR(0)(\mu) &= \frac{-1}{2ik} \int_{\mathbb{R}} \mu'(x) \partial_x w^+ w^+ dx = \frac{-1}{2} \int_{\mathbb{R}} \mu'(x) e^{2ikx} dx = ik \int_{\mathbb{R}} \mu(x) e^{2ikx} dx \\ dT(0)(\mu) &= \frac{-1}{2ik} \int_{\mathbb{R}} \mu'(x) \partial_x w^+ w^- dx = \frac{-1}{2} \int_{\mathbb{R}} \mu'(x) dx = 0. \end{aligned} \quad (2.16)$$

Since the real part of $x \mapsto e^{2ikx}$ is even while its imaginary part is odd, we see that it is easy to find functions μ_1, μ_2 satisfying the relations (2.8), which ensure that $dR(0) : \mathcal{C}_0^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ is onto.

On the other hand, we find that $dT(0)$ vanishes. It means that a perturbation of order ε of the reference strip gives a transmission coefficient T^ε such that $T^\varepsilon - 1$ is in $O(\varepsilon^2)$. This looks appealing for perfect transmission. The problem is that since $dT(0)$ vanishes, one cannot use this term which has a linear dependence with respect to the perturbation of the reference strip to cancel the whole (non linear) expansion of T^ε via the resolution of the fixed point problem. As a consequence, our technique fails to design perfectly invisible defects.

Finally, let us mention that the formal calculus above can be rigorously justified by proving error estimates. To proceed, one method consists in rectifying the boundary of Ω^ε using “almost identical” diffeomorphisms to transform the perturbed domain into the original geometry Ω (see *e.g.* [294, Chap. 7, §6.5]). Then one can prove the estimate, for ε small enough,

$$\|u^\varepsilon - (u_0 + \varepsilon u_1)\|_{H^1(\Omega_L \setminus \bar{\omega})} \leq C\varepsilon^2,$$

where $C > 0$ is a constant independent of ε , $\Omega_L := (-L; L) \times (0; 1)$ and ω is a neighborhood of $\text{supp}(\mu) \times \{1\}$.

2.2.4 Numerical implementation

The theoretical approach presented above leads very naturally to an algorithm to construct numerically non reflecting perturbations. Let us describe the strategy.

The main idea consists in solving the fixed point equation

$$\vec{\tau} = G^\varepsilon(\vec{\tau})$$

(see (2.9)) using an iterative procedure. First we choose μ_0, μ_1, μ_2 satisfying (2.7)–(2.8) once and for all. Then we start with $\vec{\tau}^0 = (0, 0)$ and for $p \in \mathbb{N}$, we set $\vec{\tau}^{p+1} = G^\varepsilon(\vec{\tau}^p)$. Using the definition of G^ε , one observes that there holds $G^\varepsilon(\vec{\tau}^p) = \vec{\tau}^p - \varepsilon^{-1}(\Re R(\varepsilon\mu^p), \Im R(\varepsilon\mu^p))$ with $\mu^p = \mu_0 + \tau_1^p \mu_1 + \tau_2^p \mu_2$. Therefore, we have

$$\vec{\tau}^{p+1} = \vec{\tau}^p - \varepsilon^{-1}(\Re R(\varepsilon\mu^p), \Im R(\varepsilon\mu^p)). \quad (2.17)$$

We stop the loop when we have $|\vec{\tau}^{p+1} - \vec{\tau}^p| \leq \eta$ where $\eta > 0$ is a small given criterion. We then define $\vec{\tau}^{\text{sol}}$ as the last value of $\vec{\tau}^p$. Then we have $R(\varepsilon\mu^{\text{sol}}) \approx 0$. If the iterative process does not converge, we try again with a smaller value of $\varepsilon > 0$. Note that at each step $p \geq 0$ of the procedure, we need to solve a scattering problem of the form

$$\left| \begin{array}{l} \text{Find } u \text{ such that } u - w^+ \text{ is outgoing and} \\ \Delta u + k^2 u = 0 \quad \text{in } \Omega^p := \Omega(\varepsilon\mu^p) \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega^p. \end{array} \right. \quad (2.18)$$

To proceed, we approximate the solution of (2.18) with a P2 finite element method in $\Omega_5^p := \{(x, y) \in \Omega^p \mid |x| < 5\}$. At $x = \pm 5$, a truncated Dirichlet-to-Neumann map with 10 terms serves as a transparent boundary condition. In particular note that at each step, it is necessary to mesh a new domain.

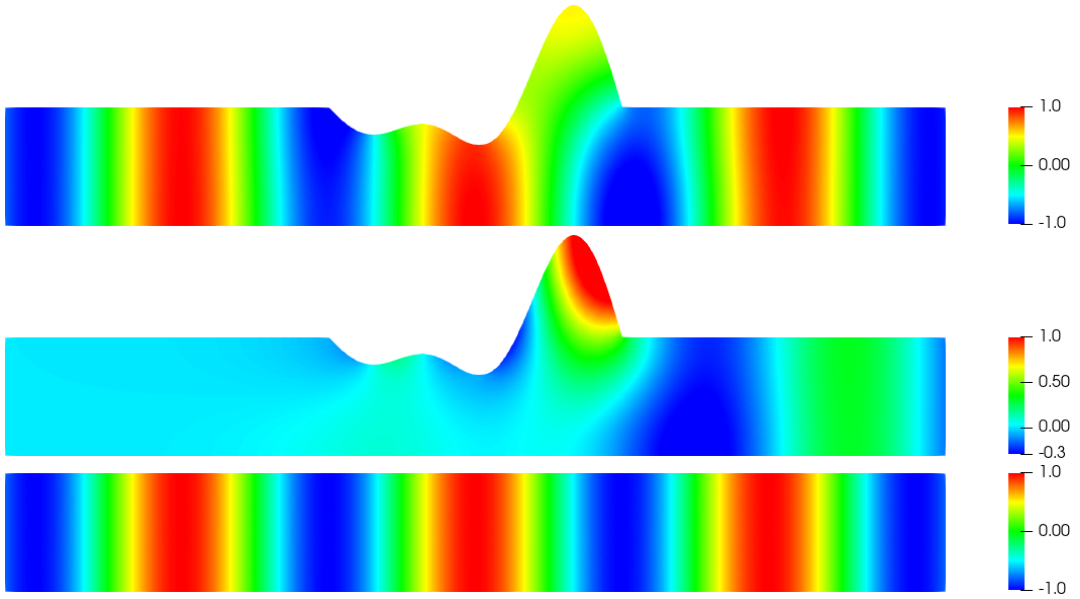


Figure 2.4: Example of non reflecting perturbation of the boundary for Problem (2.1). Top: $\Re u^\varepsilon$. Middle: $\Re u_\varepsilon$. Bottom: $\Re w^+$ in the reference strip. Here $k = 0.8\pi$.

In Figure 2.4, we display a first example of non reflecting perturbation of the geometry. As expected the scattered field is exponentially decaying as $x \rightarrow -\infty$. Interestingly, since there is only one complex coefficient to cancel, the iterative procedure to solve the fixed point equation converges though ε is not that small. This allows us to get not so small non reflecting perturbations. Computations have been implemented with **FreeFem++**¹ In Figure 2.5, we give a second example of non reflecting perturbation for which the defect lies entirely in the region $y < 1$. Because of this property, we can symmetrize it with respect to the line $y = 1$ to create a non reflecting obstacle completely embedded in the duct (see Figure 2.6). These techniques are very convenient to create projects mixing physical intuition, theoretical aspects and the use of the finite element method. I have used them several times in my teaching at École Polytechnique.

¹FreeFem++, <https://freefem.org/>.

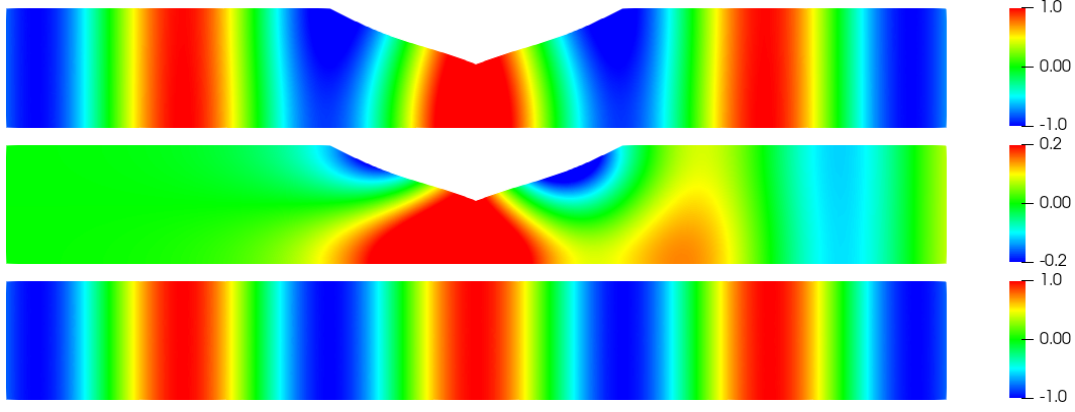


Figure 2.5: Example of non reflecting perturbation of the boundary for Problem (2.1). Top: $\Re u^\varepsilon$. Middle: $\Re u_s^\varepsilon$. Bottom: $\Re w^+$ in the reference strip. Here $k = 0.8\pi$.

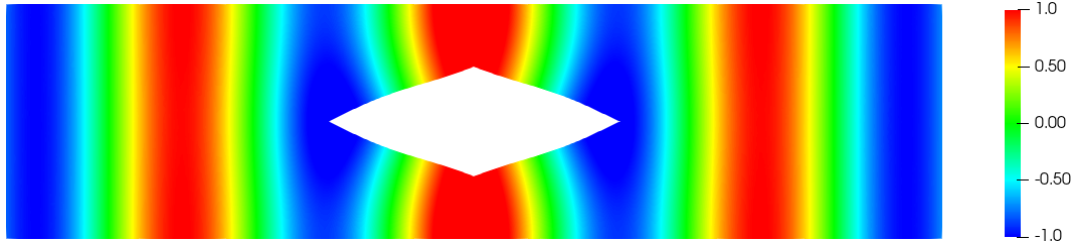


Figure 2.6: Example of non reflecting obstacle. Here $k = 0.8\pi$.

We showed how to construct non reflecting smooth perturbations of the reference strip $\mathcal{S}$. Below we wish to explain how to design invisible non smooth perturbations of $\mathcal{S}$. The terminology smooth/not smooth here does not refer to the regularity of the domain but to the form of the asymptotic expansion involved in the asymptotic procedure. In the non smooth case, the field u^ε exhibits rapid variations in a neighborhood of the perturbation that must be caught with adapted variables. In that situation, the asymptotics of the scattering coefficients is in general more complicated to obtain. The interesting point is that it can bring new useful terms, for example non zero differentials for T . In the next two sections, we consider two types of non smooth perturbations of the reference strip.

2.3 Non reflecting groups of small obstacles

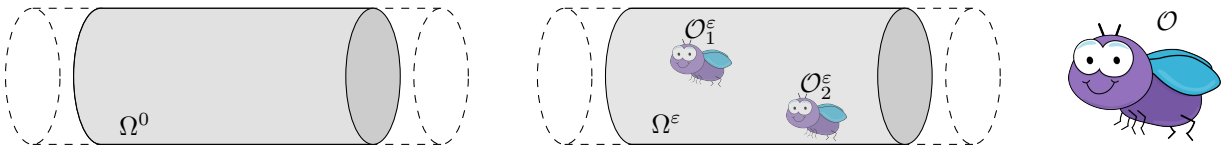


Figure 2.7: Unperturbed waveguide (left), perturbed waveguide with two flies (middle), set $\mathcal{O}$ (right).

We adapted the method presented in the previous section to hide groups of small obstacles instead of smooth perturbations of the boundary in [165]. Let us describe the results. To simplify the asymptotic analysis, we decided to work in 3D with Dirichlet boundary conditions. Note that in

2D, the Green function of the Laplace operator, whose behavior dictates the form of the asymptotic expansions as ε tends to zero, has a logarithmic singularity which does not appear in 3D. Let

$$\Omega^0 := \{\mathbf{x} = (x, y) \mid x \in \mathbb{R} \text{ and } y \in \omega\}$$

be a cylinder of $\mathbb{R}^3$ whose transverse section $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary (see Figure 2.7 left). Consider $\mathcal{O} \subset \mathbb{R}^3$ a bounded domain with Lipschitz boundary and for M_1 a point located in Ω^0 , set, for ε small,

$$\mathcal{O}_1^\varepsilon := \{\mathbf{x} \in \mathbb{R}^3 \mid \varepsilon^{-1}(\mathbf{x} - M_1) \in \mathcal{O}\}.$$

Finally define the perturbed waveguide

$$\Omega^\varepsilon := \Omega^0 \setminus \overline{\mathcal{O}_1^\varepsilon}.$$

To illustrate the point, we can think of $\mathcal{O}_1^\varepsilon$ as a fly located in the waveguide. To begin with, and to simplify the exposition, for the moment we assume that there is only one fly and not a group. The problem we consider writes

$$\begin{cases} \Delta u^\varepsilon + k^2 u^\varepsilon = 0 & \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases} \quad (2.19)$$

We fixed the wavenumber $k > 0$ such that only two modes $w^\pm$ can propagate in Ω^ε . They are not the same as the $w^\pm$ in (2.3). In particular, due to the Dirichlet boundary condition, they depend on the transverse variable y . However the ideas are similar. As in (2.4), Problem (2.19) admits a solution with the expansion

$$u^\varepsilon = \begin{cases} w^+ + R^\varepsilon w^- + \tilde{u}^\varepsilon & \text{for } x \leq -d \\ T^\varepsilon w^+ + \tilde{u}^\varepsilon & \text{for } x \geq d, \end{cases}$$

where $R^\varepsilon, T^\varepsilon \in \mathbb{C}$ and $\tilde{u}^\varepsilon$ decays exponentially at infinity. Here $d > 0$ is such that $\Omega^\varepsilon = \Omega^0$ for $|x| \geq d$. The first step in the approach is to compute an asymptotic expansion of u^ε as ε tends to zero. This is a rather long work that we will not present here (one may consult the reference [352, §2.2] for more details). Let us simply stress that it appears that u^ε has a rapid variation in a neighborhood of the obstacle to satisfy the constraint of being null on $\partial\mathcal{O}_1^\varepsilon$. This rapid variation must be caught with adapted variables. Then it is necessary to work both with some inner field and outer field expansions of u^ε that we match in some intermediate region. This is the method of matched asymptotic expansions which is well documented in the literature (see the monographs [283, 352]). Adapting the approach presented in (2.15) to turn the expansion for u^ε into an expansion for $R^\varepsilon, T^\varepsilon$, at the end of the calculus, when ε tends to zero, we obtain

$$R^\varepsilon = 0 + \varepsilon 4i\pi \text{cap}(\mathcal{O}) w^+(M_1)^2 + O(\varepsilon^2), \quad T^\varepsilon = 1 + \varepsilon 4i\pi \text{cap}(\mathcal{O}) |w^+(M_1)|^2 + O(\varepsilon^2).$$

Here $\text{cap}(\mathcal{O})$ stands for the capacity of the domain $\mathcal{O}$, a constant which appears classically in asymptotic analysis. It is given by

$$\text{cap}(\mathcal{O}) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} |\nabla P|^2 dz$$

where P is the potential satisfying

$$\begin{cases} \Delta P = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ P = 1 & \text{on } \partial\mathcal{O} \end{cases}$$

which decays at infinity. Note that one can prove that P admits the decomposition

$$P(z) = \frac{\text{cap}(\mathcal{O})}{|z|} + O(|z|^{-2}) \quad \text{as } |z| \rightarrow +\infty.$$

An important point for our study is that one has always $\text{cap}(\mathcal{O}) > 0$. Additionally, one finds that w^+ does not vanish in Ω^0 . From these two properties, we infer that one single small obstacle cannot even be non reflecting: whatever the choice for M_1 or for the shape $\mathcal{O}$, $\mathcal{O}_1^\varepsilon$ will always generate a reflection whose amplitude is of order ε .

Let us add a second small obstacle

$$\mathcal{O}_2^\varepsilon := \{\mathbf{x} \in \mathbb{R}^3 \mid \varepsilon^{-1}(\mathbf{x} - M_2) \in \mathcal{O}\}$$

in the waveguide, centered at the point $M_2 \in \Omega^0$ with $M_2 \neq M_1$. Still denoting by R^ε , T^ε the scattering coefficients in this new geometry, we obtain the expansions, as $\varepsilon \rightarrow 0^+$,

$$R^\varepsilon = 0 + \varepsilon 4i\pi \text{cap}(\mathcal{O}) \sum_{j=1}^2 w^+(M_j)^2 + O(\varepsilon^2), \quad T^\varepsilon = 1 + \varepsilon 4i\pi \text{cap}(\mathcal{O}) \sum_{j=1}^2 |w^+(M_j)|^2 + O(\varepsilon^2). \quad (2.20)$$

Observe that the interactions between the small objects do not appear at order ε , only at higher orders. The interesting thing is that now, using the known expression of w^+ , we can find positions M_1, M_2 of the flies such that

$$\sum_{j=1}^2 w^+(M_j)^2 = 0. \quad (2.21)$$

In that case, we have a perturbation of Ω^0 of order ε which produces a reflection in $O(\varepsilon^2)$. As mentioned above, this is almost no reflection. But by working a bit harder, we can achieve more. Pick M_1, M_2 such that relation (2.21) is satisfied. What we have shown in [165] is that we can slightly perturb the position of one fly to get $R^\varepsilon = 0$ exactly. More precisely, for $\tau \in \mathbb{R}^3$, define $M_1^{\varepsilon\tau} = M_1 + \varepsilon\tau$ and

$$\mathcal{O}_1^{\varepsilon\tau} := \{\mathbf{x} \in \mathbb{R}^3 \mid \varepsilon^{-1}(\mathbf{x} - M_1^{\varepsilon\tau}) \in \mathcal{O}\}.$$

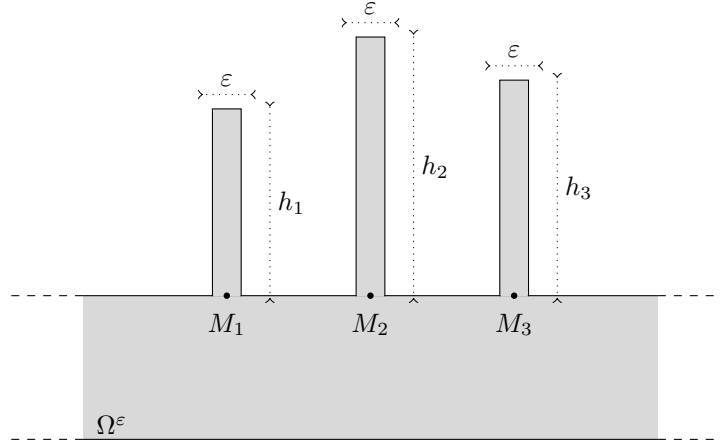
We have shown that there is τ , which is defined as the solution of a fixed point problem similar to (2.9), such that the reflection coefficient in the corresponding waveguide is zero. Note that since $\tau \in \mathbb{R}^3$, we have enough degrees of freedom to cancel one single complex number.

Concerning perfect invisibility, *i.e.* obtaining $T^\varepsilon = 1$, we see from (2.20) that again this approach is inoperative. Indeed, whatever the position of the small flies or their number, we always have a phase shift for the transmitted wave.

In [165], we have also studied what happens at higher wavenumbers k . In that situation, as can be seen in the particular case (2.2), more modes can propagate. The reflection and transmission coefficients then become respectively reflection and transmission matrices $\mathcal{R}^\varepsilon$, $\mathcal{T}^\varepsilon$. We have proved that by working with a sufficiently large number of flies, we can cancel exactly the whole reflection matrix. The idea is the same as above. First we compute an asymptotic expansion of $\mathcal{R}^\varepsilon$ as $\varepsilon \rightarrow 0$. Then we find positions $M_1, M_2, \dots, M_p$ of the flies to cancel the term of order ε in the expansion of $\mathcal{R}^\varepsilon$. Then by slightly perturbing the position of one group of flies by solving a fixed point problem in $\mathbb{R}^q$, for a certain q depending on the number of propagating modes, we get $\mathcal{R}^\varepsilon = 0$. The higher the number of propagating modes, the more flies we need.

To conclude this section, let us mention that in this work, the main difficulty mathematically consists in proving error estimates in the asymptotic expansions which are uniform with respect to the parameter τ in a closed ball to justify that the map appearing in the fixed point problem is indeed a contraction for ε small enough.

2.4 Perfect transmission

Figure 2.8: Geometry of Ω^ε .

The methods presented in the two previous sections fail to provide examples of waveguides where $T = 1$ (perfect transmission without phase shift). This led us to doubt the possibility of achieving perfect invisibility. For this reason, in the article [73], for the acoustic problem (2.1) with Neumann boundary conditions, by establishing some sort of energy identities, we first proved negative results showing that for a given obstacle, one cannot have $T = 1$ for wavenumbers k too small. Then we realized that by adapting the strategy presented in Section 2.3, by considering other types of singular perturbations, $T = 1$ can be reached. Let us describe the setting.

Consider the waveguide pictured in Figure 2.8. It is made of the reference strip $\mathcal{S} = \mathbb{R} \times (0; 1)$ to which we have glued at the points $M_1 := (x_1, 1)$, $M_2 := (x_2, 1)$, $M_3 := (x_3, 1)$, thin ligaments of width $\varepsilon > 0$ small and heights respectively equal to h_1, h_2, h_3 . Again we fix $k \in (0; \pi)$ so that only the modes $w^\pm$ can propagate. The first step is to compute an asymptotic expansion of the scattering coefficients as ε , the width of the thin ligaments, tends to zero. Again, this is a rather long work that we will not present here, the reason being that the fields vary rapidly in a region around the M_n , $n = 1, 2, 3$. By using the method of matched asymptotic expansions, we have been able to show that when the h_n are such that $kh_n \notin \{\pi/2 + \pi\mathbb{N}\}$, when ε tends to zero, we have

$$\begin{aligned} R^\varepsilon &= 0 + \varepsilon \left(ik \sum_{n=1}^3 (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2) \\ T^\varepsilon &= 1 + \varepsilon \left(ik \sum_{n=1}^3 \tan(kh_n) \right) + O(\varepsilon^2). \end{aligned} \quad (2.22)$$

Remark 2.4.1 When $kh_n \in \{\pi/2 + \pi\mathbb{N}\}$, resonant phenomena appear in the ligaments and the asymptotics (2.22) are no longer valid. This regime will be exploited in §2.8.

The crucial point to observe in (2.22) compared to the other perturbations of the reference strip above is the appearance of the term of order ε in the expansion of T^ε . Its imaginary part is non zero and can change sign according to the value of the h_n . Thus by considering another type of perturbation of $\mathcal{S}$, we have been able to obtain a non zero $dT(0)$.

By using that $(w_+(M_n))^2 = e^{2ikx_n}$, we can find positions and heights of the ligaments such that $R^\varepsilon = O(\varepsilon^2)$ and $T^\varepsilon - 1 = O(\varepsilon^2)$. Then perturbing slightly the h_n around these particular values, by solving a fixed point problem in $\mathbb{R}^3$ similar to (2.9), we can achieve

$$R^\varepsilon = 0 \quad \text{and} \quad \Im m T^\varepsilon = 0 \quad (2.23)$$

in the new geometry. But due to conservation of energy, we must have $|R^\varepsilon|^2 + |T^\varepsilon|^2 = 1$. Therefore in case (2.23), we have either $T^\varepsilon = -1$ or $T^\varepsilon = 1$. Since expansion (2.22) ensures that T^ε is close to one for ε small enough, we deduce that necessarily $T^\varepsilon = 1$. Thus perfect transmission can be achieved.

Initially we have to control two complex numbers (R^ε and T^ε), so *a priori* we need four real degrees of freedom. But due to the constraints of conservation of energy, three parameters are sufficient. This explains why three ligaments are involved here. We could also have probably worked with only two ligaments, by perturbing their heights and the distance between them.

This approach can be implemented numerically very naturally. First we set the M_n and the h_n to kill the terms of order ε in (2.22). Then we tune slightly the length of the ligaments by solving the corresponding fixed point problem iteratively as in (2.17). At each step of the iterative procedure, we solve a scattering problem in a new geometry. It is as if one was pressing on trumpet's valves to obtain $T^\varepsilon = 1$. In Figure 2.9 we display a geometry obtained with this method. A bit far from the geometric perturbation so that evanescent terms can be neglected, we obtain the same field as in the reference strip. Again, note that numerically the iterative procedure converges though ε , the width of the ligaments, is not that small.

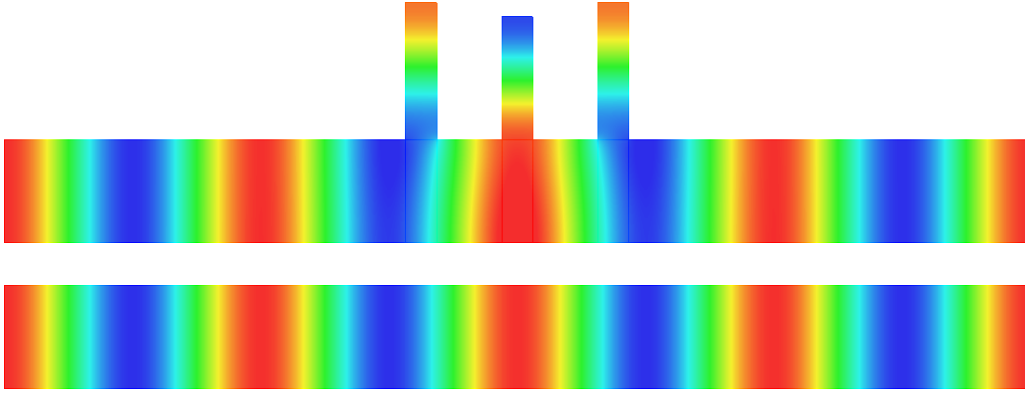


Figure 2.9: Top: real part of u^ε in the geometry obtained at the end of the iterative procedure. Bottom: real part of w^+ in the reference strip.

We adapted this work with S.A. Nazarov and J. Taskinen in [174] to design non reflecting perturbations of the bottom of a canal for a water-wave problem. This setting is a bit different from the acoustic problem (2.1). In particular propagating modes are localized at the surface. But we can adapt the approach above and we have been able to remove some assumptions made on the wavenumber in [83] when dealing with smooth perturbations of the bottom of the canal. In Figure 2.10, we display the kind of non reflecting defects we construct. Here we play with the position of two of the three thin tunnels. Let us emphasize that it was important to work with three tunnels and not just two. The reason is that $|R^\varepsilon|$ is invariant with respect to the coordinates change $(x, y) \mapsto (x - h, y)$ for $h \in \mathbb{R}$. As a consequence, perturbing the position of two tunnels boils down to modifying the distance between them and offers only one degree of freedom, which is not enough to control the complex quantity R^ε . On the other hand, this type of geometric perturbation does not allow one to get $T^\varepsilon = 1$.

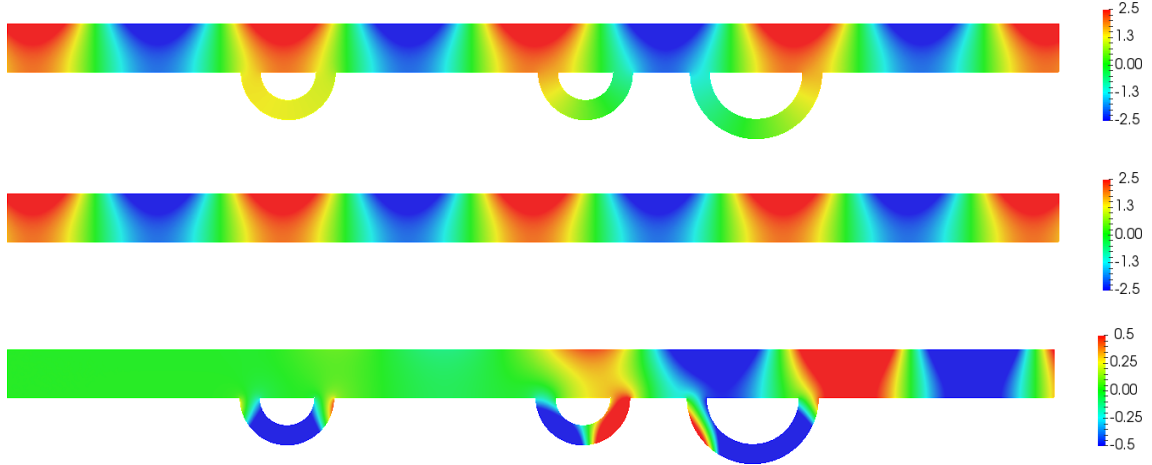


Figure 2.10: Real parts of u^ε (top), w^+ (middle) and $u^\varepsilon - w^+$ (bottom) for a water-wave problem. The three thin (of width $\varepsilon > 0$) curved channels appearing in the top and bottom pictures have been designed to be non reflecting for an incident right-going surface wave.

2.5 Continuation method

The different methods above allow us to construct perturbations of the reference strip or of the reference cylinder which are invisible, in a broad sense (non reflecting or perfectly invisible), at a given wavenumber k . With these techniques, we can actually impose invisibility not only at a given k but also for a finite number of different k . We emphasize however that these methods cannot be used to achieve invisibility for a continuum of k . In the numerical results we presented, we were interested in controlling only R and in Figure 2.9, $\Im m T$ at one k . As a consequence, we had very few constraints and for this reason, the fixed point algorithm converges with not so small values of ε which allowed us to obtain rather large invisible defects. When the number of constraints increases, for example, when working at higher k so that there are more scattering coefficients to control, in practice we find that ε must be chosen smaller to have convergence of the algorithm. A natural idea is to try to reiterate the procedure to obtain larger invisible defects: once an invisible obstacle has been constructed, we can consider it as a new starting configuration and perturb it while keeping the same scattering coefficients. This continuation method has been one of the topics of the PhD of A. Bera that I supervised in collaboration with A.-S. Bonnet-Ben Dhia. He decided to stop after two and a half years before writing the manuscript but we obtained interesting results that led to the article [49] and that we present now.

We work with a penetrable obstacle and consider the problem

$$\begin{cases} \Delta u + k^2 n^2 u &= 0 & \text{in } \mathcal{S} = \mathbb{R} \times (0; 1) \\ \partial_\nu u &= 0 & \text{on } \partial \mathcal{S}, \end{cases}$$

where n is a function of $L^\infty(\mathcal{S})$ which is equal to one outside of a bounded domain. Set $\rho := n^2 - 1$ whose support coincides with the obstacle.

In [49], we first derive general formulas to compute the differential of the scattering coefficients, which are gathered in the scattering matrix $\mathbb{S}$, with respect to the material index (which is a function). An important aspect is that when starting from a perturbed situation ($n \neq 1 \Leftrightarrow \rho \neq 0$), the formulas we obtain are not as explicit as in (2.16). For this reason, it is harder to show that the differentials of the scattering coefficients are onto in the appropriate spaces as needed to implement the continuation method. Proving certain of these results is one of the outcomes of [49].

In Figures 2.11–2.12, we create non reflecting obstacles for $k = 7 \in (2\pi; 3\pi)$. In this case, three modes can propagate in the waveguide and we have to cancel 6 complex terms (because $\mathbb{S}$ is symmetric). In Figure 2.11, we display the sequence of non reflecting obstacles that we obtain by starting from $\rho \equiv 0$. Though constraints are quite numerous, reiterating nine times the fixed point algorithm, we observe that we can get non reflecting ρ with a relatively high contrast (see the last picture of Figure 2.11).

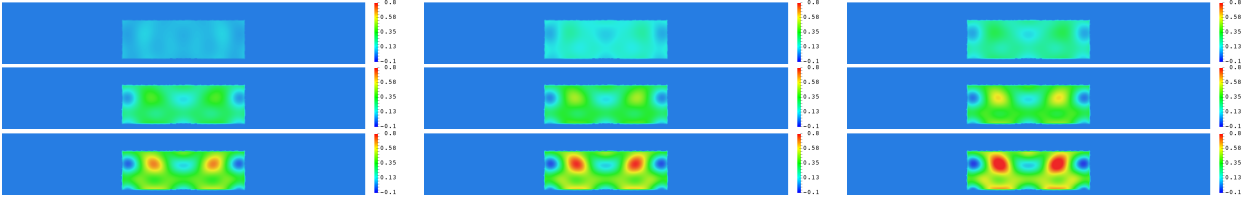


Figure 2.11: Evolution of the non reflecting parameter ρ .

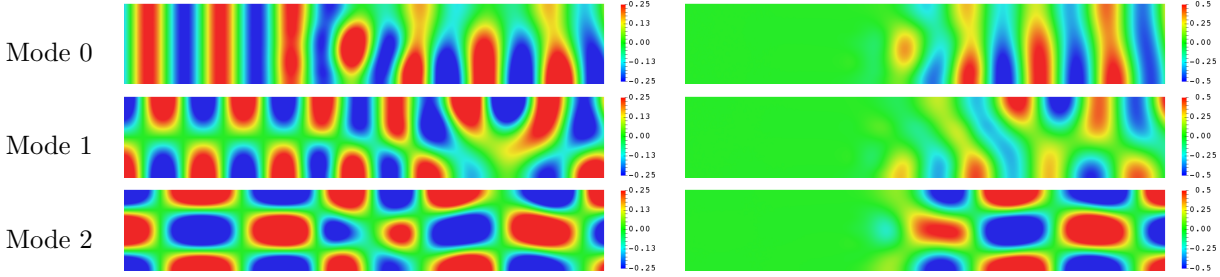


Figure 2.12: We choose k so that three modes can propagate. The line $j \in \{1, 2, 3\}$ corresponds to the scattering of the right-going incident mode w_{j-1}^+ . The left (resp. right) picture represents the real part of the total field u_{j-1}^+ (resp. scattered field $u_{j-1}^+ - w_{j-1}^+$) for the ρ of Figure 2.11.

In [49], we also study relative invisibility. Can we construct two different obstacles which have the same scattering coefficients? We showed that the continuation method can be used for this purpose. However, since the scattering matrix has a strong structure, it is symmetric and unitary, the relations to impose to get the desired scattering coefficients should be cleverly chosen. Actually we already had a glimpse of that when constructing perfectly invisible obstacles. More precisely, in Section 2.4 we did not try to get directly $T - 1 = 0$. This fails because the corresponding differential is not onto. Instead it is better to impose $R = 0$ and $\Im m T = 0$. In [49], we gave a general strategy to select the relations one should impose to obtain a given target for $\mathbb{S}$. We illustrate these results in Figures 2.13 and 2.14. There we exhibit two different indices ρ for which the corresponding scattering matrices are the same. We work with $k = 0.8\pi \in (0; \pi)$, that is in monomode regime². In this case, the scattering matrix is of size 2×2 . For the initial ρ of Figure 2.13 (see the top left picture), we find $T \approx 0.82 + 0.52i$. Since $\Re T \neq 0$, we showed that we can do unique continuation with the functional F such that $F(\rho) = (\Re R, \Im m R, \Im m T)$. In the last line of Figure 2.13, as expected, we observe that the scattered field by the two indices have the same behavior at infinity. For the Figure 2.14, the initial ρ is equal to zero (see the top left picture). In other words, there is no penetrable obstacle. But there is a defect in the walls of the waveguide and this defect has been chosen so that the transmission coefficient vanishes (see [173] for the explanation). This can be seen on the first picture of the second line which represents the total field for a right-going incident wave. We indeed note that it is exponentially decaying as $x \rightarrow +\infty$. The top right picture represents another setting, with a penetrable obstacle which has been designed so that the scattering matrix remains the same. In that case it is necessary to work with a different F . The important message here is that the choice of the functional is crucial when imposing relative invisibility.

²We call monomode regime the regime of wavenumbers such that only one mode can propagate.

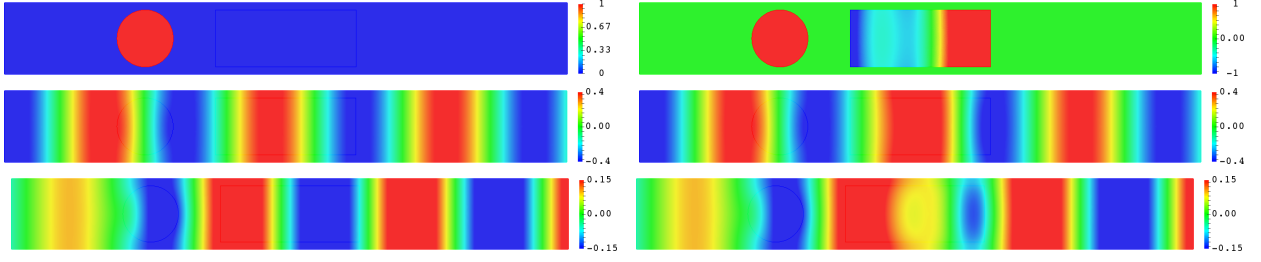


Figure 2.13: Two different indices ρ with the same scattering matrices. The first line represents the two indices. In the second (resp. third) line, we display the real part of the total field u^+ (resp. scattered field $u^+ - w^+$) for each of the two indices.

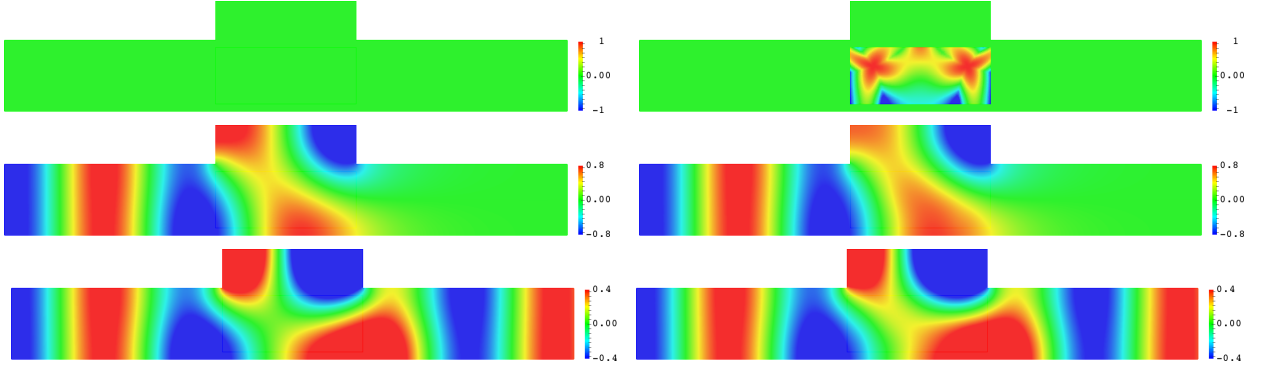


Figure 2.14: Two different indices ρ with the same scattering matrices. The first line represents the two indices. In the second (resp. third) line, we display the real part of the total field u^+ (resp. scattered field $u^+ - w^+$) for each of the two indices.

In [49] we also investigate how to impose additional constraints on the invisible obstacles we construct. In the next two experiments, we show that we can impose ρ (and so n) to be piecewise constant. In both cases, we start from $\rho \equiv 0$. In Figures 2.15–2.16, we construct piecewise constant perfectly invisible parameters ρ . We work with $k = 0.8\pi \in (0; \pi)$. Since we cancel both R and $\Im m T$, we need to work with at least three inclusions. In Figures 2.17–2.18, we construct piecewise constant non reflecting ρ with $k = 7 \in (2\pi; 3\pi)$. In this case, three modes can propagate in the waveguide. Since we have to cancel 6 complex coefficients, that is 12 real coefficients, we need to have at least 12 parameters to tune.



Figure 2.15: Piecewise constant perfectly invisible parameter ρ .



Figure 2.16: Real parts of the total field u^+ (left) and of the scattered field $u^+ - w^+$ (right) for the ρ of Figure 2.15.

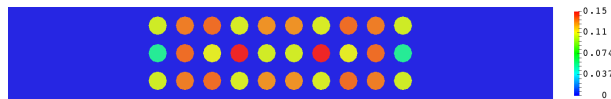


Figure 2.17: Piecewise constant non reflecting parameter ρ .

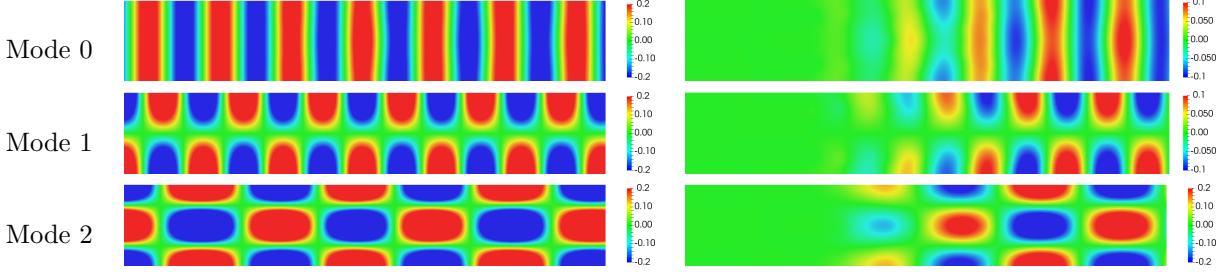


Figure 2.18: Here k is chosen so that three modes can propagate. The line $j \in \{1, 2, 3\}$ corresponds to the scattering of the right-going incident mode w_{j-1}^+ . The left (resp. right) picture represents the real part of the u_{j-1}^+ (resp. $u_{j-1}^+ - w_{j-1}^+$) for the ρ of Figure 2.17.

Let us mention that though numerically we can iterate the continuation method quite easily, theoretically we are unable to prove a priori estimates on the possible values for ε and nothing prevents ε from decreasing at each step of the process. Therefore we cannot guarantee that the resulting invisible obstacle be as large as desired.

2.6 Waveguides with finite length branches

All the approaches presented above to achieve invisibility rely on perturbative ideas. To implement them, it is crucial to start from a setting where one has invisibility (for example, the reference strip). Moreover, though we showed that we can use continuation techniques, these methods lead *a priori* to the construction of small invisible perturbations. From this, a natural question was “how to create large invisible defects?”. In this section, we present a first technique to achieve this goal which has been developed in [173].

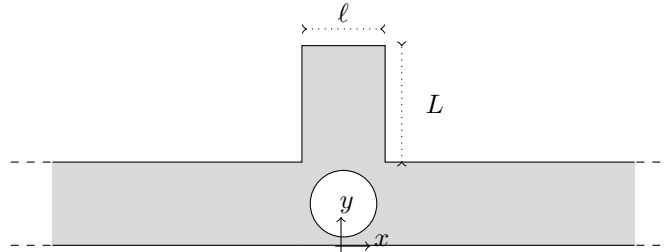


Figure 2.19: Geometry of Ω_L .

2.6.1 Setting

Consider some waveguide Ω_0 which coincides with the strip $\mathcal{S} = \mathbb{R} \times (0; 1)$ outside of some bounded domain. We assume that Ω_0 is symmetric with respect to the vertical axis, *i.e.* that there holds

$$\Omega_0 = \{(-x, y) \mid (x, y) \in \Omega_0\}.$$

For $\ell > 0$, $L > 0$, define the branch of finite length $\mathcal{B}_L = (-\ell/2; \ell/2) \times [1; 1 + L]$ and set

$$\Omega_L := \Omega_0 \cup \mathcal{B}_L \quad (2.24)$$

(see Figure 2.19). We assume that Ω_L is connected with Lipschitz boundary. In Ω_L , we study the problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_L \\ \partial_\nu u = 0 & \text{on } \partial\Omega_L. \end{cases} \quad (2.25)$$

We fix $k \in (0; \pi)$ so that only the two waves $w^\pm$ with $w^\pm(x, y) = e^{\pm ikx}$ can propagate in Ω_L . As in (2.4), the scattering of the right-going wave by the structure leads us to introduce the solution of (2.25) admitting the expansion

$$u = \begin{cases} w^+ + R w^- + \tilde{u} & \text{for } x \leq -d \\ T w^+ + \tilde{u} & \text{for } x \geq d \end{cases} \quad (2.26)$$

(here $d > 0$ is such that $\Omega_0 = \mathcal{S}$ for $|x| \geq d$). Below we fix $\ell > 0$ and explain how to show that for certain $L > 0$ we have $R = R(L) = 0$ (zero reflection). To proceed, we exploit the symmetry of the geometry with respect to the (Oy) axis.

2.6.2 Half-waveguide problems

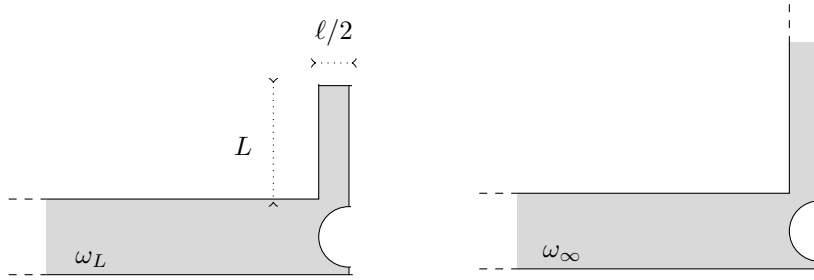


Figure 2.20: Domains ω_L (left) and ω_∞ (right).

Define the half-waveguide

$$\omega_L := \{(x, y) \in \Omega_L \mid x < 0\} \quad (2.27)$$

(see Figure 2.20 left). Introduce the problem with Neumann boundary conditions

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } \omega_L \\ \partial_\nu v = 0 & \text{on } \partial\omega_L \end{cases} \quad (2.28)$$

as well as the problem with mixed boundary conditions

$$\begin{cases} \Delta V + k^2 V = 0 & \text{in } \omega_L \\ \partial_\nu V = 0 & \text{on } \partial\omega_L \cap \partial\Omega_L \\ V = 0 & \text{on } \Sigma_L := \partial\omega_L \setminus \partial\Omega_L. \end{cases} \quad (2.29)$$

Problems (2.28) and (2.29) admit respectively the solutions

$$v = w^+ + R_N w^- + \tilde{v}, \quad (2.30)$$

$$V = w^+ + R_D w^- + \tilde{V}, \quad (2.31)$$

where $R_N, R_D \in \mathbb{C}$ and $\tilde{v}, \tilde{V} \in H^1(\omega_L)$. Due to conservation of energy, one has

$$|R_N| = |R_D| = 1 \quad (2.32)$$

(since there is only one output in ω_L , all the energy propagated by the incident wave is backscattered). Now, direct inspection shows that for the function u introduced in (2.26), we have

$$u(x, y) = \frac{v(x, y) + V(x, y)}{2} \quad \text{in } \omega_L, \quad u(x, y) = \frac{v(-x, y) - V(-x, y)}{2} \quad \text{in } \Omega_L \setminus \overline{\omega_L}$$

(possibly up to a term which is exponentially decaying at $\pm\infty$ if there are trapped modes at the given wavenumber k). We deduce that the scattering coefficients R, T appearing in the decomposition (2.26) of u are such that

$$R = \frac{R_N + R_D}{2} \quad \text{and} \quad T = \frac{R_N - R_D}{2}. \quad (2.33)$$

2.6.3 Main analysis

Now, we study the asymptotic behavior of $R = R(L)$, $T = T(L)$ as $L \rightarrow +\infty$. To proceed, we use (2.33) and work with $R_N = R_N(L)$, $R_D = R_D(L)$. The behaviors of $R_N(L)$, $R_D(L)$ as $L \rightarrow +\infty$ depend on the properties of the equivalents of Problems (2.28), (2.29) set in the limit geometry ω_∞ obtained from ω_L making formally $L \rightarrow +\infty$ (see Figure 2.20 right). In particular, the number of propagating waves existing in the vertical branch

$$\mathfrak{b}_\infty := \{(-\ell/2; 0) \times [1; +\infty)\} \cap \omega_\infty$$

plays a crucial role.

★ Assume that $\ell \in (0; \pi/k)$. Then for Problem (2.29), no mode can propagate in $\mathfrak{b}_\infty$ (due to the mixed boundary conditions). With this property, we can show the asymptotic expansion, as $L \rightarrow +\infty$,

$$R_D(L) = R_D^\infty + \dots \quad (2.34)$$

where $R_D^\infty \in \mathcal{C}(0, 1) := \{z \in \mathbb{C} \mid |z| = 1\}$ is a constant and the dots correspond to a remainder which is exponentially small as $L \rightarrow +\infty$.

For Problem (2.28) (with Neumann boundary conditions), one propagating mode exists in $\mathfrak{b}_\infty$. Because of the reflection of this mode on the wall at $y = L$, the coefficient $R_N(L)$ does not converge as $L \rightarrow +\infty$. More precisely, we can prove that it admits the expansion

$$R_N(L) = R_N^{\text{asy}}(L) + \dots$$

where $R_N^{\text{asy}}(L)$ is a term whose dependence with respect to L is explicit and which runs periodically on $\mathcal{C}(0, 1)$ as $L \rightarrow +\infty$. Again the dots stand for a remainder which is exponentially small as $L \rightarrow +\infty$.

Imagine that we want to achieve $R(L) = 0$ (zero reflection). According to (2.33), we must impose $R_N(L) = -R_D(L)$. Relations (2.32) guarantee that for all $L > 0$, both $R_N(L)$ and $R_D(L)$ are located on the unit circle $\mathcal{C}(0, 1)$. But $R_D(L)$ tends to a constant $R_D^\infty \in \mathcal{C}(0, 1)$ while $R_N(L)$ runs continuously on $\mathcal{C}(0, 1)$ as $L \rightarrow +\infty$. This proves the existence of $L_\star$ such that $R_N(L_\star) = -R_D(L_\star)$ and so $R(L_\star) = 0$. Actually since $L \mapsto R_N^{\text{asy}}(L)$ is periodic, this shows the existence of an unbounded sequence of L , whose terms tend to a periodic sequence, such that we have zero reflection in the corresponding waveguide.

Numerically, we can approximate the solution u introduced in (2.26) by solving Problem (2.25) as explained after (2.18). In Figure 2.21, we work with $k = 3$ in a particular geometry (see the right picture) for which $\ell = 1$ and vary $L \in (1; 9)$. We display the corresponding scattering coefficients in the complex plane. In accordance with the theory above, we observe that for $L > 0$, asymptotically R and T run on circles of radius $1/2$ passing through zero. In Figure 2.21, we also represent the real part of $u - w^+$ in a particular geometry where $R = 0$. As expected, the scattered field is exponentially decaying as $x \rightarrow -\infty$. To identify this geometry, we simply select one particular L corresponding to a peak in the graph of $L \mapsto -\ln |R(L)|$.

In Figure 2.22, we display an L-shaped waveguide where $R = 0$. The geometry is unbounded in the left and bottom directions. We play with the length of the oblique branch. This framework is not exactly the one described above. However, due to the symmetry, it can be dealt with in a completely similar way.

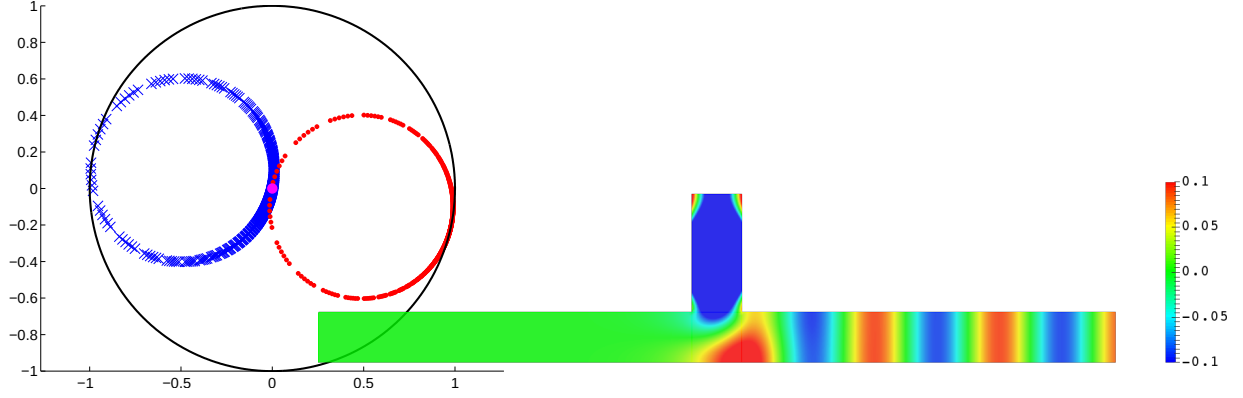


Figure 2.21: Left: R ($\times$) and T ($\bullet$) in the complex plane for $L \in (1; 9)$ (the geometry is the one of the right picture). Due to conservation of energy, R and T are located inside the unit disk. Right: real part of $u - w^+$ for a particular L_* such that $R(L_*) = 0$ (zero reflection). Here $k = 3$ and $\ell = 1$.

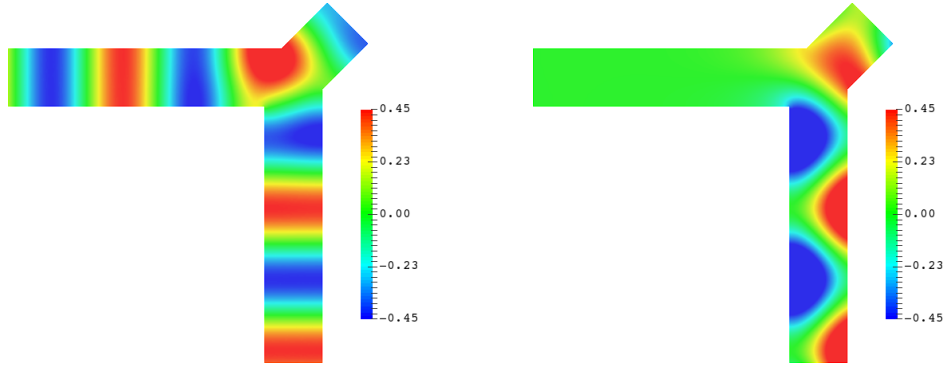


Figure 2.22: Real parts of u (left) and $u - w^+$ (right) in an L-shaped waveguide where the length of the oblique branch has been tuned to achieve $R = 0$ (zero reflection).

The analysis above also shows that there is an almost periodic sequence of L such that $R_N(L) = R_D(L)$ and so $T(L) = 0$. This is an interesting situation referred to as zero transmission. In that case, that we have not yet discussed, though the waveguide is not obstructed, the propagation of an incident wave is stopped and energy is completely backscattered ($|R| = 1$). To identify these situations, we simply select the L corresponding to peaks in the graph of $L \mapsto -\ln |T(L)|$. In Figure 2.23, we display geometries where $T = 0$. Note that according to the decomposition (2.26), in this case the transmitted field is exponentially decaying.

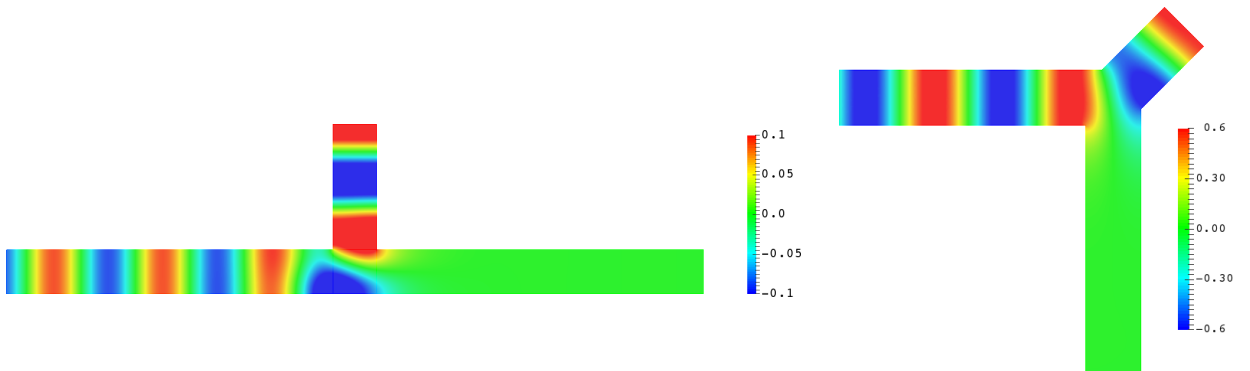


Figure 2.23: Geometries where $T = 0$ (zero transmission).

★ When $\ell \in (\pi/k; 2\pi/k)$, *i.e.* for larger vertical branches, both for Problems (2.29) and (2.28), one

propagating mode exists in $\mathfrak{b}_\infty$. Then we can prove the asymptotic expansions, as $L \rightarrow +\infty$,

$$R_N(L) = R_N^{\text{asy}}(L) + \dots, \quad R_D(L) = R_D^{\text{asy}}(L) + \dots, \quad (2.35)$$

where $R_N^{\text{asy}}(L)$, $R_D^{\text{asy}}(L)$ are explicitly known coefficients which run periodically on $\mathcal{C}(0,1)$ with periods $\Pi_N < \Pi_D$. This is enough to conclude that $R = 0$ for an unbounded sequence of L . This also shows that we have $T = 0$ for another sequence of L . However, compared to the case $\ell \in (0; \pi/k)$, the behavior of R and T in the complex plane can be much more erratic, especially when Π_N/Π_D is not a rational number (see Figure 2.24).

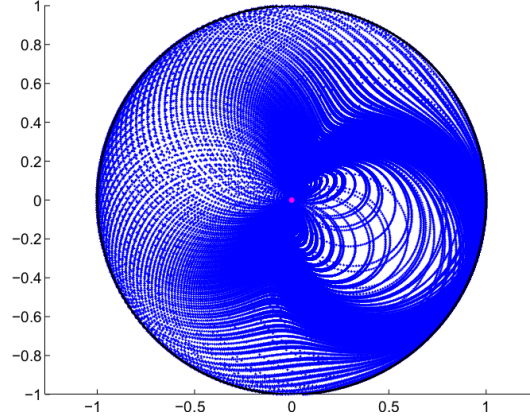


Figure 2.24: Coefficient R in the geometry of Figure 2.25. Here $k = 3$, $\ell = 1.7$ and L varies in $(1; 99)$.

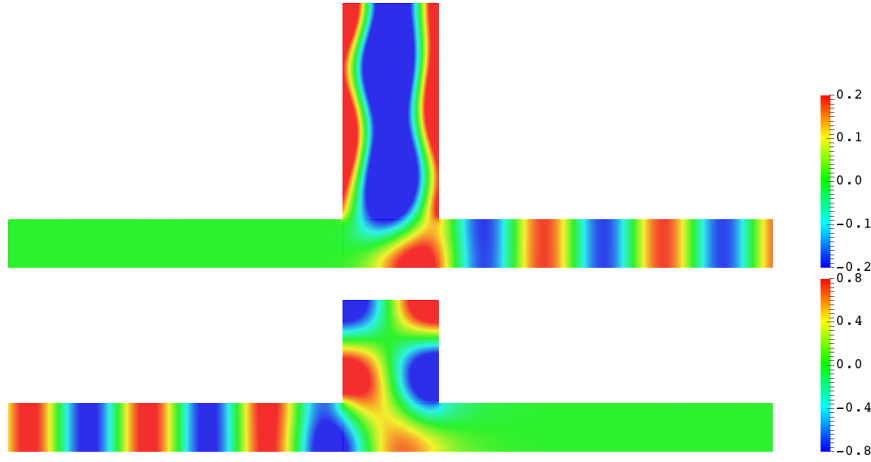


Figure 2.25: Top: real part of $u - w^+$ in a geometry where $R = 0$ (zero reflection). Bottom: real part of u in a geometry where $T = 0$ (zero transmission).

2.6.4 Comments

Let us conclude this section with a few comments.

i) To obtain perfect invisibility, *i.e.* $T = 1$, from formula (2.33), we see that we must impose both $R_N = 1$ and $R_D = -1$. This requires additional efforts but we have shown in [173] that by working with three vertical branches as represented in Figure 2.26, this can be achieved. Let us emphasize that compared to the methods based on thin ligaments leading to the numerics of Figure 2.9, though the picture looks quite similar, the approach is very different. In particular, in Figure 2.26 large defects could be present in the waveguide but not in Figure 2.9. On the other hand, in

Figure 2.26 the geometry must be symmetric with respect to the vertical while it is not the case in Figure 2.9. Finally, the technique providing Figure 2.9 relies on asymptotic analysis with thin ligaments. Figure 2.26 uses asymptotic analysis for long branches.

ii) We have considered only 2D problems with Neumann boundary condition but higher dimension with other boundary conditions can be dealt with using exactly the same procedure. The analysis we have presented works for the monomode regime $k \in (0; \pi)$. It seems complicated to extend it to configurations where several propagating modes exist in the horizontal waveguide.

iii) Mathematically, the main difficulty consists in proving error estimates to justify the asymptotic expansions (2.34)–(2.35). Constructing the expansion is quite natural and the consistency analysis is rather straightforward. However establishing stability estimates is a difficult task, in particular because the geometry changes. To proceed, as in §1.5.1, it is necessary to work with the very useful technique of overlapping cut-off functions in adapted weighted Sobolev spaces. For more details, we refer the reader to [352, Chap. 5, §5.6, Thm. 5.6.3].

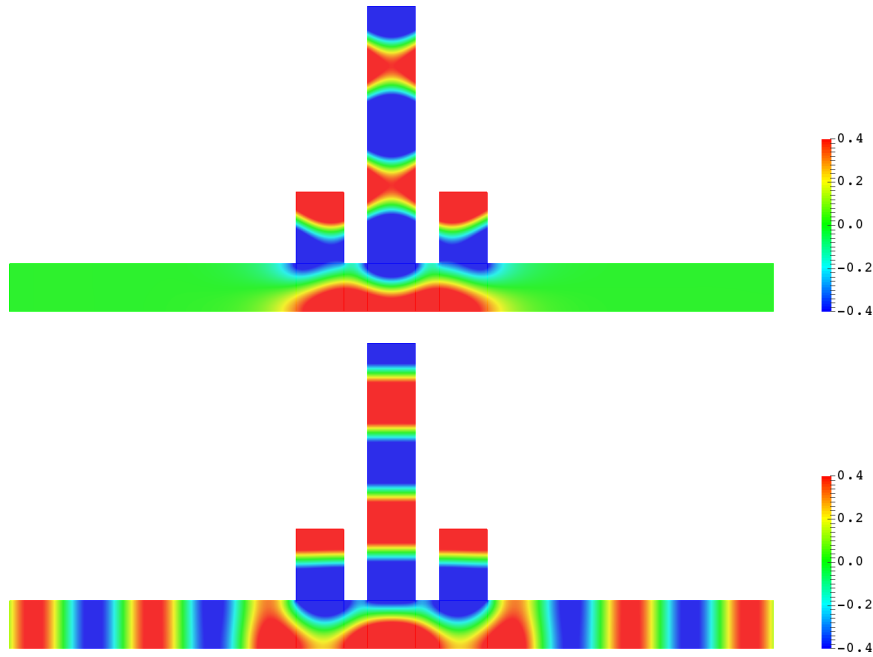


Figure 2.26: Geometry where $T = 1$ (perfect invisibility). Top: $\Re(u - w^+)$. Bottom: $\Re u$.

2.6.5 Simple examples of perfect invisibility and trapped modes

With V. Pagneux, we adapted the technique presented above to construct very simple examples of geometries where we have perfect invisibility. Let us explain the approach presented in [176].

Assume that k is fixed in $(0; \pi)$. Set $\ell := \pi/k > 1$ and

$$\Omega_L := \mathcal{S} \cup \mathcal{B}_L \quad (2.36)$$

with $\mathcal{S} = \mathbb{R} \times (0; 1)$ and $\mathcal{B}_L = (-\ell/2; \ell/2) \times [1; 1 + L]$ (see Figure 2.27). Observe that compared to (2.24), we choose a particular width for the vertical branch and moreover, we do not allow for the presence of obstacles inside the reference strip. Then all the analysis presented in §2.6.2 is valid in that geometry. Therefore we have the formulas

$$R = \frac{R_N + R_D}{2} \quad \text{and} \quad T = \frac{R_N - R_D}{2},$$

where R_N , R_D appear respectively in the decompositions (2.30), (2.31).

Now for the particular geometry considered here, one observes that $V := w^+ - w^- = 2i \sin(kx)$ is a solution to (2.29) for all $L > 0$. Note that this property is based on the fact that $\pi/(2k)$, the width of the bounded branch of ω_L , coincides with the quarter wavelength of the waves $w^\pm$. By uniqueness of the definition of the coefficient R_D in (2.31), we deduce that

$$R_D = R_D(L) = -1 \quad \text{for all } L > 0.$$

On the other hand, one can show as above that as $L \rightarrow +\infty$, $R_N(L)$ runs continuously on the unit circle $\mathcal{C}(0,1)$ (remember the constraint (2.32)) and that we have $|R_N(L) - R_N^{\text{asy}}(L)| \leq C e^{-cL}$ for certain $c, C > 0$ where $L \mapsto R_N^{\text{asy}}(L)$ runs periodically on $\mathcal{C}(0,1)$ with a period of π/k . Therefore from (2.33), we infer that as $L \rightarrow +\infty$, $T(L)$ runs continuously on $\mathcal{C}(1/2, 1/2)$, almost periodically with a period of π/k . As a consequence, there is an unbounded sequence of L such that we have $T(L) = 1$. In Figure 2.27, we give an example of geometry which has been tuned so that $T = 1$.

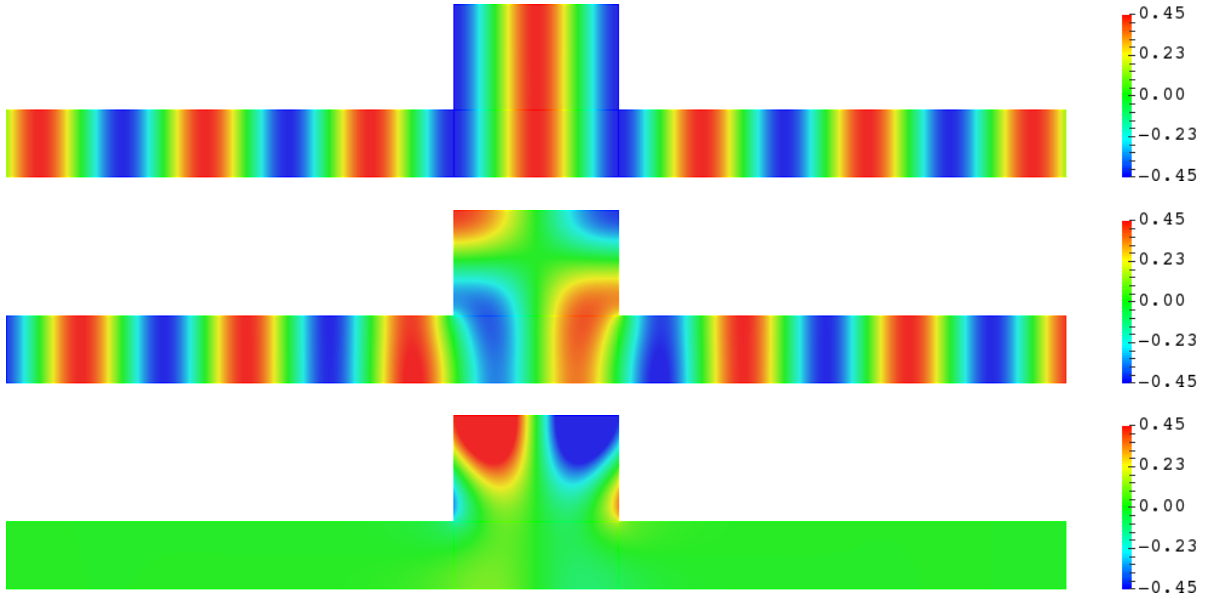


Figure 2.27: $\Re u$ (top), $\Im u$ (middle) and $\Im(u - w^+)$ (bottom) in a geometry tuned so that $T = 1$.

In the analysis above, a key ingredient is that

$$w^+ - w^- = 2i \sin(kx) \tag{2.37}$$

solves Problem (2.29) in ω_L with mixed boundary conditions for all $L > 0$. This is true in the ω_L defined from the Ω_L introduced in (2.36) but it is easy to see that this occurs in other domains. For example, this is satisfied as well in the half-waveguide defined from the geometry of Figure 2.28 where the steps have width π/k and arbitrary heights. In this setting, then we vary the height of the central branch to achieve $T = 1$.

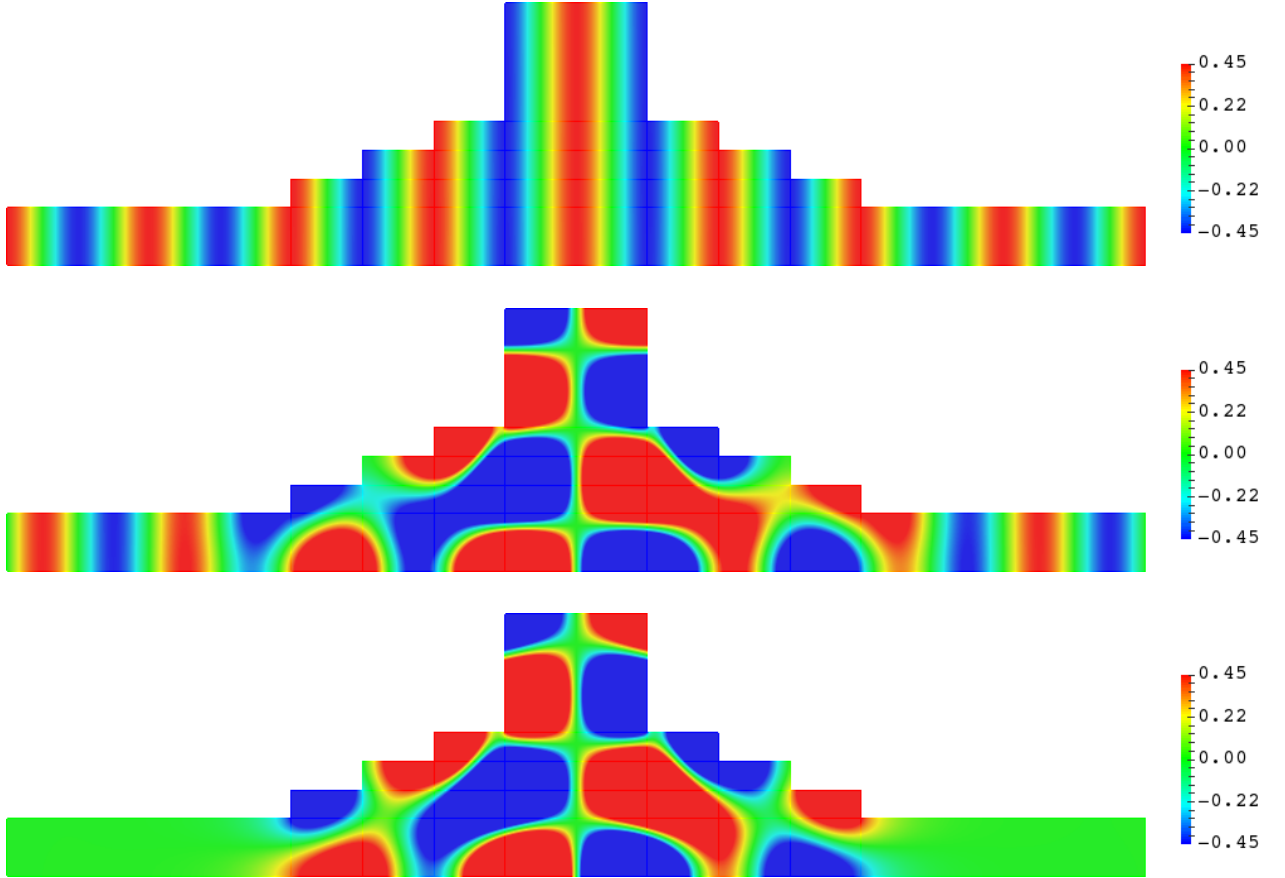


Figure 2.28: $\Re u$ (top), $\Im u$ (middle) and $\Im(w - w^+)$ (bottom) in a geometry tuned so that $T = 1$.

In [176], we were also concerned with so-called trapped modes. We remind the reader that trapped modes are non zero solutions of the homogeneous problem (2.25) which are of finite energy: they decay exponentially at infinity and are in H^1 . They play a key role in physical systems and they have been widely studied, in particular much more than “invisible modes”. We refer the reader for example to [461, 224, 225, 207, 340, 375, 388, 184, 410] for corresponding references. More precisely, in [176] we were interested in trapped modes associated with eigenvalues embedded in the continuous spectrum. Let us mention that for the Laplace operator with homogeneous Neumann boundary conditions that we consider here, the continuous spectrum, which is due to the existence of propagating modes, coincides with the half-line $[0; +\infty)$. Such trapped modes are also called Bound States in the Continuum (BSCs or BICs) in quantum mechanics (see for example [438, 359, 239] as well as the recent review [278]). Such objects are difficult to observe. In particular they are unstable with respect to the geometry and a small perturbation transforms them in general into quasi-normal modes associated with complex resonances [18]. A classical method to exhibit trapped modes corresponding to eigenvalues embedded in the continuous spectrum consists in considering waveguides that are symmetric with respect to the (Ox) axis [224, 207, 340, 410]. This is interesting because it allows one to decouple symmetric and skew-symmetric modes. Then working with a sufficiently large resonator, one can ensure that an eigenvalue exists below the continuous spectrum for the problem with mixed boundary conditions satisfied by the skew-symmetric component of the field. Back to the original waveguide, this guarantees the existence of an eigenvalue embedded in $[0; +\infty)$. In [176] we proposed a simple alternative mechanism to construct trapped modes associated with an eigenvalue embedded in the continuous spectrum that we present now.

To proceed, we work with the notion of augmented scattering matrix, a convenient tool in-

roduced by S.A. Nazarov in [383, 293, 373, 376] allowing one to detect the presence of trapped modes. Note that the classical scattering matrix, which describes only the contributions on the propagating modes, is useless for this task. We work in the half-waveguide ω_L introduced in (2.27) and for $k \in (0; \pi)$, consider the problem (2.28) with Neumann boundary conditions. It admits the scattering solution

$$v = w^+ + R_N w^- + \tilde{v}$$

with $R_N \in \mathcal{C}(0, 1)$ and $\tilde{v} \in H^1(\omega_L)$ and so the scattering matrix is simply a scalar. Renormalize the $w^\pm$ by setting $W_0^\pm := w^\pm / \sqrt{2k}$ and define the “wave packets”

$$W_1^\pm(x, y) = \frac{1}{\sqrt{2\beta_1}} (e^{-\beta_1 x} \pm i e^{\beta_1 x}) \sqrt{2} \cos(\pi y),$$

where we recall that $\beta_1 = \sqrt{\pi^2 - k^2}$. Observe that both W_1^+ and W_1^- are exponentially growing as $x \rightarrow -\infty$. Here the normalisation coefficients are chosen so that the matrix defined in (2.39) is unitary. As shown in [383, 293, 373, 376], Problem (2.28) admits solutions with the expansions

$$\begin{aligned} v_0 &= W_0^+ + s_{00} W_0^- + s_{01} W_1^- + \tilde{v}_0 \\ v_1 &= W_1^+ + s_{10} W_0^- + s_{11} W_1^- + \tilde{v}_1 \end{aligned} \quad (2.38)$$

where $\tilde{v}_0, \tilde{v}_1$ have fast decay in $O(e^{\sqrt{4\pi^2 - k^2}x})$ when $x \rightarrow -\infty$. They allow us to define the augmented scattering matrix

$$\mathcal{S} := \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \in \mathbb{C}_{2 \times 2}. \quad (2.39)$$

As for the usual scattering matrix, one can show that $\mathcal{S}$ is unitary ($\mathcal{S} \bar{\mathcal{S}}^\top = \text{Id}_{2 \times 2}$) and symmetric ($s_{10} = s_{01}$). The following result (see *e.g.* [376, Thm. 2]) provides an algebraic criterion to guarantee the existence of trapped modes.

Lemma 2.6.1 *If $s_{11} = -1$, then v_1 is a trapped mode for Problem (2.28) set in ω_L .*

Remark 2.6.2 *Note that $s_{11} = -1$ is only a sufficient criterion of existence of trapped modes. Indeed the geometry ω_L can support trapped modes for Problem (2.28) with $s_{11} \neq -1$. In this case, this trapped mode must decay as $O(e^{\sqrt{4\pi^2 - k^2}x})$ when $x \rightarrow -\infty$.*

Proof. If $s_{11} = -1$, since $\mathcal{S}$ is unitary, then $s_{10} = 0$. In such a situation, according to (2.38), we have $v_1(x, y) = (2i/\sqrt{\beta_1}) e^{\beta_1 x} \cos(\pi y) + O(e^{\sqrt{4\pi^2 - k^2}x})$ as $x \rightarrow -\infty$. This shows that $v_1 \not\equiv 0$ belongs to $H^1(\omega_L)$. In other words, v_1 is a trapped mode. ■

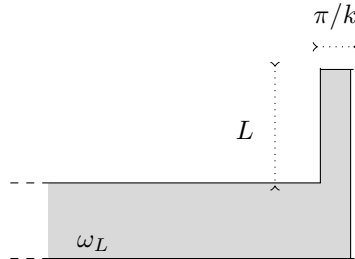


Figure 2.29: Geometry ω_L supporting trapped modes for certain $L > 0$.

Now we consider the particular geometry, for $L > 0$,

$$\omega_L := (-\infty; 0) \times (0; 1) \cup (-\pi/k; 0) \times [1; 1 + L] \quad (2.40)$$

(see Figure 2.29). In ω_L , we observe that $W_0^+ + W_0^- = \sqrt{2/k} \cos(kx)$ is a solution to (2.28) for all $L > 0$. From the uniqueness of the definition of the coefficients s_{mn} in (2.38), we deduce that for all $L > 0$, we have $s_{00}(L) = 1$ and $s_{01}(L) = s_{10}(L) = 0$. Since $\mathcal{S}$ is unitary, we infer that $|s_{11}(L)| = 1$ for all $L > 0$. In other words, $s_{11}(L)$ is located on the unit circle in the complex plane. But working as above, one can show that as $L \rightarrow +\infty$, $s_{11}(L)$ gets closer and closer to an asymptotic term which runs periodically on the unit circle. Additionally, s_{11} depends continuously on L . This guarantees the existence of an unbounded sequence of L such that $s_{11}(L) = -1$ and so, from Lemma 2.6.1, such that trapped modes exist for Problem (2.28) at the given k . By symmetry with respect to the vertical axis, we directly get trapped modes in a waveguide with two unbounded branches. Besides, we can enlarge the class of geometries considered in (2.40) similarly to what was explained after (2.37). For example, in Figure 2.30, we represent a trapped mode corresponding to an eigenvalue embedded in the continuous spectrum. In this geometry, the central and secondary steps have respectively widths $2\pi/k$ and π/k . We tuned the height of the central branch while the height of the secondary steps is chosen arbitrarily.

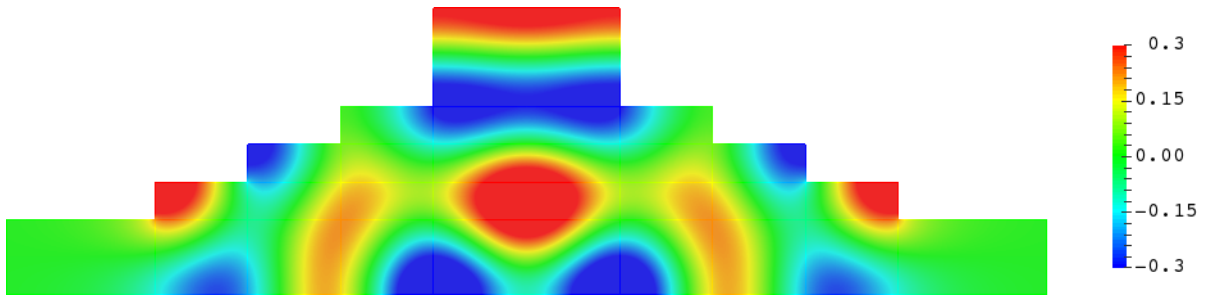


Figure 2.30: Real part of a trapped mode associated with an eigenvalue embedded in the continuous spectrum.

Let us mention that the techniques presented in this paragraph are specific to the Neumann boundary condition. More precisely, we exploit that the first propagating mode does not depend on the y variable. This cannot be adapted to the Dirichlet boundary condition.

2.6.6 From zero transmission to trapped modes in waveguides

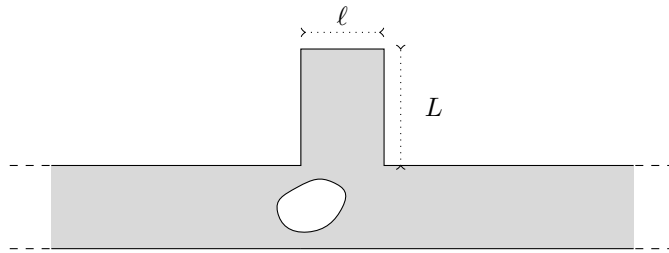


Figure 2.31: Non symmetric waveguide Ω_L with a branch of length $L > 0$.

In the article [177], we studied the propagation of acoustic waves in geometries as in Figure 2.31, *i.e.* with a long vertical branch of length $L > 0$, but without the assumption of symmetry with respect to (Oy) . For $k \in (0; \pi)$, the scattering matrix is of size 2×2 and has the form

$$\mathbb{S} := \begin{pmatrix} R_+ & T \\ T & R_- \end{pmatrix}.$$

Here R_+ (resp. R_-) is the reflection coefficient associated with a right-going (resp. left-going) incident plane wave. This scattering matrix is unitary ($\mathbb{S}\mathbb{S}^\top = \text{Id}_{2 \times 2}$) and symmetric. In that situation,

we can still obtain an asymptotic expansion of $\mathbb{S}$ as $L \rightarrow +\infty$ and show that asymptotically, the scattering coefficients run on circles. However, compared to the situation where Ω_L is symmetric with respect to the vertical axis, in general the reflection coefficients $R_{\pm}(L)$ do not go through zero as $L \rightarrow +\infty$. For the transmission coefficient however, the situation is different. Let us detail this. First, for T , we obtain the expansion, as $L \rightarrow +\infty$

$$T(L) = T^{\text{asy}}(L) + \dots \quad (2.41)$$

where $T^{\text{asy}}(L)$ is an asymptotic term whose expression is explicit and the dots correspond to terms which become exponentially small as $L \rightarrow +\infty$. Interestingly, we can show that $L \mapsto T^{\text{asy}}(L)$, which is a periodic function, passes through zero as $L \rightarrow +\infty$ (see [177, Prop. 2.1]). This is quite extraordinary because this is true for any obstacle. But this is not enough to guarantee that $L \mapsto T(L)$ passes through zero. With this, from (2.41), *a priori* we only deduce that $L \mapsto T(L)$ passes as close to zero as desired when $L \rightarrow +\infty$. However we have shown in [177, Thm. 2.1] that the strong structure of the scattering matrix forces $L \mapsto T(L)$ to pass exactly through zero. Let us explain the mechanism. Since $\mathbb{S}$ is unitary, we have the identity

$$R_+(L)\overline{T(L)} + T(L)\overline{R_-(L)} = 0$$

for all $L > 0$. Now assume that $L \mapsto T(L)$ passes very close to zero but not through zero on a small interval $[L_1; L_2]$. Then the phase of $L \mapsto T(L)$ runs on an interval of length approximately equal to π (which is not the case if $L \mapsto T(L)$ passes exactly through zero). We deduce that

$$R_+(L)/\overline{R_-(L)} = -T(L)/\overline{T(L)}$$

runs rapidly on the unit circle $\mathcal{C}(0, 1)$. This is incompatible with the error estimates derived for $R_{\pm}(L)$ which impose to $R_{\pm}(L)$ to stay close to $R_{\pm}^{\text{asy}}(L)$, some terms which do not move much on $[L_1; L_2]$.

In [177], we also explained how, from waveguides where $T = 0$, we can create geometries supporting trapped modes. Roughly speaking, the idea is the following. When $T = 0$, one has a mirror effect: all the energy of the incident field is backscattered. By placing face to face two mirrors and by tuning the distance between them, we can trap waves.

2.7 Playing with Fano resonances

In the previous section, we described a first mechanism to create large non reflecting or invisible defects of the reference strip. Let us present other ideas. The important point is that we have to act strongly on scattering coefficients while keeping control of their values. To proceed, in the next two sections, we will work with resonant phenomena. First, we explain how to exploit Fano resonances together with symmetry considerations to construct large non reflecting obstacles in monomode regime.

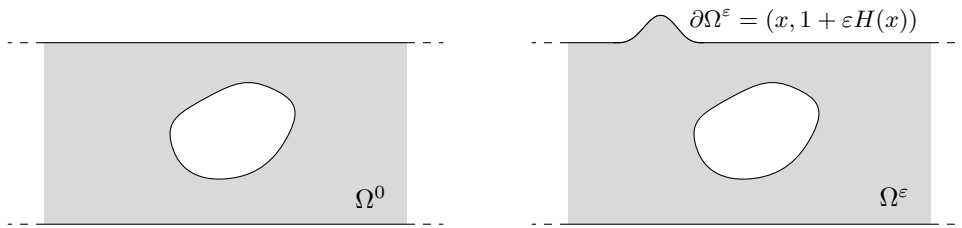


Figure 2.32: Original waveguide Ω^0 (left) and perturbed geometry Ω^ϵ (right).

The Fano resonance is a classical phenomenon that arises in many situations in physics (see [228] for the seminal paper and [357, 346] for recent reviews). For the presentation of simple models

in optics, we also refer the reader to [226, 227]. For our particular concern, it appears as follows. Assume that the geometry of our waveguide is characterized by a real parameter ε . Below, ε will be the amplitude of a local perturbation of the walls (see Figure 2.32 right). To simplify notation, set $\lambda := k^2$ so that Problem (2.1) writes

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega^\varepsilon \\ \partial_\nu u = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases} \quad (2.42)$$

Assume that trapped modes exist for Problem (2.42) with $\varepsilon = 0$ and $\lambda = \lambda^0 \in (0; \pi^2)$. Then, for $\varepsilon \neq 0$ small, the scattering matrix $\mathbb{S}$, which is of size 2×2 in monomode regime, exhibits a rapid change for real λ varying in a neighborhood of λ^0 .

In [166], with S.A. Nazarov, we exploited this rapid change together with symmetry considerations of the geometry to provide examples of waveguides where $R = 0$ or $T = 0$ (exactly). This was also the subject of the 3A internship at École Polytechnique of L. Scaglione that I have supervised during the Covid period. The main part of the work was dedicated to prove rigorously that the scattering matrix is not smooth at $(\varepsilon, \lambda) = (0, \lambda^0)$. Our study shares similarities with [446, 445, 447, 3]. In these articles, probably for the first time, the authors analyze rigorously the Fano resonance phenomenon in gratings in electromagnetism, a context close to ours, via techniques of complex analysis and by means of tools of the theory of analytic functions of several variables. Moreover in [445], under certain symmetry assumptions, they show examples of total transmission and total reflection for this problem. In [166], we proved similar results following a different path. We worked with the augmented scattering matrix $\mathcal{S}$ that we already used in (2.39). The important point is that $\mathcal{S}$ is smooth at $(\varepsilon, \lambda) = (0, \lambda^0)$. Then from the relation existing between $\mathcal{S}$ and the usual scattering matrix $\mathbb{S}$, this allow us to understand the behavior of $\mathbb{S}$ at $(\varepsilon, \lambda) = (0, \lambda^0)$.

With this approach, we never work with complex λ , that is, we never deal with the meromorphic extension of $\mathbb{S}$. In particular, we do not use the fact that when the geometry is perturbed from $\varepsilon = 0$ to $\varepsilon \neq 0$, in general the eigenvalue λ^0 embedded in the continuous spectrum becomes a complex resonance, that is a pole of the meromorphic extension of $\mathbb{S}$ [18, 481]. We think that this alternative method to analyze the Fano resonance can be of interest. Thus it allows one to obtain relatively explicit formulas with respect to the features of the geometry. It could also allow one to deal with singular perturbations of the problem (for example if one adds some small non penetrable obstacles in the waveguide as in [165]). For computation of complex resonances and numerical investigations of the Fano resonance phenomenon in waveguides, we refer the reader to [214, 140, 220, 265, 273, 266]; for results concerning the existence of trapped modes associated with eigenvalues embedded in the continuous spectrum, see *e.g.* [461, 224, 225, 207, 340, 375].

To illustrate more precisely the Fano resonance phenomenon, let us work on a 1D toy problem.

2.7.1 A 1D toy problem

Unperturbed case



Figure 2.33: A 1D geometry.

Consider the geometry

$$\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3 \quad \text{with } \Omega_1 := (-\infty; 0) \times \{0\}, \Omega_2 := \{0\} \times (0; 1), \Omega_3 := (0; 1) \times \{0\}$$

(see Figure 2.33). For a function φ defined in Ω , set $\varphi_j := \varphi|_{\Omega_j}$, $j = 1, 2, 3$. Working in suitable coordinates, we can see the Ω_j as 1D domains. Similarly to (2.42), we study the problem

$$-\varphi_j'' = k^2 \varphi_j \quad \text{in } \Omega_j, \quad j = 1, 2, 3, \quad (2.43)$$

with the conditions

$$\begin{cases} \varphi_1(0) = \varphi_2(0) = \varphi_3(0), \\ \varphi_1'(0) = \varphi_2'(0) + \varphi_3'(0), \\ \varphi_2'(1) = \varphi_3'(1) = 0. \end{cases} \quad (2.44)$$

Thus at the junction point O , we impose continuity of the field and conservation of the flux (Kirchhoff law) and we work with Neumann boundary conditions at the ends of the ligaments. It can be shown that this is a good model to describe the properties of the acoustic problem in the 2D domain obtained by thickening the above 1D graph. We are interested in the scattering of the rightgoing incident wave $x \mapsto e^{ikx}$. We denote by φ the corresponding total field. For the simple problem considered here, the radiation condition boils down to assuming that φ writes as $\varphi(x) = e^{ikx} + R e^{-ikx}$ in Ω_1 , where $R \in \mathbb{C}$ is the reflection coefficient. Using the two boundary conditions of (2.43)–(2.44), we are led to look for a solution φ such that

$$\varphi_1(x) = e^{ikx} + R e^{-ikx}, \quad \varphi_2(y) = a \cos(k(y-1)), \quad \varphi_3(x) = b \cos(k(x-1)),$$

where $a, b \in \mathbb{C}$. Writing the transmission conditions at the junction point O , we obtain that R, a, b must solve the system

$$\mathbb{M}(k)\Phi = F \quad \text{with } \mathbb{M}(k) := \begin{pmatrix} 1 & -\cos k & 0 \\ 0 & \cos k & -\cos k \\ i & \sin k & \sin k \end{pmatrix}, \quad \Phi := \begin{pmatrix} R \\ a \\ b \end{pmatrix}, \quad F := \begin{pmatrix} -1 \\ 0 \\ i \end{pmatrix}. \quad (2.45)$$

One finds

$$\det \mathbb{M}(k) = (2 \sin k + i \cos k) \cos k.$$

Therefore this system (and so Problem (2.43)–(2.44) with the above mentioned radiation condition) is uniquely solvable if and only if $k \notin (2\mathbb{N} + 1)\pi/2$. When $k \in (2\mathbb{N} + 1)\pi/2$, the kernel of Problem (2.43)–(2.44) coincides with $\text{span}(\varphi_{\text{tr}})$ where φ_{tr} is the trapped mode such that

$$\varphi_{\text{tr}}(x) = \begin{cases} 0 & \text{in } \Omega_1 \\ \sin(ky) & \text{in } \Omega_2 \\ -\sin(kx) & \text{in } \Omega_3. \end{cases}$$

On the other hand, for any $k > 0$, one can check that System (2.45) (and so Problem (2.43)–(2.44)) admits a solution because $F \in (\ker {}^t\mathbb{M})^\perp$ (this can be verified by an explicit calculus). Moreover, the coefficient R is always uniquely defined (even when $k \in (2\mathbb{N} + 1)\pi/2$) and such that

$$R(k) = \frac{\cos(k) + 2i \sin(k)}{\cos(k) - 2i \sin(k)}. \quad (2.46)$$

The map $k \mapsto R(k)$ is π -periodic and $|R(k)| = 1$. The latter relation, which is due to conservation of energy, guarantees that $R(k) = e^{i\theta(k)}$ for some phase $\theta(k) \in \mathbb{R}/(2\pi\mathbb{Z})$. In Figure 2.34, we represent the map $k \mapsto \theta(k)$ for $k \in (0; \pi)$. We observe that it has a smooth behavior, also around the value $k = \pi/2$ for which trapped modes exist for Problem (2.43)–(2.44). Thus we see that the scattering is insensitive to the existence of trapped modes.

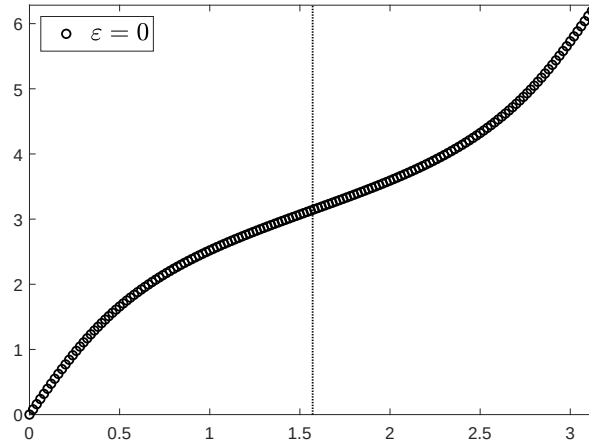


Figure 2.34: Map $k \mapsto \theta(k)$. The vertical black dotted line indicates the value $k = \pi/2$.

It is also interesting to give a spectral description of the properties of the problem. Denote by A the unbounded operator of $L^2(\Omega)$ such that

$$A\varphi = -\varphi_j'' \quad \text{in } \Omega_j, \quad j = 1, 2, 3, \quad (2.47)$$

with domain

$$D(A) = \{\varphi \in L^2(\Omega) \mid \varphi_j \in H^2(\Omega_j) \text{ for } j = 1, 2, 3 \text{ and } \varphi \text{ satisfies (2.44)}\}. \quad (2.48)$$

Classically, see *e.g.* [54], one shows that A is a selfadjoint operator. Therefore its spectrum $\sigma(A)$ is real. One can prove that $\sigma(A)$ coincides with $[0; +\infty)$. More precisely, we have $\sigma_{\text{ess}}(A) = [0; +\infty)$ where $\sigma_{\text{ess}}(A)$ denotes the essential spectrum of A . By definition, $\sigma_{\text{ess}}(A)$ corresponds to the set of $\lambda \in \mathbb{C}$ for which there exists a so-called singular sequence, that is a sequence $(u^{(m)})$ of functions of $D(A)$ such that $\|u_m\|_{L^2(\Omega)} = 1$, $(u^{(m)})$ converges weakly to 0 in $L^2(\Omega)$ and $((A - \lambda)u^{(m)})$ converges strongly to 0 in $L^2(\Omega)$. The fact that $\sigma_{\text{ess}}(A) = [0; +\infty)$ is directly related to the existence of the propagating modes $e^{\pm ikx}$ for all $k > 0$. On the other hand, the above calculations ensure that A also has point spectrum corresponding to eigenvalues, *i.e.* values of λ such that $\ker(A - \lambda \text{Id}) \neq \{0\}$. More precisely, one has $\sigma_p(A) = \{(2n+1)^2\pi^2/4, n \in \mathbb{N}\}$. Let us emphasize that these eigenvalues are embedded in the essential spectrum (see the illustration of Figure 2.35).

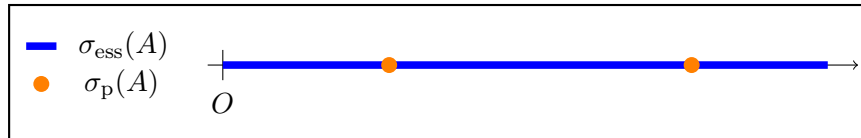


Figure 2.35: Spectrum of A in the complex plane.

Perturbed case

Now we consider the same problem in the perturbed geometry $\Omega^\varepsilon := \Omega_1 \cup \Omega_2 \cup \Omega_3^\varepsilon$ with $\Omega_3^\varepsilon := (0; 1 + \varepsilon) \times \{0\}$ and $\varepsilon \in \mathbb{R}$ small. We denote with a superscript ε all the above quantities. In Ω^ε , the resolution of the previous scattering problem leads to solve the system

$$\mathbb{M}^\varepsilon(k)\Phi^\varepsilon = F \quad \text{with } \mathbb{M}^\varepsilon(k) := \begin{pmatrix} 1 & -\cos k & 0 \\ 0 & \cos k & -\cos(k(1 + \varepsilon)) \\ i & \sin k & \sin(k(1 + \varepsilon)) \end{pmatrix}, \quad \Phi^\varepsilon := \begin{pmatrix} R^\varepsilon \\ a^\varepsilon \\ b^\varepsilon \end{pmatrix}. \quad (2.49)$$

The vector F is the same as in (2.45). We find

$$\det \mathbb{M}^\varepsilon(k) = \sin(k(2 + \varepsilon)) + i \cos k \cos(k(1 + \varepsilon)).$$

Therefore we find that for $\varepsilon \neq 0$ small, the determinant of $\mathbb{M}^\varepsilon$ does not vanish for a given range of $k > 0$. As a consequence, Problem (2.43)–(2.44) set in Ω^ε has a unique solution. One obtains

$$R^\varepsilon(k) = \frac{\cos k \cos(k(1 + \varepsilon)) + i \sin(k(2 + \varepsilon))}{\cos k \cos(k(1 + \varepsilon)) - i \sin(k(2 + \varepsilon))}.$$

Again, we have $|R^\varepsilon(k)| = 1$ (conservation of energy) so we can write $R^\varepsilon(k) = e^{i\theta^\varepsilon(k)}$ for some $\theta^\varepsilon(k) \in \mathbb{R}/(2\pi\mathbb{Z})$. Note that $\theta^0 = \theta$ where θ appears after (2.46). The map $k \mapsto \theta^\varepsilon(k)$ is displayed in Figure 2.36 for several values of ε (see also the alternative representation in Figure 2.37). We observe that for $\varepsilon \neq 0$, the curve $k \mapsto \theta^\varepsilon(k)$ has a fast variation for k close to $\pi/2$. The variation is even faster as $\varepsilon \neq 0$ gets small. On the other hand, for $\varepsilon = 0$, as represented in Figure 2.34, the curve $k \mapsto \theta^0(k)$ has a very smooth behavior. We emphasize that for $(\varepsilon, k) = (0, \pi/2)$, as mentioned above, trapped modes exist for Problem (2.43)–(2.44).

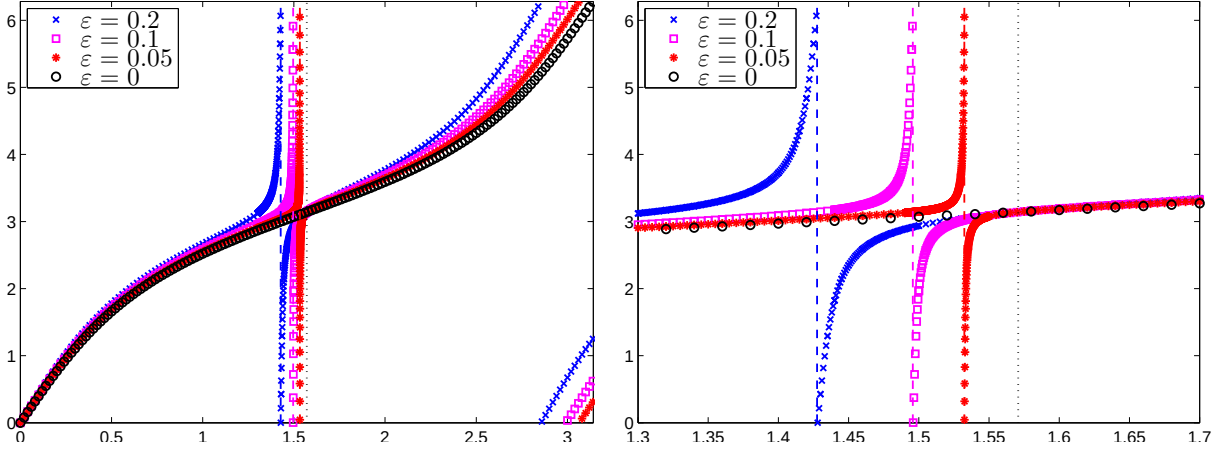


Figure 2.36: Maps $k \mapsto \theta^\varepsilon(k)$ for several values of ε . The right picture is a zoom on the left picture around $k = \pi/2$ (marked by the vertical black dotted line). The vertical colored dashed lines indicate the values of k such that $\theta^\varepsilon(k) = 0$.

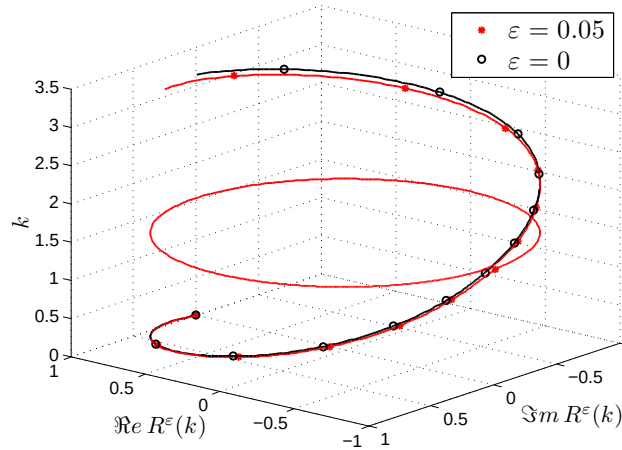


Figure 2.37: Parametric curves $k \mapsto (\Re R^\varepsilon(k), \Im R^\varepsilon(k))$ for $k \in (0, \pi)$.

In order to study the variations of the reflection coefficient with respect to the frequency and the geometry, we define the map $\mathcal{R} : \mathbb{R}^2 \rightarrow \mathbb{C}$ such that

$$\mathcal{R}(\varepsilon, k) = \frac{\cos(k) \cos(k(1 + \varepsilon)) + i \sin(k(2 + \varepsilon))}{\cos(k) \cos(k(1 + \varepsilon)) - i \sin(k(2 + \varepsilon))}. \quad (2.50)$$

With such a notation, we have $R^\varepsilon(k) = \mathcal{R}(\varepsilon, k)$ and $R(k) = \mathcal{R}(0, k)$. For all $k \in (0; \pi)$, there holds $\lim_{\varepsilon \rightarrow 0} \mathcal{R}(\varepsilon, k) = \mathcal{R}(0, k)$. Now assume that the frequency and the geometry are related by some prescribed law in a neighborhood of the point $(\varepsilon, k) = (0, \pi/2)$ corresponding to a setting supporting trapped modes. For example, assume that

$$k = \pi/2 + \varepsilon k'$$

for a given $k' \in \mathbb{R}$. Then for $k' \neq -\pi/4$, starting from expression (2.50), we find as $\varepsilon \rightarrow 0$ the expansion

$$\mathcal{R}(\varepsilon, \pi/2 + \varepsilon k') = -1 + \varepsilon \left(\frac{-2ik'(\pi + 2k')}{\pi + 4k'} \right) + O(\varepsilon^2). \quad (2.51)$$

Note that we have $\mathcal{R}(0, \pi/2) = -1$.

For $k' = -\pi/4$, the asymptotic behavior as ε tends to zero is more surprising. Indeed, for

$$k = \pi/2 - \varepsilon\pi/4 + \varepsilon^2\mu$$

with $\mu \in \mathbb{R}$ (we take $k' = -\pi/4$ but allow for some freedom at higher order), we obtain

$$\mathcal{R}(\varepsilon, \pi/2 - \varepsilon\pi/4 + \varepsilon^2\mu) = g(\mu) + O(\varepsilon) \quad \text{with} \quad g(\mu) = \frac{\pi^2 + i(32\mu - 4\pi)}{\pi^2 - i(32\mu - 4\pi)}. \quad (2.52)$$

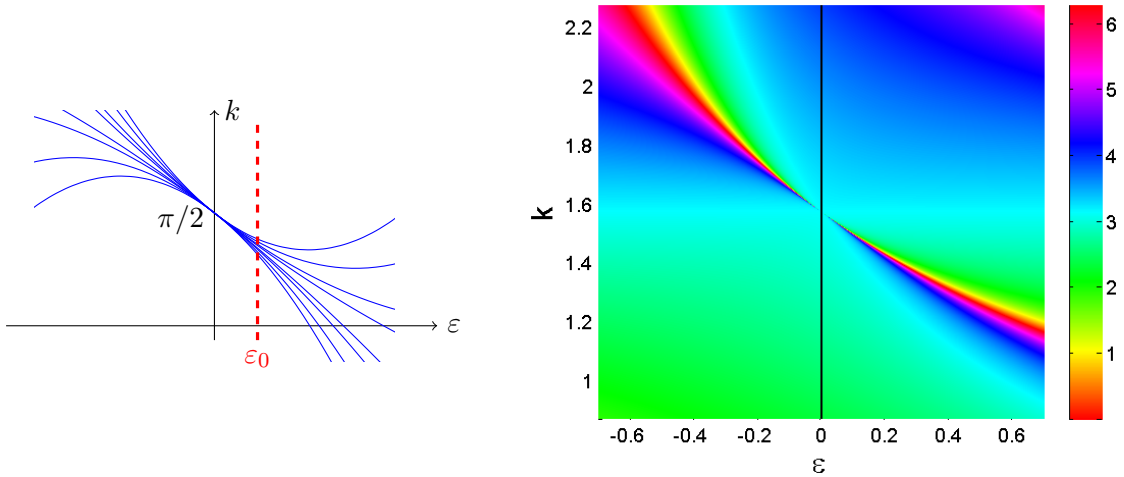


Figure 2.38: Left: parabolic paths $\{(\varepsilon, \pi/2 - \varepsilon\pi/4 + \varepsilon^2\mu), \varepsilon \in (-1; 1)\} \subset \mathbb{R}^2$ for several values of μ . According to μ , the limit of the coefficient $\mathcal{R}(\varepsilon, \pi/2 - \varepsilon\pi/4 + \varepsilon^2\mu)$ defined in (2.50) as $\varepsilon \rightarrow 0$ is different. Right: the colors indicate the phase of $\mathcal{R}(\varepsilon, k)$. This phase is valued in $[0; 2\pi)$.

Classical results concerning the Möbius transform (see *e.g.* [271, Chap. 5]) guarantee that g is a bijection between $\mathbb{R}$ and $\mathcal{C}(0, 1) \setminus \{-1 + 0i\}$ ($\mathcal{C}(0, 1)$ is the unit circle of the complex plane). Thus for any $z_0 \in \mathcal{C}(0, 1)$, we can find a path $\{\gamma(s), s \in (0; 1)\} \subset \mathbb{R}^2$ such that

$$\lim_{s \rightarrow 1} \gamma(s) = (0, \pi/2) \quad \text{and} \quad \lim_{s \rightarrow 1} \mathcal{R}(\gamma(s)) = z_0$$

(see Figure 2.38 right). In other words, by tuning cleverly the frequency and the amplitude of the perturbation, we can get any desired value for the reflection coefficient on the unit circle. Besides, this proves that the map $\mathcal{R}(\cdot, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{C}$ is not continuous at $(0, \pi/2)$. Finally, this shows also that for $\varepsilon_0 \neq 0$ small fixed (see the vertical red dashed line in Figure 2.38 left), the curve $k \mapsto \mathcal{R}(\varepsilon_0, k)$ must exhibit a rapid change. Indeed, μ varying in $[-C\varepsilon_0^{-1}; C\varepsilon_0^{-1}]$ for some arbitrary $C > 0$ (which is only a small change for k) leads to a large change for $\mathcal{R}(\varepsilon_0, \pi/2 - \varepsilon_0\pi/4 + \varepsilon_0^2\mu)$. This is exactly what we observed in Figure 2.36.

Let us return to the spectral description of the properties of the problem. Denote by A^ε the analog of the operator A introduced in (2.47)–(2.48) in the perturbed geometry Ω^ε . Since we made a perturbation in a bounded region, it can be shown that A^ε has the same essential spectrum as A , *i.e.* $\sigma_{\text{ess}}(A^\varepsilon) = [0; +\infty)$ (again this is directly due to the existence of the propagating modes $e^{\pm ikx}$ for all $k > 0$). On the other hand, the above computations guarantee that for $\varepsilon > 0$ small, A^ε has no eigenvalue, *i.e.* $\sigma_p(A^\varepsilon) = \emptyset$. This gives the picture of Figure 2.39 for $\sigma(A^\varepsilon)$. A natural question then is: what happened to the eigenvalues of A when the geometry has been perturbed? The answer is that they became so-called complex resonances. These are values of $k^2 \in \mathbb{C}$ with $\Im m k < 0$ such that there is a solution to

$$-\varphi_j'' = k^2 \varphi_j \quad \text{in } \Omega_j^\varepsilon, \quad j = 1, 2, 3,$$

with the conditions (2.44), admitting the expansion $\varphi_1(x) = e^{-ikx}$ in Ω_1 (generalized outgoing behavior). Note that such generalized eigenfunctions are exponentially growing as $x \rightarrow -\infty$ and so do not belong to $L^2(\Omega^\varepsilon)$. In the literature, they are often referred to as quasi-normal modes or leaky modes. Observe that they cannot exist for $\Im m k > 0$ because they would be exponentially decaying as $x \rightarrow -\infty$ and so would belong to $L^2(\Omega^\varepsilon)$. But this is impossible because the operator is selfadjoint and therefore cannot have eigenvalues in $\mathbb{C} \setminus \mathbb{R}$.

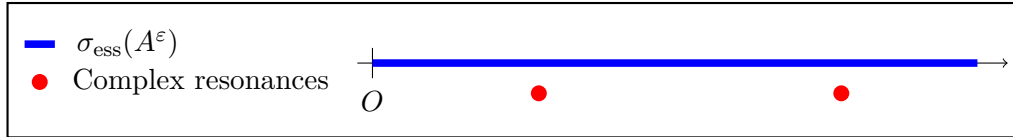


Figure 2.39: Spectrum of A^ε in the complex plane.

Let us stress that this conversion of eigenvalues into complex resonances when one perturbs the geometry is crucially related to the fact that we are considering eigenvalues which are embedded in the essential spectrum. In particular, it does not occur in bounded domains when no essential spectrum exists. Let us illustrate this with a concrete example. Denote by B the Neumann Laplacian on the interval $(0; 1)$ such that

$$\begin{cases} D(B) &= \{\varphi \in H^2(0; 1) \mid \partial_x \varphi(0) = \partial_x \varphi(1) = 0\} \\ B\varphi &= -\varphi''. \end{cases}$$

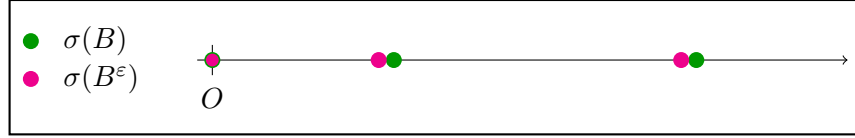
The operator B has only discrete spectrum (no essential spectrum), namely

$$\sigma(B) = \{n^2\pi^2, n \in \mathbb{N}\}$$

and the corresponding eigenfunctions are the φ_n such that $\varphi_n(x) = \cos(n\pi x)$. Now if we denote by B^ε the same Neumann Laplacian in the perturbed geometry $(0; 1 + \varepsilon)$, we find

$$\sigma(B^\varepsilon) = \{n^2\pi^2/(1 + \varepsilon)^2, n \in \mathbb{N}\}$$

and the corresponding eigenfunctions are the φ_n^ε such that $\varphi_n^\varepsilon(x) = \cos(n\pi x/(1 + \varepsilon))$. In other words, the eigenvalues of B have been only slightly shifted on the real axis (see Figure 2.40), they did not turn into complex resonances.

Figure 2.40: Spectra of B and B^ε in the complex plane.

On the other hand, not all geometric perturbations of Problem (2.47)–(2.48) convert embedded eigenvalues into complex resonances. For example, if both Ω_2, Ω_3 are perturbed into $\Omega_2^\varepsilon = \{0\} \times (0; 1 + \varepsilon)$, $\Omega_3^\varepsilon = (0; 1 + \varepsilon) \times \{0\}$, one can check that embedded eigenvalues remain embedded eigenvalues for the perturbed problem. Somehow, we need to break the symmetry as in (2.49), to create some coupling between the trapped modes and the scattering properties of the problem. In that case, complex resonances located close to the real axis have a strong impact on the scattering.

2.7.2 Fano resonance in 2D waveguides

Let us come back to Problem (2.42) in 2D. We assume that Ω^0 is such that the Neumann Laplacian has a simple eigenvalue $\lambda_0 \in (0; \pi^2)$ (geometric multiplicity equal to one). We perturb the geometry from some smooth compactly supported profile function H with amplitude $\varepsilon \geq 0$ as in Figure 2.32 right. We denote by Ω^ε the new waveguide and $\mathbb{S}(\varepsilon, \lambda)$, $T(\varepsilon, \lambda)$, $R_\pm(\varepsilon, \lambda)$ the scattering matrix/coefficients in the geometry Ω^ε at frequency λ . For short, we set $\mathbb{S}^0 = \mathbb{S}(0, \lambda^0)$, $T^0 = T(0, \lambda^0)$, $R_\pm^0 = R_\pm(0, \lambda^0)$. Introduce u_0 an eigenfunction associated with λ^0 such that $\|u_0\|_{L^2(\Omega)} = 1$. Decomposition in Fourier series guarantees that as $|x| \rightarrow +\infty$, we have the expansion

$$u_0(x, y) = K_\pm e^{-\sqrt{\pi^2 - \lambda^0}|x|} \cos(\pi y) + \dots$$

where $K_\pm \in \mathbb{C}$ and the $\dots$ correspond to terms which decay as $O(e^{-\sqrt{4\pi^2 - \lambda^0}|x|})$ as $|x| \rightarrow +\infty$. In [166, Thm. 5.1], the following result is proved.

Theorem 2.7.1 *Assume that $(K_+, K_-) \neq (0, 0)$. There is a quantity $\ell(H) \in \mathbb{R}$, which depends linearly on the profile function H , such that when $\varepsilon \rightarrow 0$,*

$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda') = \mathbb{S}^0 + O(\varepsilon) \quad \text{for } \lambda' \neq \ell(H), \quad (2.53)$$

and, for any $\mu \in \mathbb{R}$,

$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\ell(H) + \varepsilon^2\mu) = \mathbb{S}^0 + \frac{\tau^\top \tau}{i\tilde{\mu} - |\tau|^2/2} + O(\varepsilon). \quad (2.54)$$

In this expression $\tau = (a, b) \in \mathbb{C} \times \mathbb{C}$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some unessential real constants A, B with $A \neq 0$.

The justification of this result is more difficult than in 1D because one cannot perform explicit calculations. It requires adapted tools (see [446, 445, 447, 3, 166]) that we will not present here. However the interpretation of the result is completely similar to that of the 1D case. Observe that (2.53) and (2.54) are respectively the analogues of (2.51) and (2.52). As explained above, Theorem 2.7.1 shows that the mapping $\mathbb{S}(\cdot, \cdot)$ is not continuous at $(0, \lambda^0)$ (setting where trapped modes exist). Moreover for ε_0 small fixed, it proves that the scattering matrix $\lambda \mapsto \mathbb{S}(\varepsilon_0, \lambda)$ exhibits a quick change in a neighborhood of $\lambda^0 + \varepsilon_0\ell(H)$: this is the Fano resonance phenomenon. When $(K_+, K_-) = (0, 0)$ a faster Fano resonance phenomenon occurs.

2.7.3 Zero reflection and zero transmission

In the sequel, to simplify we denote by $\mathbb{S}^\varepsilon(\mu)$, $T^\varepsilon(\mu)$, $R_\pm^\varepsilon(\mu)$ the values of $\mathbb{S}$, T , $R_\pm$ in Ω^ε at the frequency $\lambda = \lambda^0 + \varepsilon\ell(H) + \varepsilon^2\mu$.

When Ω^ε is symmetric with respect to the vertical axis, we can proceed as in §2.6.2 and decompose the problem into two half-waveguide problems with Neumann/Dirichlet boundary conditions at $x = 0$. Let us denote by $R_N^\varepsilon(\mu)/R_D^\varepsilon(\mu)$ the corresponding reflection coefficients. Then from (2.33), we have

$$R_+^\varepsilon(\mu) = R_-^\varepsilon(\mu) = \frac{R_N^\varepsilon(\mu) + R_D^\varepsilon(\mu)}{2} \quad \text{and} \quad T^\varepsilon(\mu) = \frac{R_N^\varepsilon(\mu) - R_D^\varepsilon(\mu)}{2}. \quad (2.55)$$

To set ideas assume that the trapped modes associated with λ^0 are even. In that case, $(\varepsilon, \lambda) \mapsto R_D^\varepsilon(\mu)$ is smooth at $(0, \lambda_0)$. As a consequence, for ε small, $\mu \mapsto R_D^\varepsilon(\mu)$ does not vary much on the unit circle $\mathcal{C}(0, 1)$ for $\mu \in (-\varepsilon^{-1/2}; \varepsilon^{-1/2})$. On the other hand, by adapting the result of Theorem 2.7.1, we established that $\mu \mapsto R_N^\varepsilon(\mu)$ runs once on $\mathcal{C}(0, 1)$ for $\mu \in (-\varepsilon^{-1/2}; \varepsilon^{-1/2})$. From formulas (2.55), this ensures that the curves $\mu \mapsto T^\varepsilon(\mu)$, $\mu \mapsto R^\varepsilon(\mu)$ for $\mu \in (-\varepsilon^{-1/2}; \varepsilon^{-1/2})$, pass exactly through zero for ε small enough. This provides examples of geometries where we have either zero reflection or zero transmission. In Figure 2.41, we give an example of geometry supporting trapped modes for Neumann problem (2.28) at a particular $\lambda = \lambda_0 \in (0; \pi^2)$. Note that the waveguide is symmetric with respect to the line of equation $y = 1/2$. Then we perturb the domain by slightly shifting vertically the disk³. Then the symmetry is broken and from the analysis above, we know that when we sweep in λ in a small neighborhood of λ_0 , there is one λ such that one observes zero reflection (Figure 2.42) and one λ such that one gets zero transmission (Figure 2.43). Note that the smaller ε , the more delicate the adjustment of λ .



Figure 2.41: Left: geometry of the half waveguide. Right: real part of a trapped mode for $\varepsilon = 0$ and $k^0 := \sqrt{\lambda^0} \approx 2.7403$. The computation has been realized by using Perfectly Matched Layers.

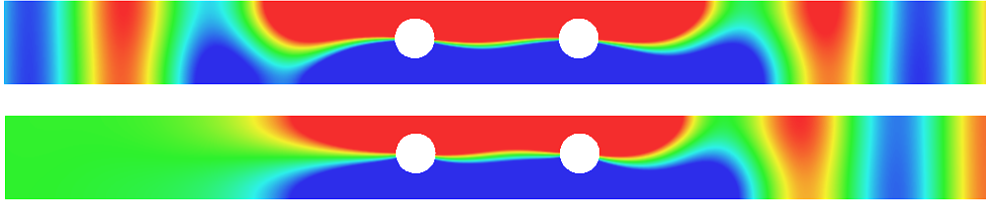


Figure 2.42: $\text{Re } u^\varepsilon$ (top) and $\text{Re } (u^\varepsilon - w^+)$ in a setting where $R_\pm^\varepsilon = 0$ ($\varepsilon = 0.05$ and $k = \sqrt{\lambda} = 2.751$).

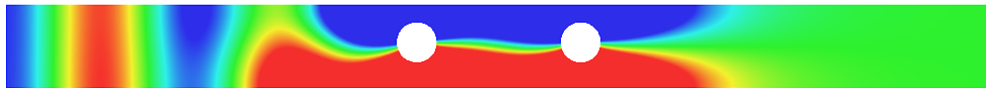


Figure 2.43: $\text{Re } u^\varepsilon$ in a setting where $T^\varepsilon = 0$ ($\varepsilon = 0.05$ and $k = \sqrt{\lambda} = 2.75495$).

When the domain Ω^ε is not symmetric with respect to the vertical axis, we cannot use the decomposition with the two half-waveguide problems. In that case, for a fixed small $\varepsilon > 0$, in general the curves $\mu \mapsto R_\pm^\varepsilon(\mu)$ do not go through zero in the complex plane and we do not observe zero

³Observe that a horizontal shifting of the position of the disk would maintain the decoupling between symmetric and skew-symmetric modes. As a consequence, the eigenvalue embedded in the continuous spectrum would remain an eigenvalue embedded in the continuous spectrum and no Fano resonance phenomenon would be observed.

reflection. However we have shown in [167] that $\mu \mapsto T^\varepsilon(\mu)$ passes through zero. This surprising result, that echoes the one of §2.6.6, was intuited in [333] using a continuation idea from a symmetric setting. In [167], we proved it rigorously using a different approach which does not require to start from a symmetric setting. Let us give the main ingredients of the proof which shares similarities with the one of §2.6.6.

Theorem 2.7.2 *Assume that $T^0 = T(0, \lambda^0) \neq 0$. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0; \varepsilon_0]$, there is $\mu \in \mathbb{R}$ such that $T^\varepsilon(\mu) = 0$.*

Proof. Theorem 2.7.1 provides the estimate

$$|T^\varepsilon(\mu) - T^{\text{asy}}(\mu)| \leq C \varepsilon \quad (2.56)$$

$$\text{with } T^{\text{asy}}(\mu) = T^0 + \frac{ab}{i\tilde{\mu} - (|a|^2 + |b|^2)/2}.$$

For any compact set $I \subset \mathbb{R}$, the constant $C > 0$ in (2.56) can be chosen independent of $\mu \in I$. We will prove first that the asymptotic term $T^{\text{asy}}(\cdot)$ always passes through zero. Then, from that and Estimate (2.56), we will deduce, due to the rigid structure of the scattering matrix, that $T^\varepsilon(\cdot)$ also passes through zero.

★ First, we study the set $\{T^{\text{asy}}(\mu), \mu \in \mathbb{R}\}$. Classical results concerning the Möbius transform guarantee that $\{T^{\text{asy}}(\mu), \mu \in \mathbb{R}\}$ coincides with $\mathcal{C}^{\text{asy}} \setminus \{T^0\}$ where $\mathcal{C}^{\text{asy}}$ is a circle passing through T^0 . Let us show that $\mathcal{C}^{\text{asy}}$ also passes through zero. One finds that $T^{\text{asy}}(\mu) = 0$ for some $\mu \in \mathbb{R}$ if and only if there holds

$$\frac{|a|^2 + |b|^2}{2} = \Re \left(\frac{ab}{T^0} \right). \quad (2.57)$$

An intermediate calculus of [166] implies $R_+^0 \bar{a} + T^0 \bar{b} = a$ and $T^0 \bar{a} + R_-^0 \bar{b} = b$. From this and the unitarity of $\mathbb{S}^0$ which imposes $R_-^0 = -\bar{R}_+^0 T^0 / \bar{T}^0$, we can obtain (2.57). Denote $\mu_\star$ the value of μ such that $T^{\text{asy}}(\mu_\star) = 0$ and for $\varepsilon > 0$, define the interval $I^\varepsilon = (\mu_\star - \sqrt{\varepsilon}; \mu_\star + \sqrt{\varepsilon})$. From (2.56), for $\varepsilon > 0$ small, we know that the curve $\{T^\varepsilon(\mu), \mu \in I^\varepsilon\}$ passes close to zero. Now, using the unitary structure of $\mathbb{S}^\varepsilon(\mu)$ as in §2.6.6, we show that this curve passes exactly through zero for ε small.

★ Assume by contradiction that for all $\varepsilon > 0$, $\mu \mapsto T^\varepsilon(\mu)$ does not pass through zero in I_ε . Since $\mathbb{S}^\varepsilon(\mu)$ is unitary, there holds $R^\varepsilon(\mu) \overline{T^\varepsilon(\mu)} + T^\varepsilon(\mu) \overline{R^\varepsilon(\mu)} = 0$ and so

$$-R^\varepsilon(\mu) / \overline{\tilde{R}^\varepsilon(\mu)} = T^\varepsilon(\mu) / \overline{T^\varepsilon(\mu)} \quad \forall \mu \in I^\varepsilon.$$

But if $\mu \mapsto T^\varepsilon(\mu)$ does not pass through zero on I^ε , one can verify that the point $T^\varepsilon(\mu) / \overline{T^\varepsilon(\mu)} = e^{2i \arg(T^\varepsilon(\mu))}$ must run rapidly on the unit circle for $\mu \in I_\varepsilon$ as $\varepsilon \rightarrow 0$. On the other hand, $R^\varepsilon(\mu) / \overline{\tilde{R}^\varepsilon(\mu)}$ tends to a constant on I_ε as $\varepsilon \rightarrow 0$. This way we obtain a contradiction. ■

Remark 2.7.3 *The fact that $\mathcal{C}^{\text{asy}}$ passes through zero is quite mysterious. Without assumption of symmetry, we do not have physical reason to explain this miracle.*

We illustrate this result in Figure 2.44. First we find that trapped modes exist for $\varepsilon = 0$ and $\sqrt{\lambda^0} \approx 1.2395 \in (0; \pi)$. Then we compute $T(\varepsilon, \lambda)$ (×) and $R(\varepsilon, \lambda)$ (•) for $\sqrt{\lambda} \in (1.2; 1.3)$ and $\varepsilon = 0.05$. As predicted, we observe that $\lambda \mapsto T(\varepsilon, \lambda)$ passes through zero around λ^0 . Finally, we display the real part of u_+ in Ω^ε for $\varepsilon = 0.05$ and $\sqrt{\lambda} = 1.2449$, a configuration where $T(\varepsilon, \lambda) \approx 0$.

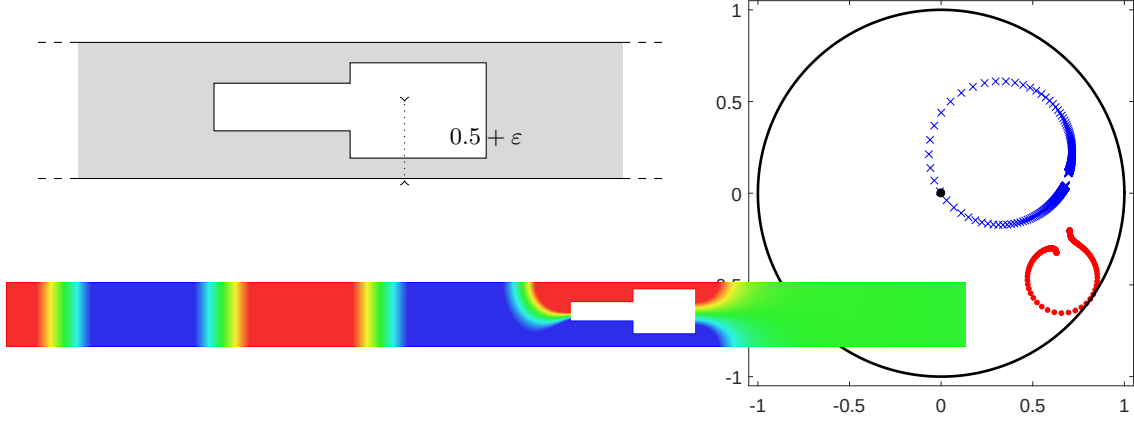


Figure 2.44: Zero transmission in a waveguide via the Fano resonance mechanism.

Finally, let us mention that everything presented here can be adapted in higher dimension, that is for waveguides in $\mathbb{R}^d$, $d \geq 3$, with a bounded transverse section. Moreover we could also have studied analogously problems with homogeneous Dirichlet boundary conditions.

2.7.4 Abnormal acoustic transmission in a waveguide with perforated screens

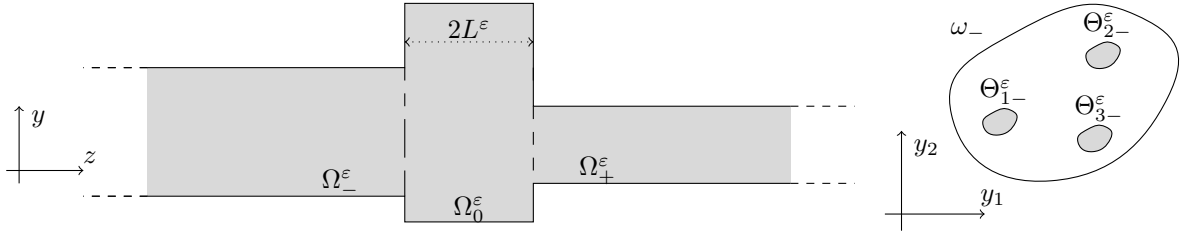


Figure 2.45: Side view of the waveguide Ω^ε (left) and picture of the left perforated screen (right).

We studied a related topic with S.A. Nazarov in [168]. In this work, we considered the propagation of the piston mode (the plane wave) in an acoustic waveguide obstructed by two screens with small holes (see Figure 2.45). In general, due to the features of the geometry, almost no energy of the incident wave is transmitted through the structure. However, by tuning carefully the distance between the two screens, which form a resonator, one can get good transmission in energy. If additionally the geometry is symmetric with respect to the vertical axis, we can reach zero reflection.

Let us mention that similar problems have been considered in [36, 209] but with only one hole and with Dirichlet boundary conditions. In [209], the authors work with decompositions in Fourier series which are hard to generalize. In [36], following [352], the authors use techniques of construction of asymptotic expansions for boundary value problems in singularly perturbed domains. We also adopt this approach. Unlike the above-mentioned publications, we employ a fine tuning procedure of the geometric shape which allows us to reveal the zero reflection phenomenon.

In Figure 2.46, we work in a symmetric domain. There is one hole of width $\varepsilon = 10^{-4}$ in the screens. We display the field u^ε for two different values of L , namely for a generic one where, as expected, T is almost zero and for some well-chosen $L = L^*$ where $R = 0$.

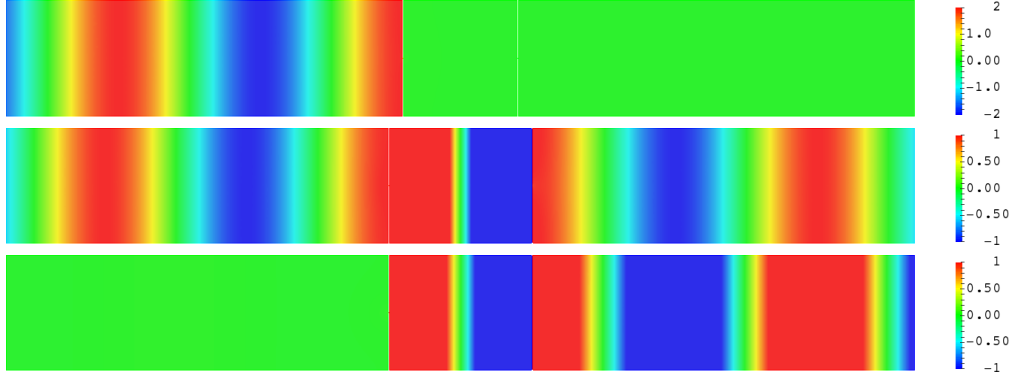


Figure 2.46: Top: $\Re u^\varepsilon$ for $L = 0.5$. Middle: $\Re u^\varepsilon$ for $L = L^*$. Bottom: $\Re (u^\varepsilon - w^+)$ for $L = L^*$.

2.8 Playing with thin resonant ligaments

2.8.1 Cloaking of a given obstacle

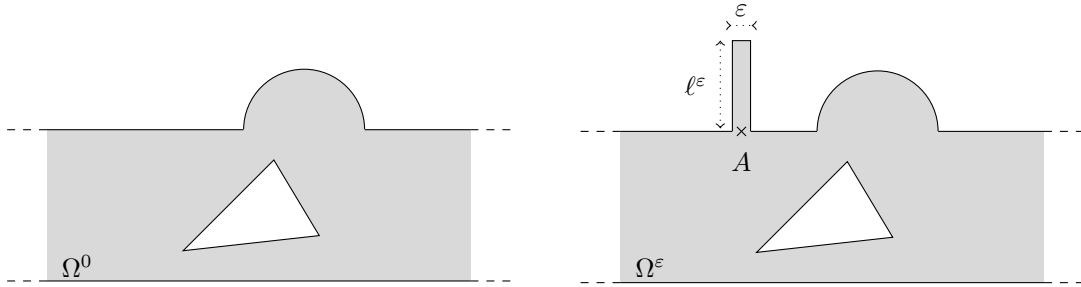


Figure 2.47: Left: initial setting. Right: geometry with one thin outer resonator.

In Sections 2.4, 2.5, we presented approaches to construct invisible defects using perturbative techniques. In Section 2.6, we explained how to create symmetric large invisible obstacles. During the post-doc of J. Heleine, in the article [161] with S.A. Nazarov, we proposed a method to cloak any given obstacle in an acoustic waveguide in monomode regime without this assumption of symmetry of the geometry. More precisely, starting from a setting where $T \neq 1$, we explain in [161] how to perturb the initial geometry to obtain a new waveguide where $T \approx 1$. Let us mention that we do not work with *ad hoc* penetrable materials as in transformation optics. Additionally, we do not add active sources in the system as people do in active cloaking [355, 463, 464, 404, 409, 145, 135]. What we realize is passive cloaking at infinity by perturbing the shape of the waveguide.

To proceed, we work with thin outer resonators, that we also call ligaments, of width $\varepsilon > 0$ small compared to the wavelength, as depicted in Figure 2.47 right. These ligaments are interesting because they are almost 1D objects, which allows us to make explicit their influence on the fields and so on the scattering coefficients. However in general, *i.e.* for most lengths, they produce only perturbations of order ε which is not sufficient to compensate for the scattering due to the initial object. But working around the resonance lengths (see (2.59)) of the resonators, we can get effects of order one. This is a key aspect in our approach which makes it in particular different from the technique presented in Section 2.4. Note that thin ligaments around resonance lengths have been studied for example in [317, 86, 338, 339, 337] in a context close to ours, namely in the analysis of the scattering of an incident wave by a periodic array of subwavelength slits. The core of our approach is based on an asymptotic expansion of the scattering solutions with respect to ε as ε tends to zero. This allows us to derive formulas for the scattering coefficients with a relatively explicit dependence on the geometric features. To obtain the expansions, we apply techniques of matched asymptotic

expansions. For related methods, we refer the reader to [37, 234, 313, 369, 235, 372, 290, 32, 73]. We emphasize that an important feature of our study distinguishing it from the previous references is that the lengths, and not only the widths, of the resonators depend on ε . This way of considering the problem, used also in the results presented in §2.7.4, is an essential ingredient to reveal the resonant phenomena. From this perspective, our work shares similarities with [275, 105, 106] (see also references therein). On the other hand, in the asymptotic procedure we observe that when ε tends to zero, everything happens as if punctual sources were located at the junction points between the resonators and the main part of the waveguide. Therefore our approach somehow has some connections with the works on active cloaking mentioned above.

Let us describe the general strategy. First, as usual in asymptotic analysis for singularly perturbed problems, we work with different ansatz depending on the region. More precisely, we consider the outer expansions

$$\begin{cases} u_+^\varepsilon(x, y) = u^0(x, y) + \dots & \text{in } \Omega^0 \\ u_+^\varepsilon(x, y) = \varepsilon^{-1}v^{-1}(y) + v^0(y) + \dots & \text{in the resonator} \end{cases} \quad (2.58)$$

where Ω^0 denotes the initial waveguide without the ligament and u^0 , v^{-1} , v^0 denote unknown functions which are independent of ε . To begin with, suppose that the ligament has a length $\ell > 0$ independent of ε . Considering the restriction of Problem (2.1) in Ω^ε to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathcal{P}_{1D}) \begin{cases} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1 + \ell) \\ v(1) = \partial_y v(1 + \ell) = 0. \end{cases}$$

An important message is that the features of $(\mathcal{P}_{1D})$ play a key role in the physical phenomena and so in the asymptotic analysis. We denote by ℓ_{res} (resonance lengths), the values of ℓ , given by

$$\ell_{\text{res}} := \pi(m + 1/2)/k, \quad m \in \mathbb{N}, \quad (2.59)$$

such that $(\mathcal{P}_{1D})$ admits a non zero solution. Note that the non zero functions solving $(\mathcal{P}_{1D})$ coincide, up to a multiplicative constant, with $\sin(k(y - 1))$.

Assume first that $\ell \neq \ell_{\text{res}}$ so that the only solution of $(\mathcal{P}_{1D})$ is zero. Then we set $v^{-1} \equiv 0$ in (2.58) and when $\varepsilon \rightarrow 0$, we can show that

$$\begin{cases} u_\pm^\varepsilon(x, y) = u_\pm + o(1) & \text{in } \Omega^0 \\ u_\pm^\varepsilon(x, y) = u_\pm(A) v_0(y) + o(1) & \text{in the resonator} \end{cases} \quad (2.60)$$

where $u_\pm$ stands for the scattering solution corresponding to an incident plane wave coming from $\mp\infty$ and $A = (x_A, 1)$ is the attachment point of the ligament. Moreover in (2.60),

$$v_0(y) = \cos(k(y - 1)) + \tan(k\ell) \sin(k(y - 1))$$

(observe that we have $\partial_y^2 v + k^2 v = 0$ in $(1; 1 + \ell)$ as well as $v_0(1) = 1$ and $\partial_y v_0(1 + \ell) = 0$). From this, we deduce that

$$R_\pm^\varepsilon = R_\pm + o(1), \quad T^\varepsilon = T + o(1),$$

where $R_\pm$, T are the scattering coefficients in the geometry without the ligament. In that situation, we see that the resonator has no influence at order ε^0 , which is not interesting for our purpose.

Assume now that $\ell = \ell_{\text{res}}$. In that case, the asymptotic analysis is more involved. At the end of the (long) procedure, we obtain, when $\varepsilon \rightarrow 0$,

$$\begin{cases} u_+^\varepsilon(x, y) = u_+(x, y) + ak\gamma(x, y) + o(1) & \text{in } \Omega^0 \\ u_+^\varepsilon(x, y) = \varepsilon^{-1}a \sin(k(y - 1)) + O(1) & \text{in the resonator} \end{cases} \quad (2.61)$$

where γ denotes the outgoing Green function such that

$$\begin{cases} \Delta\gamma + k^2\gamma = 0 & \text{in } \Omega \\ \partial_\nu\gamma = \delta_A & \text{on } \partial\Omega. \end{cases}$$

Here δ_A is the Dirac delta distribution supported at A . Moreover in (2.61), a is such that

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1} \ln |\varepsilon| + C_\Xi}$$

where Γ and C_Ξ are some constants which depend only on the initial geometry Ω^0 . Observe that in (2.61), the field blows up as $O(\varepsilon^{-1})$ in the resonator, which is directly related to the fact that there is a complex resonance close to the considered real k . From (2.61), we obtain

$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \quad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

This time the thin resonator has an influence at order ε^0 . Let us make a small variant by assuming that the length of the ligament is equal to $\ell^\varepsilon = \ell_{\text{res}} + \varepsilon\eta$, where $\eta \in \mathbb{R}$ is a parameter that we set as we wish. In that situation the analysis is very similar to the previous case and when $\varepsilon \rightarrow 0$, we find

$$\begin{cases} u_+^\varepsilon(x, y) = u_+(x, y) + a(\eta)k\gamma(x, y) + o(1) & \text{in } \Omega^0 \\ u_+^\varepsilon(x, y) = \varepsilon^{-1}a(\eta) \sin(k(y-1)) + O(1) & \text{in the resonator} \end{cases} \quad (2.62)$$

with

$$a(\eta)k = -\frac{u_+(A)}{\Gamma + \pi^{-1} \ln |\varepsilon| + C_\Xi + \eta}.$$

This gives

$$R_+^\varepsilon = R_+^{\text{asy}}(\eta) + o(1), \quad T^\varepsilon = T^{\text{asy}}(\eta) + o(1) \quad (2.63)$$

with

$$R_+^{\text{asy}}(\eta) := R_+ + ia(\eta)u_+(A)/2, \quad T^{\text{asy}}(\eta) := T + ia(\eta)u_-(A)/2.$$

Thus, not only the resonator has an influence at order ε^0 , but additionally the latter depends on the choice made for η . We get something similar to what has been shown in Section 2.7 (see in particular Figure 2.38): for all $\eta \in \mathbb{R}$, the resonator tends to the 1D segment $(1; 1 + \ell_{\text{res}})$, but depending on the choice of η , the limit of the corresponding scattering coefficients is not the same. As a consequence, the scattering coefficients, considered as functions of the two variables (ε, ℓ) , are not continuous at the point $(0, \ell_{\text{res}})$. Moreover, for ε_0 fixed small, varying slightly ℓ around ℓ_{res} , which corresponds for example to sweep $\eta \in [-\varepsilon_0^{-1/2}; \varepsilon_0^{-1/2}]$, we obtain a large variation for $R_+^{\varepsilon_0}$, T^{ε_0} (again see the illustration of Figure 2.38). More precisely, working with the Möbius transform, one shows that the sets

$$\{R_+^{\text{asy}}(\eta), \eta \in \mathbb{R}\}, \quad \{T^{\text{asy}}(\eta), \eta \in \mathbb{R}\}$$

where $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, coincide with circles. We deduce that asymptotically, when $\varepsilon \rightarrow 0$, when perturbing the length of the ligament around ℓ_{res} , the quantities R_+^ε , T^ε run on circles. Interestingly for our purpose, the features of these circles depend on A , the attachment point of the ligament.

In view of achieving zero reflection, using the expansions of $u_\pm(A)$ far from the obstacle, we establish the following statement:

Proposition 2.8.1 *Assume that $R_+ \neq 0$ and $T \neq 0$. There are some $A = (x_A, 1)$ such that there exists η such that $R_+^{\text{asy}}(\eta) = 0$. In that case, when $\varepsilon \rightarrow 0$, we have $R_+^\varepsilon = 0 + o(1)$.*

Remark 2.8.2 Note that we exclude the case $R_+ = 0$ because in this situation we already have zero reflection in the initial geometry and there is no need for adding a resonator. In the case $T = 0$, i.e. $|R_+| = 1$ due to conservation of energy, the most challenging situation, our approach does not work. However, one possibility to get zero reflection is to add first one or several resonators to obtain a transmission coefficient quite different from zero. And then to add another well-tuned resonator to kill the reflection. Let us mention that this strategy is also interesting when T is small but non zero because in this case achieving almost zero reflection with only one resonator is quite unstable.

Remark 2.8.3 It is important to emphasize that compared to what we presented in the previous sections, we do not reach exactly $R_+^\varepsilon = 0$ but simply get $R_+^{\text{asy}}(\eta) = 0$. In other words, there is a residue due to the error in the expansion. This analysis encourages us to take ε as small as possible. However when ε becomes small, the amplitude of the field in the resonator gets very high and the tuning procedure of the features of the ligament is very sensitive. Thus one must find a compromise between small reflection and robustness with respect to perturbations of the geometry.

In Figures 2.48–2.49, we consider the scattering of the right-going plane wave by a sound hard obstacle (the fish). We have added a well-tuned ligament to obtain almost zero reflection. In Figure 2.49, we observe that by working with a smaller ε , as expected we can reduce the reflection.

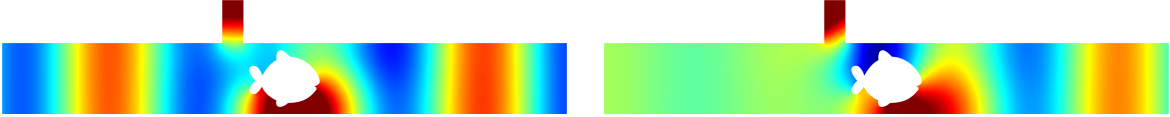


Figure 2.48: Real parts of u^ε (left) and of $u^\varepsilon - w^+$ (right). The length of the resonator is tuned to get almost zero reflection. Here $\varepsilon = 0.3$.

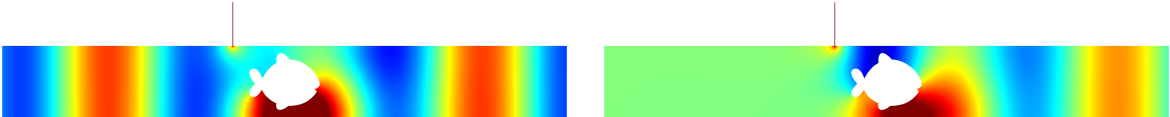


Figure 2.49: Same quantities as in Figure 2.48 but with a thinner resonator. Here $\varepsilon = 0.01$.

Once almost zero reflection has been obtained, it remains to compensate for the phase shift (recall that $R_+^{\text{asy}}(\eta) = 0$ only implies $|T_+^{\text{asy}}(\eta)| = 1$). To proceed, one method consists in coupling the previous waveguide with what we call a phase-shifter. This is a device where one has zero reflection and any prescribed phase. We have shown that such phase shifters can be designed by working with two well-tuned resonant ligaments added to the reference strip $\mathbb{R} \times (0; 1)$. At the end, we obtain $T^\varepsilon \approx 1$ with three well-tuned resonant ligaments.

On the other hand, by exploiting again the results of the asymptotic analysis (2.62), (2.63), we have established that we can get $T^\varepsilon \approx 1$ with only two well-tuned ligaments. Let us assess the degrees of freedom which are involved. We wish to control two complex coefficients, R^ε , T^ε , and so four real parameters. The relation of conservation of energy imposes one constraint. As a consequence, there are three real degrees of freedom. In our strategy here, we play with the two lengths of the resonators and with the distance between them.

In Figures 2.50, 2.51, we cloak two different obstacles/defects by working with two resonant ligaments. In each case, on the first line we display the field corresponding to the scattering of a right-going plane wave without the resonators. The setting of Figure 2.51 is particularly challenging because the initial transmission coefficient is very small. To restore a good transmission, we observe that we have to excite strongly the resonances. Practically, this is probably a limitation because we can imagine that dissipation will then become important.

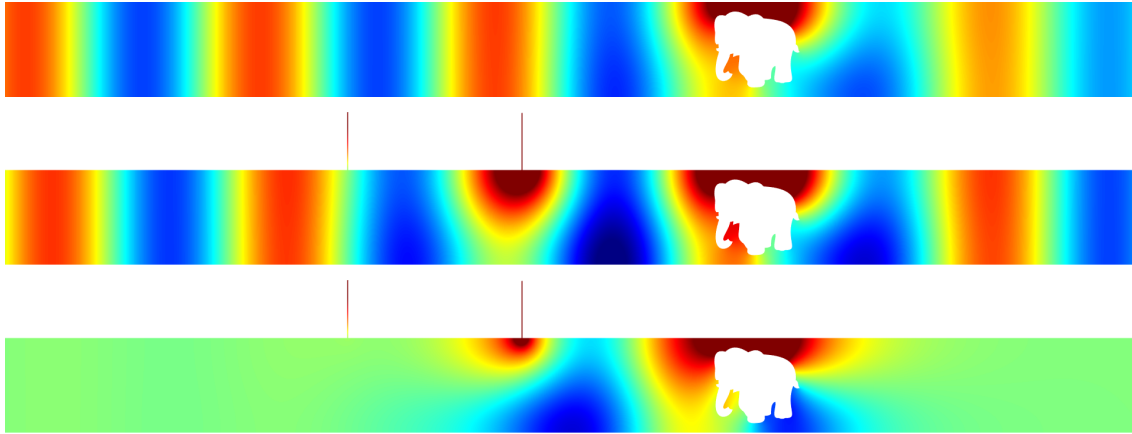


Figure 2.50: Real parts of u_+ (top), u^ε (middle) and $u^\varepsilon - w^+$ (bottom). The resonators are tuned to get $T^\varepsilon \approx 1$. Here $\varepsilon = 0.01$.

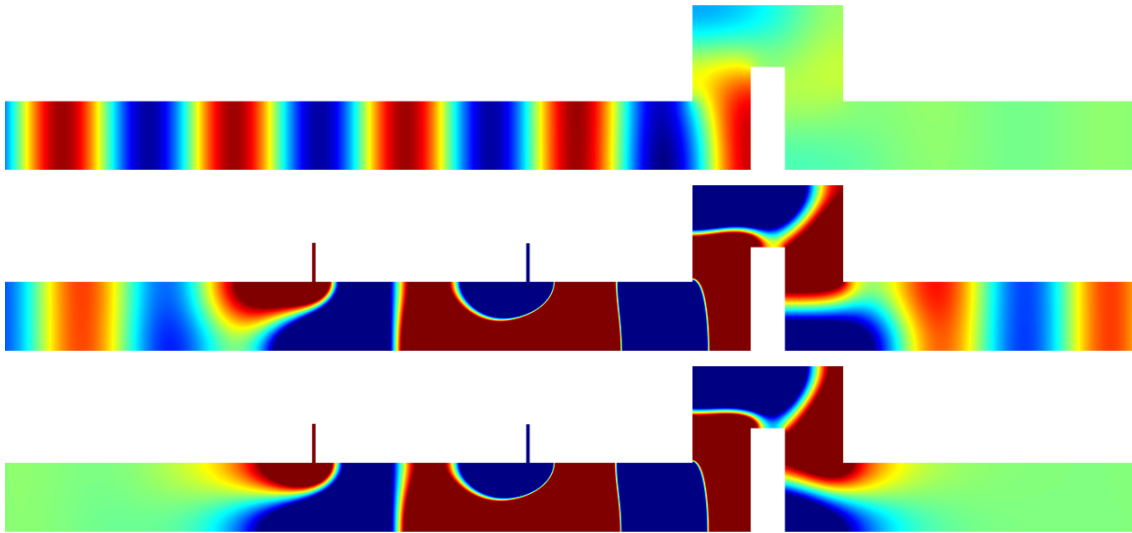


Figure 2.51: Real parts of u_+ (top), u^ε (middle) and $u^\varepsilon - w^+$ (bottom). The resonators are tuned to get $T^\varepsilon \approx 1$. Here $\varepsilon = 0.05$.

2.8.2 Energy distributor

These resonant ligaments offer a very effective way of acting on scattering of waves. We have used them to design other devices that may be of interest for applications and which are not directly related to invisibility.

First, in the article [169], with S.A. Nazarov, we adapted the approach presented above to design acoustic energy distributors. More precisely, in [169] we consider a waveguide with three unbounded channels Π_0 , Π_1 , Π_2 . In this setting, Π_0 plays the role of input/output channel while Π_1 , Π_2 are only output channels (see Figure 2.52). Our goal was to find a geometry where the energy of an incoming wave is almost completely transmitted and additionally we can control the ratio of energy transmitted in the two other output channels. A device similar to the one that we wish to create is the acoustic power divider [319, 15, 395, 223]. The difference is that in our case, we want to be able to control the ratio of energy transmitted in the two output channels. Such devices are interesting for applications not only in acoustics but also for example in optics [423, 233] or in radio electronics [180] (see the literature concerning radio diplexers).

To construct these particular waveguides, it has been proposed to work with so-called zero-index materials or similar metamaterials, see for example [223] and the references therein. However, from our understanding, these materials are still hard to handle in practice.

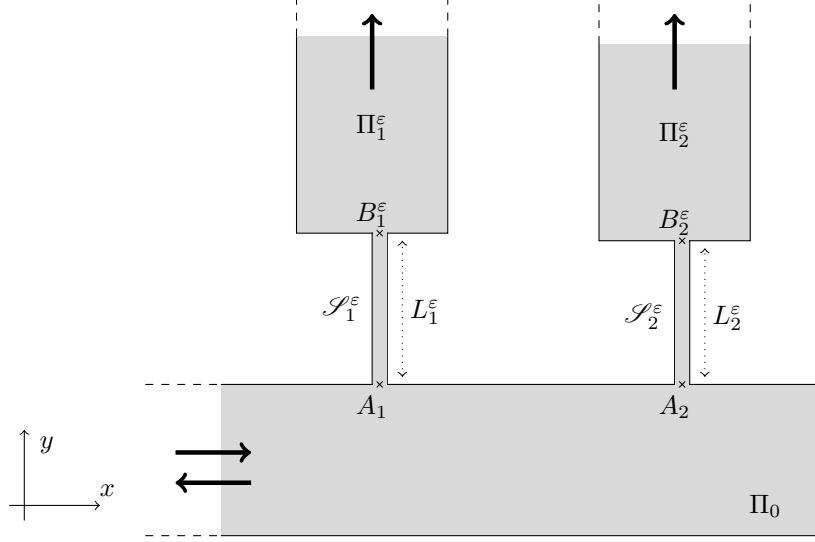


Figure 2.52: Geometry of the waveguide Ω^ε .

In [169], we work in homogeneous waveguides Ω^ε with thin ligaments of width $\varepsilon > 0$ as illustrated in Figure 2.52. In Ω^ε , Problem (2.1) admits a solution with the expansion

$$u^\varepsilon(x, y) = \begin{cases} e^{ikx} + R^\varepsilon e^{-ikx} + \dots & \text{in } \Pi_0 \\ T_1^\varepsilon e^{iky} + \dots & \text{in } \Pi_1 \\ T_2^\varepsilon e^{iky} + \dots & \text{in } \Pi_2, \end{cases}$$

where $R^\varepsilon, T_1^\varepsilon, T_2^\varepsilon \in \mathbb{C}$ and the dots stand for terms which are exponentially decaying at infinity. In general, due to the geometric features, almost no energy passes through the ligaments and $|R^\varepsilon| \approx 1$, $T_1^\varepsilon \approx 0$, $T_2^\varepsilon \approx 0$ (see the first line of Figure 2.53). Therefore it may seem a bit paradoxical to consider such domains to achieve complete transmission. However working around the resonance lengths of the ligaments as in §2.8.1, this can be realized. The strategy is similar to the one of §2.8.1. First we compute asymptotic expansions of u^ε , and so of $R^\varepsilon, T_1^\varepsilon, T_2^\varepsilon$, as ε tends to zero. In these expansions, the features of the geometry appear quite explicitly. Then we exploit these representations to find positions and lengths of the ligaments to cancel the first term in the asymptotics of R^ε and to control the ratio $|T_1^\varepsilon/T_2^\varepsilon|$. In Figure 2.53 second line, we represent two geometries which have been tuned to achieve almost zero reflection and to send energy either in the channel 1 or 2.

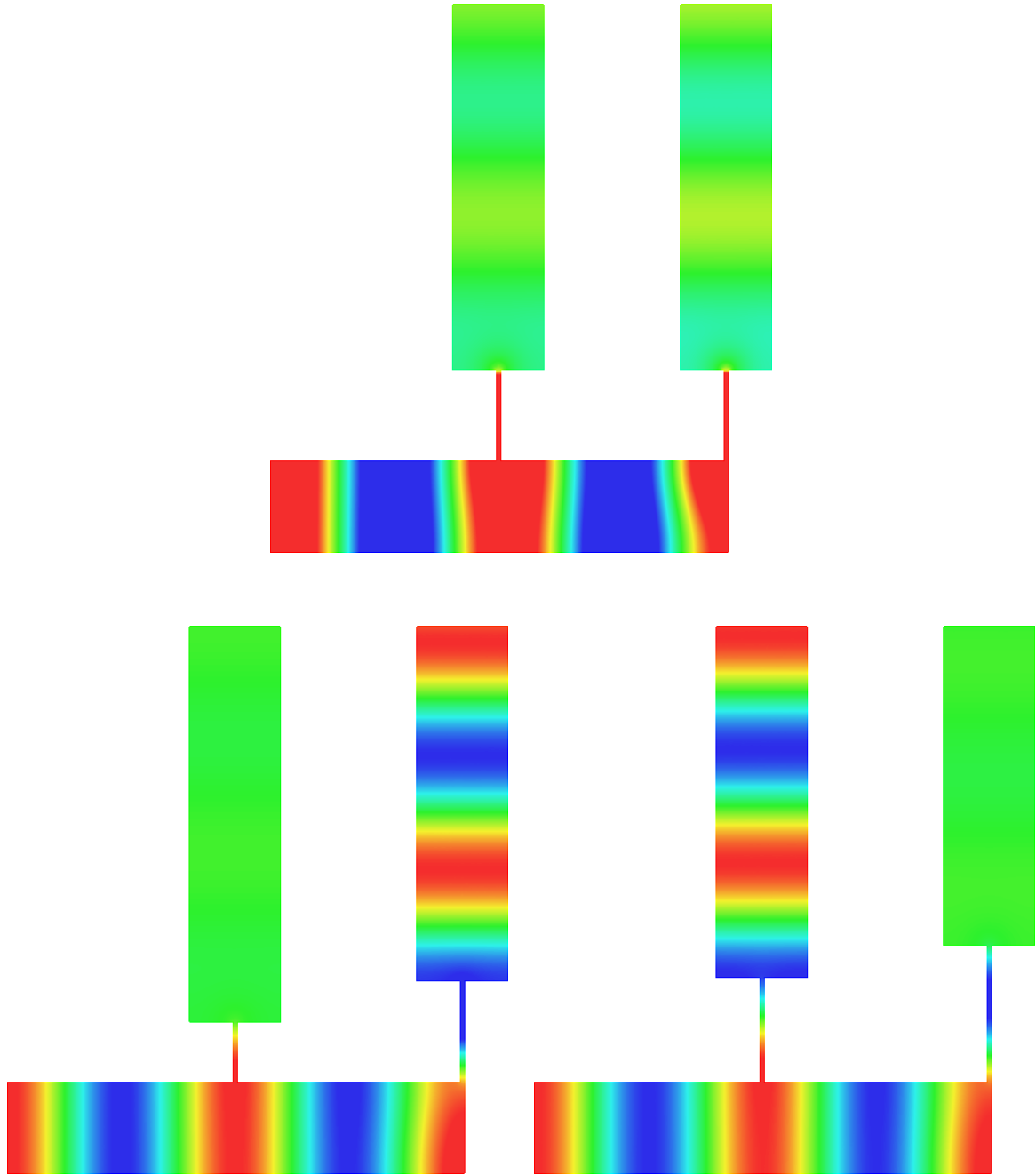


Figure 2.53: First line: without particular tuning of the lengths of the ligaments, the energy of the incoming wave is almost completely backscattered. Second line, left: setting where $R^\varepsilon \approx 0$, $T_1^\varepsilon \approx 0$ and $|T_2^\varepsilon| \approx 1$. Second line, right: setting where $R^\varepsilon \approx 0$, $|T_1^\varepsilon| \approx 1$ and $T_2^\varepsilon \approx 0$.

2.8.3 Mode converter

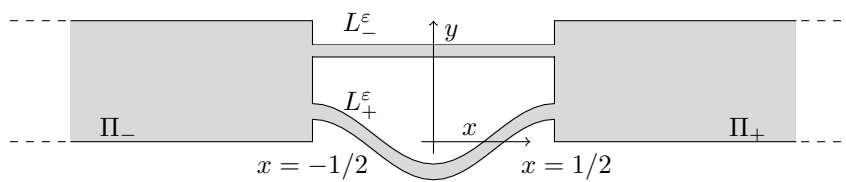


Figure 2.54: Geometry of the waveguide Ω^ε with two thin ligaments.

In [162], the third opus of the trilogy [161, 169, 162], with J. Heleine and S.A. Nazarov, we exploit the above ideas to design mode converters which are classical devices in wave physics which appear for example in optics [276, 144, 301, 343, 406] or in the field of microwaves [418, 251]. We work with a wavenumber $k > 0$ such that two modes, say modes 1 and 2, can propagate in the waveguide. When the mode 1 or the mode 2 is incoming in the structure, in general it produces a reflection on the modes 1, 2 and a transmission on the modes 1, 2. The goal of [162] is to construct waveguides which exhibit two main features at the given wavenumber. First, we want the reflection coefficients to be null so that the energy of any incident field is completely transmitted. Secondly, we wish the energy of the incident mode 1 to be transmitted only on the mode 2 and vice versa to guarantee the mode conversion.

In order to obtain such devices, techniques of optimization have been considered for example in [318, 347, 326, 328, 329, 327]. However, again, the functionals involved in the analysis are non convex and unsatisfactory local minima exist. Moreover, these methods do not allow the user to control the main features of the shape compared to the approach we propose.

To construct mode converters, we work in waveguides as in Figure 2.54 made of two truncated channels connected by thin ligaments of width $\varepsilon > 0$. As in §2.8.2, in general almost no energy passes through the ligaments (see Figure 2.55) and it might not seem relevant at first sight to exploit them to get almost complete transmission. However by tuning correctly the positions and lengths of the ligaments, we explain in [162] how to obtain the desired effect. In Figures 2.56, 2.57, 2.58, we represent examples of geometries which have been tuned to achieve mode conversion. The two pictures represent the scattering of each of the two right-going propagating modes.

Let us mention that for this application, we need to control four reflection coefficients and four transmission coefficients. However since the scattering matrix is unitary and symmetric, we simply have to act on five complex coefficients. But still this problem is more challenging than the ones of §2.8.1 and §2.8.2. In the approach, the symmetry with respect to the vertical axis plays an important role. It allows us to decouple the action of the two ligaments. More precisely, as in §2.6.2, we can decompose the problem into two half-waveguide problems with Dirichlet or Neumann boundary conditions at the end of the truncated ligaments. Then one ligament is tuned to be resonant for the first problem while the other is resonant for the second problem.

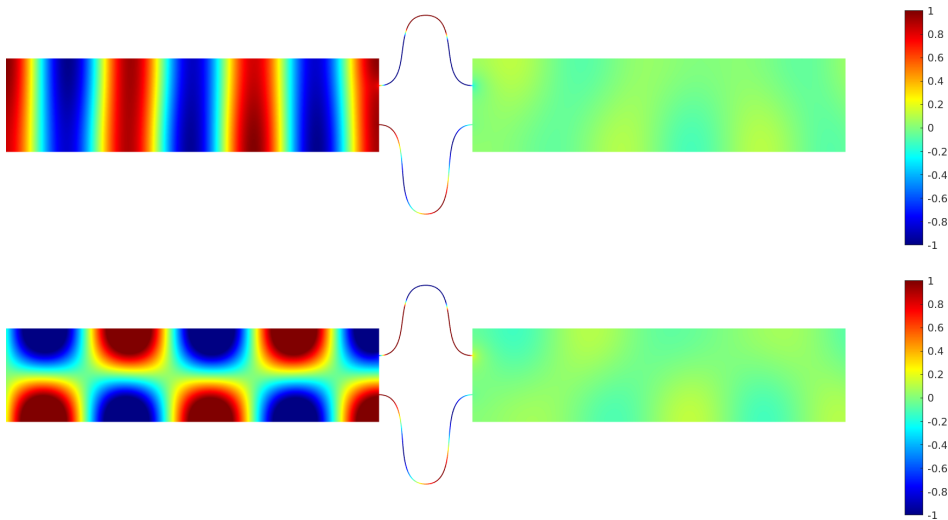


Figure 2.55: Real parts of u_1^ε (top) and u_2^ε (bottom) for $\varepsilon = 0.01$. Here the lengths of the ligaments are not particularly selected to get mode conversion and we observe mainly a total reflection.

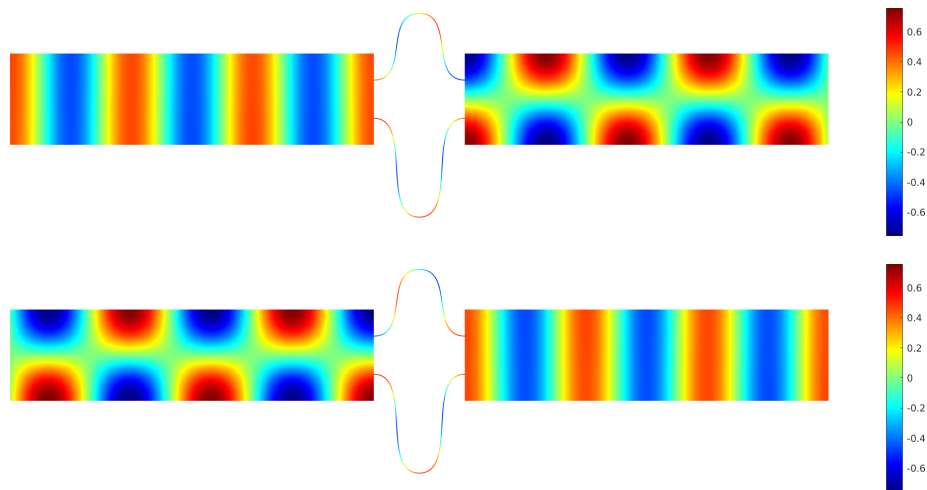


Figure 2.56: Real parts of u_1^ε (top) and u_2^ε (bottom) for $\varepsilon = 0.01$. The lengths of the ligaments have been tuned to get mode conversion.

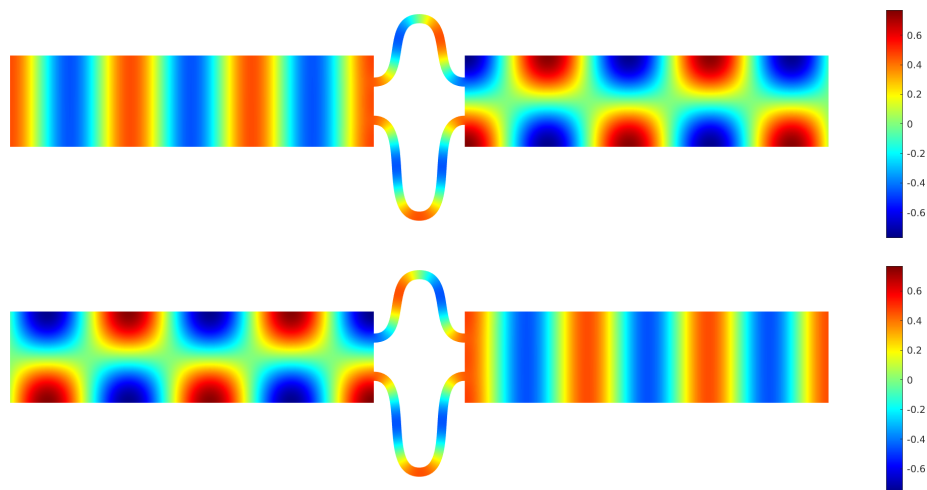


Figure 2.57: Real parts of u_1^ε (top) and u_2^ε (bottom) for $\varepsilon = 0.1$. The lengths of the ligaments have been tuned to get mode conversion.

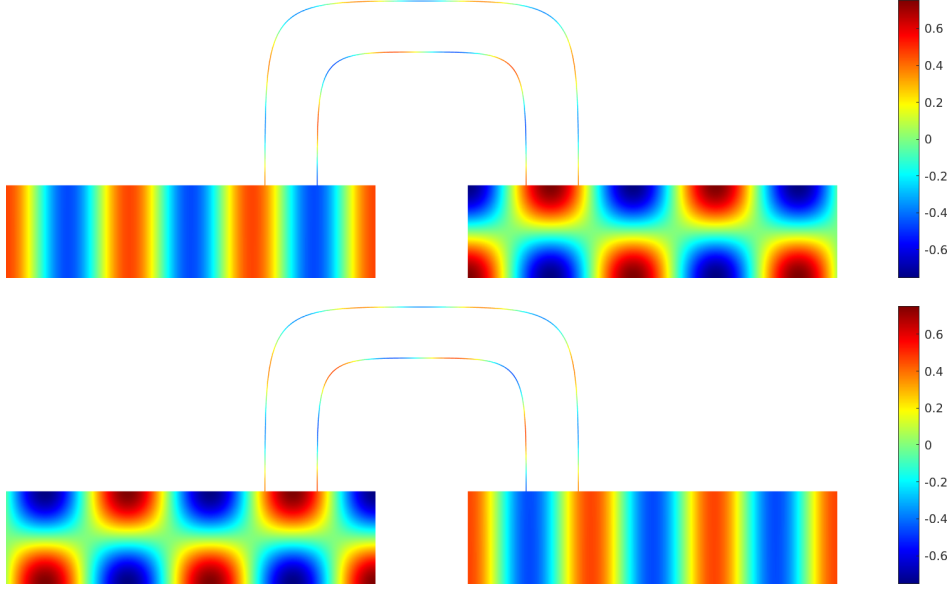


Figure 2.58: Real parts of u_1^ε (top) and u_2^ε (bottom). The lengths of the ligaments have been tuned to get mode conversion. Note that in these simulations, unlike the ones of Figures 2.56–2.57, the ligaments are attached to the upper wall.

2.8.4 Perfect dissipator

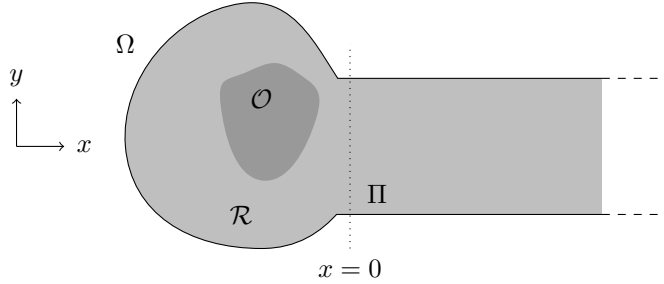


Figure 2.59: Waveguide Ω with a dissipative inclusion.

In [163], again during the post-doctoral position of J. Heleine, with S.A. Nazarov and J. Taskinen, we were interested in the propagation of acoustic waves in a (half) waveguide which contains a bounded penetrable inclusion made of a dissipative material as represented in Figure 2.59. More precisely, here Ω is a domain which coincides with the half-strip $(0; +\infty) \times (0; 1)$ outside of a bounded region. To model the dissipative inclusion, introduce $\mathcal{O}$ a non empty open set with smooth boundary such that $\overline{\mathcal{O}} \subset \Omega$ and $b \in \mathcal{C}^\infty(\overline{\mathcal{O}})$ a positive function that we extend by zero in $\Omega \setminus \overline{\mathcal{O}}$. Then we consider the problem

$$\begin{cases} \Delta u^\eta + k^2(1 + i\eta b)u^\eta &= 0 & \text{in } \Omega \\ \partial_\nu u^\eta &= 0 & \text{on } \partial\Omega, \end{cases} \quad (2.64)$$

where $\eta \geq 0$ is a scalar parameter. We fix $k \in ((J-1)\pi; J\pi)$ so that J modes $w_0^\pm, \dots, w_{J-1}^\pm$ (see (2.2)) can propagate. Generally speaking, for an incident wave propagating in such a structure, a part of the incoming energy is scattered at infinity while the other part is dissipated in the inclusion. More precisely, we establish that the scattering matrix $\mathbb{S}^\eta \in \mathbb{C}^{J \times J}$, which is symmetric for all $\eta \geq 0$, satisfies

$$\mathbb{S}^\eta \overline{\mathbb{S}^\eta}^\top + 2k^2 \eta \mathbb{B}^\eta = \mathbb{I}. \quad (2.65)$$

Here $\mathbb{I}$ is the identity matrix of $\mathbb{C}^{J \times J}$ while $\mathbb{B}^\eta = (\mathbb{B}_{jk}^\eta)_{0 \leq j, k \leq J-1}$ is the positive definite Hermitian matrix such that

$$\mathbb{B}_{jk}^\eta = \int_{\mathcal{O}} b u_j^\eta \overline{u_k^\eta} dx, \quad (2.66)$$

u_j^η being the solution of (2.64) corresponding to the scattering of the mode w_j^- coming from the right. From (2.65), we deduce that $\mathbb{S}^\eta$ is unitary when $\eta = 0$, *i.e.* without dissipation, but not when $\eta > 0$.

The original motivation for this study comes from the design of perfect absorbers also known as Coherent Perfect Absorbers (CPA) or time-reversed lasers [178, 344, 470]. These are particular configurations where all the energy of the incident field is dissipated in the inclusion ($\mathbb{S}^\eta$ is the null matrix). Numerical and experimental strategies to construct such perfect absorbers have been proposed in the physical community, for example in [178, 354, 436, 288, 475] (see also references therein).

In [163], our goal was to consider the mathematical properties of the corresponding problem. In this work, first we prove that the scattering solutions are well-defined for Problem (2.64) and establish identity (2.65). From (2.65), to get a perfect absorber, a natural idea might be to work with a large value of η , that is with what one might think to be a very dissipative inclusion. However we prove that this is a wrong intuition because when η tends to $+\infty$, the dissipative inclusion behaves like a sound soft obstacle. As a consequence, $\mathbb{S}^\eta$ converges to a unitary matrix and not to the zero matrix.

More precisely in [163], we compute an asymptotic expansion of the scattering solutions u_j^η when η tends to zero (this is a rather direct result) and when η tends to $+\infty$ (this is the first main result of the paper). We obtain the following two statements:

Proposition 2.8.4 *As η tends to zero, we have the expansion*

$$\mathbb{S}^\eta = \mathbb{S}^0 - k^2 \eta \mathbb{B}^0 \mathbb{S}^0 + \tilde{\mathbb{S}}^\eta.$$

Here $\mathbb{B}^0$ is the positive definite Hermitian matrix introduced in (2.66) with $\eta = 0$ and the remainder $\tilde{\mathbb{S}}^\eta$ is such that $\|\tilde{\mathbb{S}}^\eta; \mathbb{C}^{J \times J}\| \leq c\eta^2$.

Denote by $\mathbb{S}^\infty$ the scattering matrix for the problem where we impose a Dirichlet condition on $\partial\mathcal{O}$ and u_j^∞ the corresponding scattering solutions.

Proposition 2.8.5 *As η tends to $+\infty$, we have the expansion*

$$\mathbb{S}^\eta = \mathbb{S}^\infty - \eta^{-1/2} \frac{e^{i\pi/4}}{k} \mathbb{E} \mathbb{S}^\infty + \tilde{\mathbb{S}}^\eta, \quad (2.67)$$

where $\mathbb{E}$ is the positive definite Hermitian matrix such that

$$\mathbb{E}_{jk} = \int_{\partial\mathcal{O}} b^{-1/2} \partial_n u_j^\infty \overline{\partial_n u_k^\infty} ds$$

and the remainder $\tilde{\mathbb{S}}^\eta$ is such that $\|\tilde{\mathbb{S}}^\eta; \mathbb{C}^{J \times J}\| \leq c\eta^{-3/4}$.

Remark 2.8.6 *In the energy identity $\mathbb{S}^\eta \overline{\mathbb{S}^\eta}^\top + 2k^2 \eta \mathbb{B}^\eta = \mathbb{I}$ (see (2.65)), one may think at first glance that the term $k^2 \eta \mathbb{B}^\eta$ becomes large when η goes to $+\infty$. However this is not true because we can show that we have $\|\mathbb{B}^\eta; \mathbb{C}^{J \times J}\| \leq c\eta^{-3/2}$. Note that this estimate is in agreement with the expansion (2.67).*

Let us mention that when η goes to $+\infty$, the field tends to zero inside the inclusion and becomes localized at its boundary. In other words, a boundary layer phenomenon appears, this is the so-called skin-effect. In order to capture it, it is necessary to work with an adapted structure for the asymptotic expansion. Such problems have already been studied in the literature (see [469, 367, 55]) and here we adapt existing techniques to our case. Very close to our topic, let us mention the article [255] (see also the related works [256, 257, 39]). There the authors consider a problem similar to ours but set in a bounded domain with an impedance boundary condition which can be seen as a low order approximation of the Sommerfeld radiation condition in free space. They propose and justify asymptotic models for highly dissipative inclusions. Their justifications mainly rely on proofs by contradiction. We proceeded a bit differently by giving a direct demonstration of the essential stability estimate necessary to establish Proposition 2.8.5.

In monomode regime, *i.e.* when $k \in (0; \pi)$ so that $J = 1$ and only the modes

$$w^\pm(x, y) = e^{\pm i k x}$$

can propagate, the scattering matrix reduces to a scalar reflection coefficient that we still denote by $\mathbb{S}^\eta \in \mathbb{C}$. In this case, identity (2.65) writes

$$|\mathbb{S}^\eta|^2 + 2k^2\eta \int_{\mathcal{O}} b |u^\eta|^2 dx = 1 \quad (2.68)$$

where u^η is the solution of (2.64) admitting the expansion

$$u^\eta = w^- + \mathbb{S}^\eta w^+ + \tilde{u}^\eta, \quad (2.69)$$

$\tilde{u}^\eta$ being exponentially decaying at infinity. Relation (2.68) imposes that $|\mathbb{S}^\eta| \in [0; 1)$ for $\eta > 0$. Moreover Propositions 2.8.4 and 2.8.5 guarantee that

$$\lim_{\eta \rightarrow 0^+} |\mathbb{S}^\eta| = \lim_{\eta \rightarrow +\infty} |\mathbb{S}^\eta| = 1.$$

Thus, the continuous function $\eta \mapsto |\mathbb{S}^\eta|$ has a global minimum at some $\eta_\star \in (0; +\infty)$. If $\mathbb{S}^{\eta_\star} = 0$, the device acts as a perfect absorber of the piston mode. However in general this minimum is not zero.

The second main outcome of [163] is the justification of a method based on the perturbation of the geometry of the waveguide to create 2D perfect absorbers in monomode regime, *i.e.* to obtain $\mathbb{S}^\eta \approx 0$ in (2.68). More precisely, we show that for any given Ω , $\mathcal{O}$ and $\eta > 0$, we can add a well designed (resonant) ligament so that in the new geometry $\mathbb{S}^\eta \approx 0$. The key idea of the approach is to glue Ω to a well-designed waveguide for which the reflection coefficient coincides with $\overline{\mathbb{S}^\eta}$. In the procedure, we exploit again the results of asymptotic analysis described in §2.8.1.

We end this part by illustrating some of the results above. In Figure 2.60, we display the field u^η for different values of η in a geometric setting where only one mode can propagate ($J = 1$). We also present the solution u^∞ of the problem with a Dirichlet boundary condition on the obstacle. In accordance with the results leading to Proposition 2.8.5, we see that u^η converges to u^∞ as η tends to $+\infty$. Interestingly, we observe that this result seems to hold true also in the situation where $\mathcal{O}$ is a rectangle (see the right column) though this case does not enter our analysis (in our work $\mathcal{O}$ has to be smooth).

In Figure 2.61, we display the eigenvalues of $\mathbb{S}^\eta$ in the complex plane for different values of η . Here the spectral parameter λ has been chosen such that five modes can propagate ($\mathbb{S}^\eta$ is of size 5×5). For $\eta = 0$, in this simple geometry we have $\mathbb{S}^0 = \mathbb{I}$ so that the five eigenvalues are equal to

one. For $\eta > 0$, we observe that the modulus of the eigenvalues of $\mathbb{S}^\eta$ is smaller than one. This was expected because for $\eta > 0$, $\mathbb{S}^\eta$ is no longer unitary, and more precisely this is in agreement with identity (2.65). Finally, we see that when η tends to $+\infty$, the eigenvalues of $\mathbb{S}^\eta$ converge to values on the unit circle. Again, this corroborates the result of Proposition 2.8.5 which guarantees that $\mathbb{S}^\eta$ tends to a unitary matrix.

Finally in Figure 2.62, we present two examples of situations (Ω and $\mathcal{O}$ are the same but η is different) where we have added a resonant ligament whose position and length have been tuned to get almost zero reflection, so that all the energy is dissipated in the inclusion.

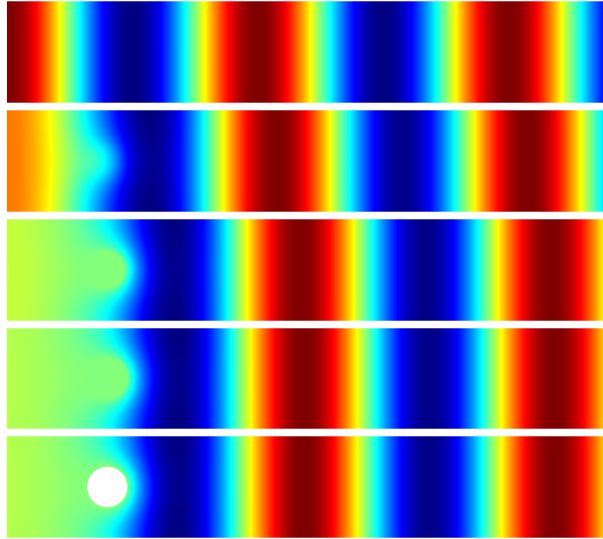


Figure 2.60: Line 1-4: real part of u^η (see (2.69)) for, respectively, $\eta = 0, 10, 1000, 10^{10}$. Last line: real part of u^∞ (solution of the problem with a Dirichlet boundary condition on the obstacle). The penetrable inclusion $\mathcal{O}$ is the disk which appears on the last line.

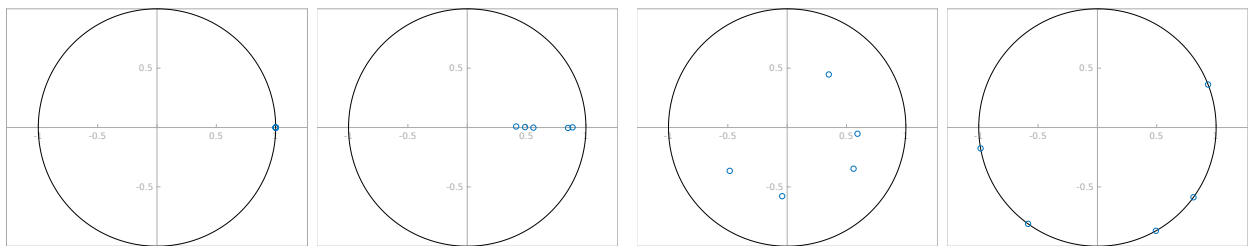


Figure 2.61: Eigenvalues of $\mathbb{S}^\eta$ in the complex plane for $\eta = 10^{-10}, 10^{-1}, 5, 10^{10}$ (from left to right). Here the spectral parameter $k = 4.8\pi \in (4\pi; 5\pi)$ has been chosen such that five modes can propagate ($\mathbb{S}^\eta$ is of size 5×5). The black circle represents the unit circle.

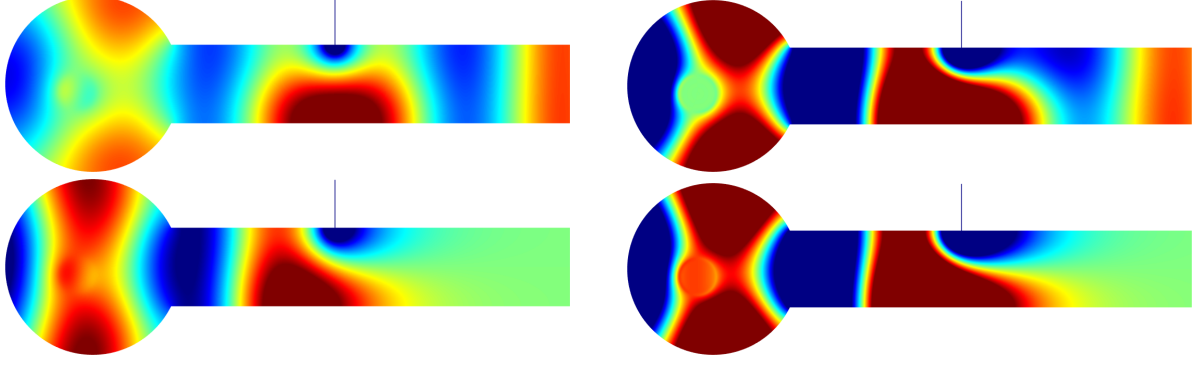


Figure 2.62: $\Re u^\eta$ (first line) and $\Re(u^\eta - w^-)$ (second line) in two situations where $R^\eta \approx 0$ (the position and the length of the ligament have been precisely tuned to get this property). We take $\eta = 10$ for the left pictures and $\eta = 1000$ for the right pictures. The dissipative inclusion is the disk which appears in green on the top right simulation.

2.8.5 Concluding remarks

Let us make a few remarks to conclude this section.

- i)* What was done in 2D here could be adapted in 3D. The asymptotic expansion would be different, in particular because the Green functions for the Laplace operator in 2D and in 3D have different singularities, but the methodology would be the same (see [380, 379] for related works).
- ii)* On the contrary, the approaches proposed above are very specific to Neumann Boundary Conditions (BCs) and cannot be adapted for Dirichlet BCs. Indeed, with Dirichlet BCs almost nothing enters into the thin ligaments. Therefore, with Dirichlet BCs, it is necessary to use other resonators with a non empty interior at the limit $\varepsilon \rightarrow 0$.
- iii)* The question of working at higher wavenumber, so that several modes can propagate, remains largely open. In this case, one has to control more scattering coefficients. A natural idea is to increase the number of thin resonators to get more means of action. In this situation, the asymptotic analysis can be adapted quite directly from what has been done above. But then we are led to work on algebraic systems which seem at first glance hard to handle. In particular, coupling effects do not facilitate the analysis.
- iv)* The strategies proposed above allow one to control the first term in the asymptotics of the scattering coefficients as ε tends to zero. For example, for cloaking, we achieve $T^\varepsilon = 1 + o(1)$ as ε tends to zero. A natural question for this application, at least from a theoretical point of view, is “can we get $T^\varepsilon = 1$ exactly?”. This may be doable by adapting some arguments presented in Section 2.4 (see [73]). Let us try to summarize the idea. By adding three non resonant thin ligaments, we could be able to cancel the term at order ε to obtain $T^\varepsilon = 1 + O(\varepsilon^2)$. To get $T^\varepsilon = 1$, we need to kill all the terms in the asymptotic and this is not simple because at order $p \geq 2$, the dependence on the geometric parameters of the thin ligaments becomes nonlinear. But we showed in Section 2.4 that one can perturb the length of the thin ligaments, with a perturbation determined by solving a fixed point problem, which permits to tackle the problem of the non linearity, to achieve $T^\varepsilon = 1$. The context of the present paragraph however is a bit different from the one of Section 2.4 and the complete justification of the scheme must be studied carefully. Results of Section 2.5 to prove the invertibility of some differentials involved in the process may be useful.
- v)* The question of mimicking (see [135] in the context of thermal cloaking) the scattering properties at infinity, that is, to perturb the geometry so that the scattering coefficients become approximately equal to those of a given obstacle, may be studied with the tools presented above.
- vi)* Assume that a thin resonator has been tuned to be resonant at a certain wavenumber $k_1 \in (0; \pi)$. Now if we work at $k_2 \neq k_1$, in general the thin resonator will only perturb the field u^ε , and so the

scattering coefficients, by a term of order ε . This is interesting because it allows us to decouple the action of the thin resonators at order 1. As a consequence, for example by working with $2p$ thin resonators, we can approximately cloak any obstacle at the discrete collection of wavenumbers $\{k_1, k_2, \dots, k_p\} \subset (0; \pi)^p$. Of course imposing invisibility for a continuum of wavenumbers is another question, which may be impossible to solve, see the related works [451, 402, 362, 139, 403].

2.9 A spectral problem characterizing zero reflection

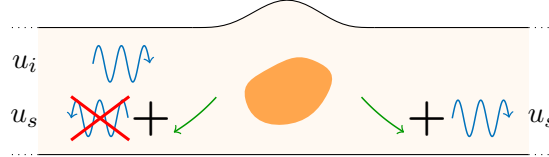


Figure 2.63: Schematic picture of a reflectionless mode. The propagating wave (blue) is not reflected. The backscattered field is purely evanescent (green).

In the works concerning invisibility that we have presented up to now, the wavenumber $k > 0$ was given and we were trying to find geometries where one has zero reflection or perfect transmission. In the article [74], with A.-S. Bonnet-Ben Dhia and V. Pagneux, we adopt a different point of view. We assume that the geometry is fixed and we look for k such that there is an incident field for which energy is completely transmitted through the waveguide. More precisely, inspired by the results obtained in [272] on a 1D $\mathcal{PT}$ -symmetric problem, we show that such reflectionless modes can be characterized as eigenfunctions of an original non-selfadjoint spectral problem. Our approach is based on the following basic observation: if for an incident wave, the backscattered field is evanescent, then the total field is ingoing in the input lead and outgoing in the output lead (see Figure 2.63). To select ingoing waves on one side of the obstacle and outgoing waves on the other side, we use complex scalings (or Perfectly Matched Layers) with imaginary parts of different signs. We prove that the real eigenvalues of the obtained spectrum correspond either to trapped modes [461, 224, 225, 207, 340, 375] (also called Bound States in the Continuum, BSCs or BICs, in quantum mechanics [438, 359, 239, 278]) or to reflectionless modes. When the geometry has certain symmetries, the new spectral problem enters the class of $\mathcal{PT}$ -symmetric problems ($\mathcal{PT}$ for Parity-Time, see Proposition 2.9.5 below).

In waveguides, complex resonances, for which corresponding “generalized” eigenfunctions are known as leaky modes or quasi normal modes, play an important role and have been intensively studied (see *e.g.* [4, 33, 448, 358, 18, 450, 481, 482, 360]). As we have seen in §2.7.2, for example they are responsible for rapid variations of the scattering coefficients when they are located close to the real axis. We emphasize that real eigenvalues and complex resonances are intrinsic and universal objects that can be computed using different approaches. Among them, let us mention methods based on complex scaling or Perfectly Matched Layers (PMLs) [50] or techniques relying on Hardy space infinite elements [273, 274]. However complex resonances fail to provide useful information on recurrent questions such as “how large is the backscattered energy?” or “is there good transmission in the system?”. The new complex spectrum we define allows us to quantify these aspects.

To make its description and the presentation of our method to compute it as simple as possible, we stick to a 2D problem in a waveguide with Neumann boundary conditions. We assume that this waveguide contains a penetrable obstacle (Figure 2.64). Everything presented here can be generalized to other types of obstacles, to other kinds of boundary conditions (Dirichlet, ...), to higher dimension (3D) and to more complex geometries (one input lead/several output leads).

2.9.1 Setting

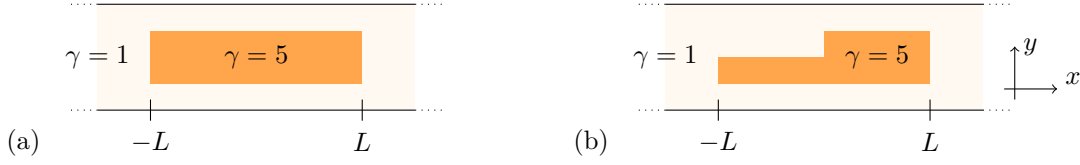


Figure 2.64: Symmetric (a) and non-symmetric (b) obstacles considered in the numerics below.

The waveguide coincides with the region $\Omega := \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1\}$ and we consider the problem

$$\begin{cases} \Delta u + k^2 \gamma u = 0 & \text{in } \Omega \\ \partial_y u = 0 & \text{on } y = 0 \text{ and } y = 1. \end{cases} \quad (2.70)$$

We assume that γ is a positive and bounded function such that $\gamma = 1$ for $|x| \geq L$ where $L > 0$ is given. In other words, the obstacle is located in the region $\Omega_L := \{(x, y) \in \Omega \mid |x| < L\}$. Pick $k \in (N\pi; (N+1)\pi)$, with $N \in \mathbb{N} := \{0, 1, \dots\}$. We use the same notation as in (2.2). Simply we renormalize the modes and set

$$w_n^\pm(x, y) = \frac{e^{\pm i\beta_n x} \varphi_n(y)}{(2|\beta_n|)^{1/2}}, \quad \beta_n := \sqrt{k^2 - n^2\pi^2}, \quad \varphi_n(y) = \begin{cases} 1 & \text{if } n = 0 \\ \sqrt{2} \cos(n\pi y) & \text{for } n > 0. \end{cases} \quad (2.71)$$

For $n = 0, \dots, N$, the wave $w_n^\pm$ propagates along the (Ox) axis from $\mp\infty$ to $\pm\infty$. On the other hand, for $n > N$, $w_n^\pm$ is exponentially growing at $\mp\infty$ and exponentially decaying at $\pm\infty$. For $n = 0, \dots, N$, we consider the scattering of the wave w_n^+ by the obstacle located in Ω . Problem (2.70) admits a solution $u_n = w_n^+ + u_n^s$ with the outgoing scattered field u_n^s written as

$$u_n^s = \sum_{p=0}^{+\infty} s_{np}^- w_p^- \quad \text{for } x \leq -L, \quad u_n^s = \sum_{p=0}^{+\infty} s_{np}^+ w_p^+ \quad \text{for } x \geq L, \quad (2.72)$$

with $(s_{np}^\pm) \in \mathbb{C}^{\mathbb{N}}$. The solution u_n is uniquely defined if and only if Trapped Modes (TMs) do not exist at the wavenumber k . We remind the reader that trapped modes are non zero functions $u \in L^2(\Omega)$ satisfying (2.70). We denote by $\mathcal{K}_t$ the set of k^2 such that TMs exist at the wavenumber k . On the other hand, the scattering coefficients $s_{np}^\pm$ in (2.72) are always uniquely defined, including for $k^2 \in \mathcal{K}_t$. In the following, we will be particularly interested in the features of the reflection matrix (whose size, determined by the number of propagating modes, depends on k)

$$R(k) := (s_{np}^-)_{0 \leq n, p \leq N} \in \mathbb{C}^{N+1 \times N+1}. \quad (2.73)$$

Definition 2.9.1 *The wavenumber $k \in (0; +\infty) \setminus \{\mathbb{N}\pi\}$ is said reflectionless if $\ker R(k) \neq \{0\}$.*

Let us explain this definition. In general, by linearity, for a right-going incident field

$$u_i = \sum_{n=0}^N a_n w_n^+, \quad (a_n)_{n=0}^N \in \mathbb{C}^{N+1}, \quad (2.74)$$

Problem (2.70) admits a solution u such that $u = u_i + u_s$ with

$$u_s = \sum_{p=0}^{+\infty} b_p^\pm w_p^\pm \quad \text{for } \pm x \geq L \quad \text{and} \quad b_p^\pm = \sum_{n=0}^N a_n s_{np}^\pm \in \mathbb{C}. \quad (2.75)$$

The above definition says that, if k is reflectionless, then there are $(a_n)_{n=0}^N \in \mathbb{C}^{N+1} \setminus \{0\}$ such that the b_p^- in (2.75) satisfy $b_p^- = 0$, $p = 0, \dots, N$. In other words, the scattered field is exponentially

decaying for $x \leq -L$. Finally notice that the corresponding total field $u = u_i + u_s$ decomposes as

$$\begin{aligned} u &= \sum_{n=0}^N a_n w_n^+ + \tilde{u} \quad \text{for } x \leq -L \\ u &= \sum_{n=0}^N t_n w_n^+ + \tilde{u} \quad \text{for } x \geq L \end{aligned} \quad (2.76)$$

where $t_n = a_n + b_n^+$ and where $\tilde{u}$ decays exponentially for $\pm x \geq L$. In other words, the total field is ingoing for $x \leq -L$ and outgoing for $x \geq L$.

In the following, we call Reflectionless Modes (RMs) the functions u satisfying (2.70) and admitting expansion (2.76). We denote by $\mathcal{K}_r$ the set of k^2 such that the wavenumber k is reflectionless. Our objective is to explain how to determine directly the set $\mathcal{K}_r$ and the corresponding RMs by solving a linear eigenvalue problem, instead of computing the reflection matrix for all values of k .

2.9.2 Classical complex scaling

As a first step, we remind briefly how to use a complex scaling to compute trapped modes. Define the unbounded operator A of $L^2(\Omega)$ such that

$$Au = -\frac{1}{\gamma} \Delta u$$

with domain

$$D(A) = \{u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega) \text{ and } \partial_y u = 0 \text{ on } \partial\Omega\}.$$

It is known that the Neumann Laplacian A is a selfadjoint operator when $L^2(\Omega)$ is endowed with the inner product $(\gamma \cdot, \cdot)_{L^2(\Omega)}$. The spectrum of A , denoted $\sigma(A)$, coincides with $[0; +\infty)$. More precisely, we have $\sigma_{\text{ess}}(A) = [0; +\infty)$ where $\sigma_{\text{ess}}(A)$ denotes the essential spectrum of A . By definition, $\sigma_{\text{ess}}(A)$ corresponds to the set of $\lambda \in \mathbb{C}$ for which there exists a so-called singular sequence $(u^{(m)})$, that is an orthonormal sequence $(u^{(m)}) \in L^2(\Omega)^{\mathbb{N}}$ such that $((A - \lambda)u^{(m)})$ converges to 0 in $L^2(\Omega)$. Besides, $\sigma(A)$ may contain eigenvalues (at most a sequence accumulating at $+\infty$) corresponding to TMs. In order to reveal these eigenvalues which are embedded in $\sigma_{\text{ess}}(A)$, one can use a complex change of variables. For $0 < \theta < \pi/2$, set $\eta = e^{i\theta}$ and define the function $\mathcal{I}_\theta : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\mathcal{I}_\theta(x) = \begin{cases} -L + (x + L)\eta & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L)\eta & \text{for } x \geq L. \end{cases} \quad (2.77)$$

For the sake of simplicity, we will use abusively the same notation $\mathcal{I}_\theta$ for the following map: $\{\Omega \rightarrow \mathbb{C} \times (0; 1), (x, y) \mapsto (\mathcal{I}_\theta(x), y)\}$. Note that with this definition, the left inverse $\mathcal{I}_\theta^{-1}$ of $\mathcal{I}_\theta$, acting from $\mathcal{I}_\theta(\Omega)$ to Ω , is equal to $\mathcal{I}_{-\theta}$. One can easily check that for all $n \geq 0$, $w_n^+ \circ \mathcal{I}_\theta$ is exponentially decaying for $x \geq L$, while $w_n^- \circ \mathcal{I}_\theta$ is exponentially decaying for $x \leq -L$. Note that here we have extended the definition of the modes $w_n^\pm$ (see (2.71)) to complex x . As a consequence, defining from expansion (2.75) the function $v_\theta = u_s \circ \mathcal{I}_\theta$, one has $v_\theta = u_s$ for $|x| < L$ and $v_\theta \in L^2(\Omega)$ (which is in general not true for u_s). Moreover v_θ satisfies the following equation in Ω :

$$\alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v_\theta}{\partial x} \right) + \frac{\partial^2 v_\theta}{\partial y^2} + k^2 \gamma v_\theta = k^2 (1 - \gamma) u_i \quad (2.78)$$

with $\alpha_\theta(x) = 1$ for $|x| < L$ and $\alpha_\theta(x) = \eta^{-1} = \bar{\eta}$ for $\pm x \geq L$. In particular, for a TM, v_θ solves (2.78) with $u_i = 0$. This leads us to consider the unbounded operator A_θ of $L^2(\Omega)$ such that

$$A_\theta v_\theta = -\frac{1}{\gamma} \left(\alpha_\theta \frac{\partial}{\partial x} \left(\alpha_\theta \frac{\partial v_\theta}{\partial x} \right) + \frac{\partial^2 v_\theta}{\partial y^2} \right) \quad (2.79)$$

with domain

$$D(A_\theta) = \{v \in H^1(\Omega) \mid A_\theta v \in L^2(\Omega) \text{ and } \partial_y v = 0 \text{ on } \partial\Omega\}.$$

Since α_θ is complex valued, the operator A_θ is not selfadjoint. For non-selfadjoint operators, several definitions exist for the essential spectrum which are in general not equivalent (see more details in [218]). We shall say that λ belongs to $\sigma_{\text{ess}}(A_\theta)$ if $A_\theta - \lambda \text{Id}$ is not a Fredholm operator. We recall below the main spectral properties of A_θ [449]:

Theorem 2.9.2 *i) There holds*

$$\sigma_{\text{ess}}(A_\theta) = \bigcup_{n \in \mathbb{N}, t \geq 0} \{n^2 \pi^2 + t e^{-2i\theta}\}. \quad (2.80)$$

ii) The spectrum of A_θ satisfies $\sigma(A_\theta) \subset \mathcal{R}_\theta^-$ with

$$\mathcal{R}_\theta^- := \{z \in \mathbb{C} \mid -2\theta \leq \arg(z) \leq 0\}.$$

iii) $\sigma(A_\theta) \setminus \sigma_{\text{ess}}(A_\theta)$ is discrete and contains only eigenvalues of finite multiplicity.

iv) Assume that $k^2 \in \sigma(A_\theta) \setminus \sigma_{\text{ess}}(A_\theta)$. Then k^2 is real if and only if $k^2 \in \mathcal{K}_t$. Moreover if v_θ is an eigenfunction associated to k^2 such that $\Im m k^2 < 0$, then $v_\theta \circ \mathcal{I}_{-\theta}$ is a solution of the original problem (2.70) whose amplitude is exponentially growing at $+\infty$ or at $-\infty$.

The interesting point is that now TMs correspond to isolated eigenvalues of A_θ , and as such, they can be computed numerically as illustrated below. Note that the elements k^2 of $\sigma(A_\theta) \setminus \sigma_{\text{ess}}(A_\theta)$ such that $\Im m k^2 < 0$, if they exist, correspond to complex resonances (quasi normal modes). Let us point out that the complex scaling is just a technique to reveal them. Indeed complex resonances are intrinsic objects defined as the poles of the meromorphic extension from $\{z \in \mathbb{C} \mid \Im m z > 0\}$ to $\{z \in \mathbb{C} \mid \Im m z \leq 0\}$ of the operator valued map $z \mapsto (\Delta + z\gamma)^{-1}$ considered in adapted weighted spaces. For more details, we refer the reader to [18].

2.9.3 Conjugated complex scaling

Now, we show that replacing the classical complex scaling by an unusual conjugated complex scaling, and proceeding as in the previous section, we can define a new complex spectrum which contains the reflectionless values $k^2 \in \mathcal{K}_r$ we are interested in. We define the map $\mathcal{J}_\theta : \Omega \rightarrow \mathbb{C} \times (0; 1)$ using the following complex change of variables

$$\mathcal{J}_\theta(x) = \begin{cases} -L + (x + L)\bar{\eta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x - L)\eta & \text{for } x \geq L, \end{cases} \quad (2.81)$$

with again $\eta = e^{i\theta}$ ($0 < \theta < \pi/2$). Note the important difference in the definitions of $\mathcal{J}_\theta$ and $\mathcal{I}_\theta$ for $x \leq -L$: η has been replaced by the conjugated parameter $\bar{\eta}$ to select the ingoing modes instead of the outgoing ones in accordance with (2.76). Now, if u is a RM associated to $k^2 \in \mathcal{K}_r$ (again we extend the definition of the modes $w_n^\pm$ to complex x), setting $w_\theta = u \circ \mathcal{J}_\theta$, one has $w_\theta = u$ for $|x| < L$ and $w_\theta \in L^2(\Omega)$ (which is not the case for u). The function w_θ satisfies the following equation in Ω :

$$\beta_\theta \frac{\partial}{\partial x} \left(\beta_\theta \frac{\partial w_\theta}{\partial x} \right) + \frac{\partial^2 w_\theta}{\partial y^2} + k^2 \gamma w_\theta = 0 \quad (2.82)$$

with $\beta_\theta(x) = 1$ for $|x| < L$, $\beta_\theta(x) = \eta$ for $x \leq -L$ and $\beta_\theta(x) = \bar{\eta}$ for $x \geq L$. This leads us to define the unbounded operator B_θ of $L^2(\Omega)$ such that

$$B_\theta w_\theta = -\frac{1}{\gamma} \left(\beta_\theta \frac{\partial}{\partial x} \left(\beta_\theta \frac{\partial w_\theta}{\partial x} \right) + \frac{\partial^2 w_\theta}{\partial y^2} \right) \quad (2.83)$$

with domain

$$D(B_\theta) = \{w \in H^1(\Omega) \mid B_\theta w \in L^2(\Omega) \text{ and } \partial_y w = 0 \text{ on } \partial\Omega\}.$$

As A_θ , the operator B_θ is not selfadjoint. Its spectral properties are summarized in the following theorem.

Theorem 2.9.3 *i) There holds*

$$\sigma_{\text{ess}}(B_\theta) = \bigcup_{n \in \mathbb{N}, t \geq 0} \{n^2 \pi^2 + t e^{-2i\theta}, n^2 \pi^2 + t e^{+2i\theta}\}. \quad (2.84)$$

ii) The spectrum of B_θ satisfies $\sigma(B_\theta) \subset \mathcal{R}_\theta$ with

$$\mathcal{R}_\theta := \{z \in \mathbb{C} \mid -2\theta \leq \arg(z) \leq 2\theta\}. \quad (2.85)$$

iii) Assume that $k^2 \in \sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$. Then k^2 is real if and only if $k^2 \in \mathcal{K}_t \cup \mathcal{K}_r$. Moreover if w_θ is an eigenfunction associated to k^2 such that $\pm \Im k^2 < 0$, then $w_\theta \circ \mathcal{J}_{-\theta}$ is a solution of (2.70) whose amplitude is exponentially growing at $\pm\infty$ and exponentially decaying at $\mp\infty$.

The important result is that isolated real eigenvalues of B_θ correspond precisely to TMs and RMs. The following proposition provides a criterion to determine whether an eigenfunction associated to a real eigenvalue of B_θ is a TM or a RM.

Proposition 2.9.4 *Assume that $(k^2, w_\theta) \in \mathbb{R} \times L^2(\Omega)$ is an eigenpair of B_θ such that $k \in (N\pi; (N+1)\pi)$, $N \in \mathbb{N}$. Set*

$$\rho(w_\theta) = \sum_{n=0}^N \left| \int_0^1 w_\theta(-L, y) \varphi_n(y) dy \right|^2 \quad (2.86)$$

where φ_n is defined in (2.71). If $\rho(w_\theta) = 0$ then $w_\theta \circ \mathcal{J}_{-\theta}$ is a TM ($k^2 \in \mathcal{K}_t$). If $\rho(w_\theta) > 0$ then $w_\theta \circ \mathcal{J}_{-\theta}$ is a RM ($k^2 \in \mathcal{K}_r$). In this case, the incident field u_i defined in (2.74) with

$$a_n = \int_0^1 w_\theta(-L, y) \varphi_n(y) dy, \quad n = 0, \dots, N,$$

yields a scattered field which decays exponentially for $x \leq -L$.

The following proposition ensures that B_θ satisfies the celebrated $\mathcal{PT}$ -symmetry property when the obstacle is symmetric with respect to the (Oy) axis. This guarantees in particular the stability of simple real eigenvalues, with respect to perturbations of the obstacle satisfying the same symmetry constraint.

Proposition 2.9.5 *Assume that γ satisfies $\gamma(x, y) = \gamma(-x, y)$ for all $(x, y) \in \Omega$. Define the operators $\mathcal{P}, \mathcal{T}$ such that $\mathcal{P}\varphi(x, y) = \varphi(-x, y)$, $\mathcal{T}\varphi(x, y) = \overline{\varphi(x, y)}$ for $\varphi \in L^2(\Omega)$. Then one has*

$$\mathcal{PT}B_\theta = B_\theta \mathcal{PT},$$

i.e. B_θ is $\mathcal{PT}$ -symmetric. This implies $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$ (if λ is an eigenvalue of B_θ , $\bar{\lambda}$ as well).

The proof is direct by observing that the β_θ defined after (2.82) satisfies $\beta_\theta(-x, y) = \overline{\beta_\theta(x, y)}$.

Finally let us mention a specific difficulty which appears in the spectral analysis of B_θ . While Theorem 2.9.2 guarantees that $\sigma(A_\theta) \setminus \sigma_{\text{ess}}(A_\theta)$ is discrete, we do not write such a statement for the operator B_θ in Theorem 2.9.3. A major difference between both operators is that $\mathbb{C} \setminus \sigma_{\text{ess}}(A_\theta)$ is connected whereas $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ has a countably infinite number of connected components. As a consequence, to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete using the analytic Fredholm theorem, it is necessary to find one λ such that $B_\theta - \lambda$ is invertible in each of the components of $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$. In general, in the presence of an obstacle, such a λ probably exists (proofs for certain classes of γ can

be obtained working as in [72]). But for this problem, we can have surprising perturbation results. Thus, if there is no obstacle ($\gamma \equiv 1$ in Ω), then there holds $\sigma(B_\theta) = \mathcal{R}_\theta$ (see (2.85)): all connected components of $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$, except the one containing the complex half-plane $\Re \lambda < 0$, are filled with eigenvalues. To show this result, observe that for $k^2 \in \sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$, the function $u \circ \mathcal{J}_\theta$, with $u(x, y) = e^{ikx}$, is a non-zero element of $\ker B_\theta$. Notice that this pathological property is also true when Ω contains a family of sound hard cracks (homogeneous Neumann boundary condition) parallel to the (Ox) axis (see the illustration of Figure 2.65).

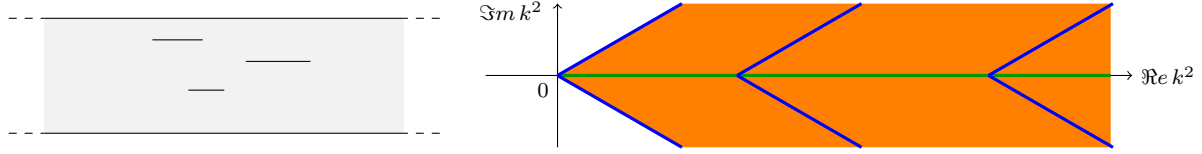


Figure 2.65: Left: waveguide with horizontal cracks. Right: the spectrum of the corresponding B_θ fills the whole sector delimited by $\sigma_{\text{ess}}(B_\theta)$. Here the blue lines represent $\sigma_{\text{ess}}(B_\theta)$.

2.9.4 Numerical experiments

Classical complex scaling: classical complex resonance modes

We first compute the spectrum of the operator A_θ defined in (2.79) with a classical complex scaling (complex resonance spectrum). For the numerical experiments, we truncate the computational domain at some distance of the obstacle and use finite elements. This corresponds to the so-called Perfectly Matched Layers (PMLs) method. We refer the reader to [292] for the numerical analysis of the error due to truncation of the waveguide and discretization. The setting is as follows. We take γ such that $\gamma = 5$ in $\mathcal{O} = (-1; 1) \times (0.25; 0.75)$ and $\gamma = 1$ in $\Omega \setminus \overline{\mathcal{O}}$ (see Figure 2.64 (a)). In the definition of the maps $\mathcal{I}_\theta, \alpha_\theta$ (see (2.77), (2.78)), we take $\theta = \pi/4$ (so that $\eta = e^{i\pi/4}$) and $L = 1$. In practice, we use a P2 finite element method in the bounded domain $\Omega_{12} := \{(x, y) \in \Omega \mid -12 < x < 12\}$ with Dirichlet boundary condition at $x = \pm 12$ (a Neumann condition would work as well). This leads us to solve the spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda, u^h) \in \mathbb{C} \times V_h \setminus \{0\} \text{ such that for all } v^h \in V_h, \\ \int_{\Omega_{12}} \alpha_\theta \partial_x u^h \partial_x v^h + \alpha_\theta^{-1} \partial_y u^h \partial_y v^h dx dy = \lambda \int_{\Omega_{12}} \gamma \alpha_\theta^{-1} u^h v^h dx dy, \end{array} \right. \quad (2.87)$$

where,

$$V_h := \left\{ v \in H^1(\Omega_{12}) \text{ such that } v|_\tau \in \mathbb{P}_2(\tau) \text{ for all } \tau \in \mathcal{T}_h \text{ and } v = 0 \text{ at } x = \pm 12 \right\}. \quad (2.88)$$

Note that to obtain (2.87), we exploited that α_θ depends only on x . In Figure 2.66 and below, we display the square root of the spectrum (k instead of k^2). The vertical marks on the real axis correspond to the thresholds $(0, \pi, 2\pi, \dots)$. In accordance with Theorem 2.9.2, we observe that $\sqrt{\sigma(A_\theta)}$ is located in the region $\sqrt{\mathcal{R}_\theta^-} = \{z \in \mathbb{C} \mid -\theta \leq \arg(z) \leq 0\}$. Moreover, the discretization of the essential spectrum $\sigma_{\text{ess}}(A_\theta)$ defined in (2.80) and forming branches starting at the threshold points appears clearly. Note that a simple calculation shows that $\sqrt{\{n^2\pi^2 + te^{-2i\theta}, t \geq 0\}}$ is a half-line for $n = 0$ and a piece of hyperbola for $n \geq 1$. This is precisely what we get. Eigenvalues located on the real axis correspond to trapped modes ($k^2 \in \mathcal{K}_t$). In the chosen setting, which is symmetric with respect to the axis $\mathbb{R} \times \{0.5\}$, one can prove that trapped modes exist [225]. On the other hand, the eigenvalues in the complex plane which are not the discretization of the essential spectrum correspond to complex resonances.

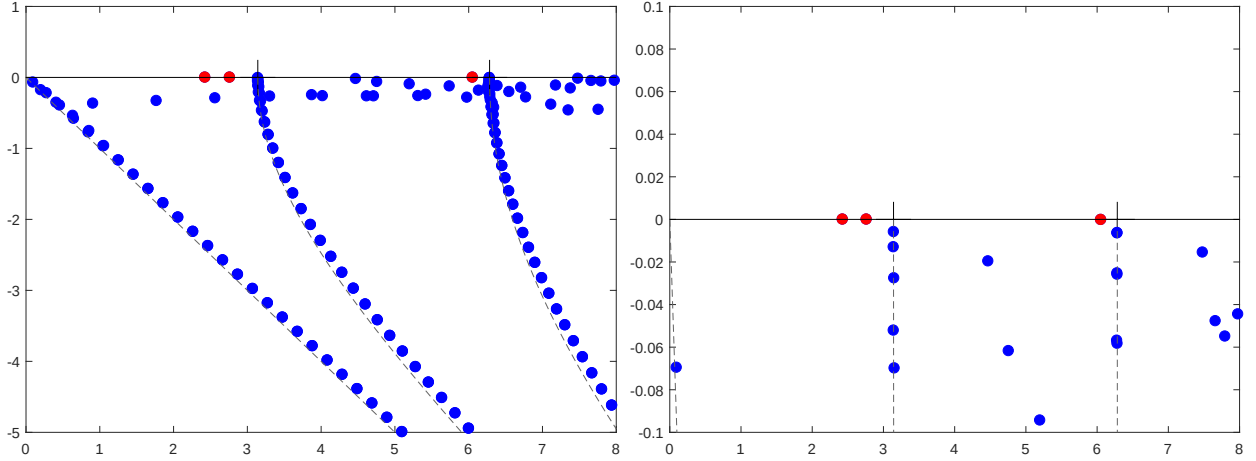


Figure 2.66: Classical complex resonances in the complex k plane corresponding to the spectrum of A_θ for a symmetric obstacle (Figure 2.64 (a)). The trapped modes are in red, the dashed lines represent the essential spectrum of A_θ (see (2.80)). The picture on the right is a zoom-in of that on the left.

Conjugated complex scaling: reflectionless modes

Now we compute the spectrum of the operator B_θ defined in (2.83) with a conjugated complex scaling. To proceed, we solve the variational spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda, u^h) \in \mathbb{C} \times V_h \setminus \{0\} \text{ such that for all } v^h \in V_h, \\ \int_{\Omega_{12}} \beta_\theta \partial_x u^h \partial_x v^h + \beta_\theta^{-1} \partial_y u^h \partial_y v^h \, dx dy = \lambda \int_{\Omega_{12}} \gamma \beta_\theta^{-1} u^h v^h \, dx dy, \end{array} \right.$$

with the same V_h as in (2.88). Note that compared to (2.87), α_θ has been replaced by β_θ (both functions are piecewise constant). First, we use exactly the same symmetric setting (see Figure 2.64 (a)) as in the previous paragraph.

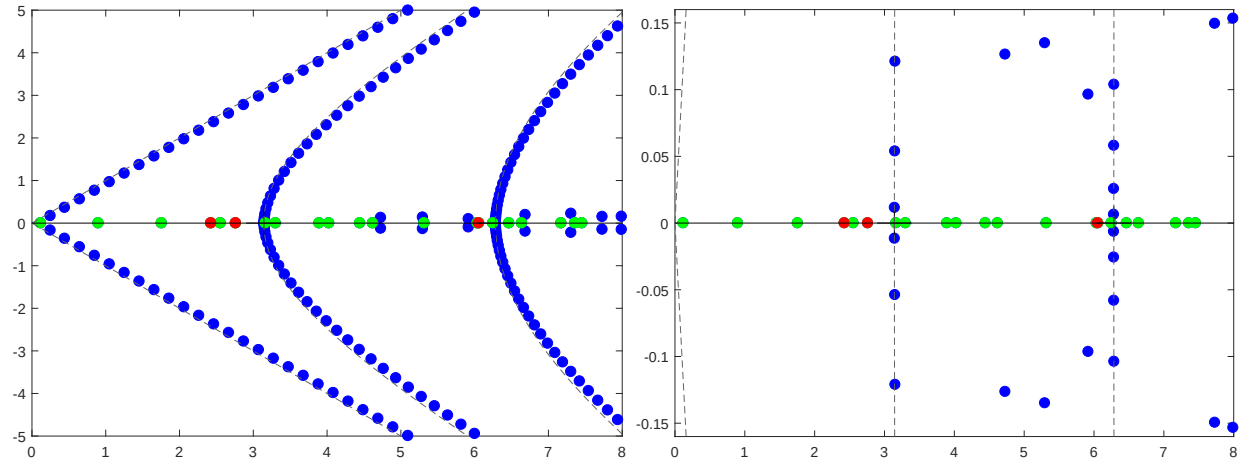


Figure 2.67: Reflectionless eigenvalues in the complex k plane corresponding to the spectrum of B_θ for a symmetric obstacle (Figure 2.64 (a)). The trapped modes in red are the same as in Figure 2.66. The reflectionless modes are in green and the dashed lines represent the essential spectrum of B_θ (see (2.84)). The picture on the right is a zoom-in of that on the left.

In Figure 2.67, we display the square root of the spectrum $\sqrt{\sigma(B_\theta)}$. Since γ satisfies $\gamma(x, y) = \gamma(-x, y)$, according to Proposition 2.9.5 we know that B_θ is $\mathcal{PT}$ -symmetric and that

therefore its spectrum is stable by conjugation ($\sigma(B_\theta) = \overline{\sigma(B_\theta)}$). This is indeed what we obtain. Note that the mesh has been constructed so that $\mathcal{PT}$ -symmetry is preserved at the discrete level. $\mathcal{PT}$ -symmetry is an interesting property in our case because it guarantees that eigenvalues located close to the real axis which are isolated (no other eigenvalue in a vicinity) are real. Therefore, according to Theorem 2.9.3, they correspond to trapped modes or to reflectionless modes. Remark that, for the same geometry, the spectrum of B_θ (Figure 2.67) contains more elements on the real axis than the spectrum of A_θ (Figure 2.66): the additional elements (green points in Figure 2.67) correspond to reflectionless modes.

In Figure 2.68 top, we represent the real part of eigenfunctions associated with seven real eigenvalues of B_θ . To obtain these pictures, we take $L = 4$ in the definition of $\mathcal{J}_\theta$ in (2.81) and we display only the restrictions of the eigenfunctions to $\Omega_L = \{(x, y) \in \Omega \mid -L < x < L\}$. We recognize two trapped modes (images 3 and 5). The other modes are reflectionless modes. In the bottom of Figure 2.68, we provide the value of the indicator function ρ defined in (2.86) for the seven eigenmodes. We have to mention that eigenmodes are normalized so that their L^2 norm is equal to one. The indicator function ρ offers a clear criterion to distinguish between trapped modes and reflectionless modes.

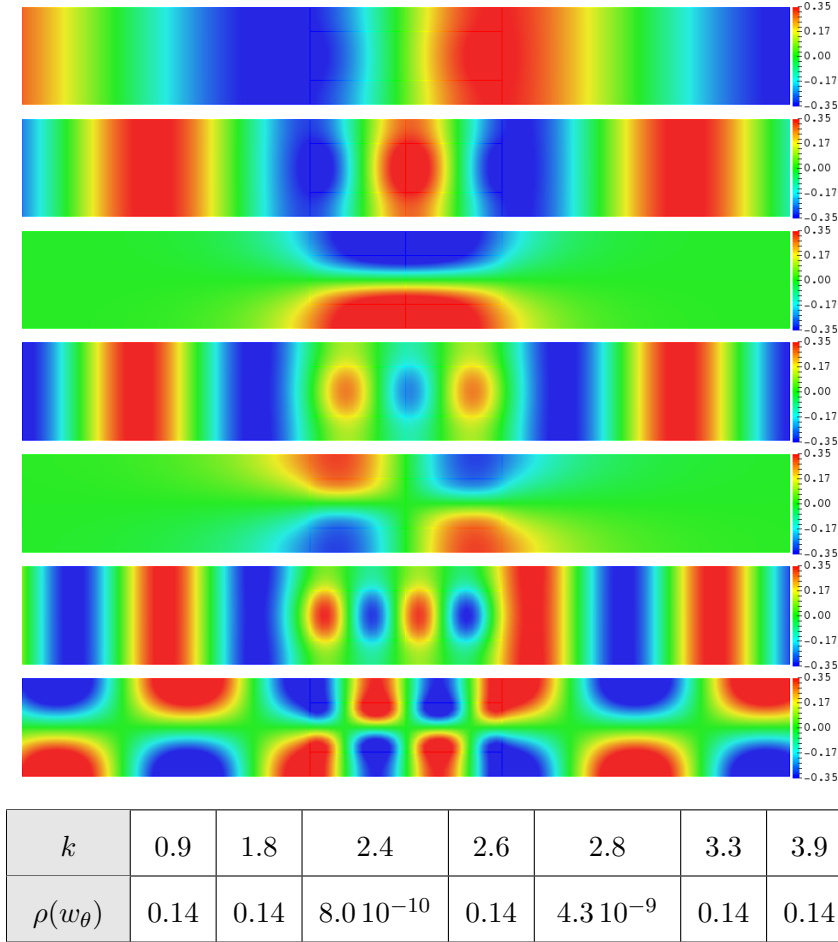


Figure 2.68: Top: real part of eigenmodes associated with real eigenvalues of B_θ from Figure 2.67. Bottom: value of k and of the indicator function ρ for each of these 7 eigenmodes. The 3rd and the 5th eigenmodes are trapped modes, the five others are reflectionless modes.

In order to inspect the scattering coefficient, we remark that for reflectionless modes associated with wavenumbers k smaller than π , the incident field u_i in (2.74) decomposes only on the piston mode $w_0^+(x, y) = e^{ikx}/\sqrt{2k}$ (monomode regime). In this case the reflection matrix $R(k)$ in (2.73)

is nothing but the usual reflection coefficient. In Figure 2.69, we thus display the modulus of this coefficient $R(k)$ with respect to $k \in (0.1; 3.1)$. As expected, we observe that R vanishes for the values of k obtained in Figure 2.68 solving the spectral problem for B_θ . Of course obtaining the curve $k \mapsto |R(k)|$ is relatively costly and it is precisely what we want to avoid by computing the reflectionless k as eigenvalues. Here it is simply a way to check our results.

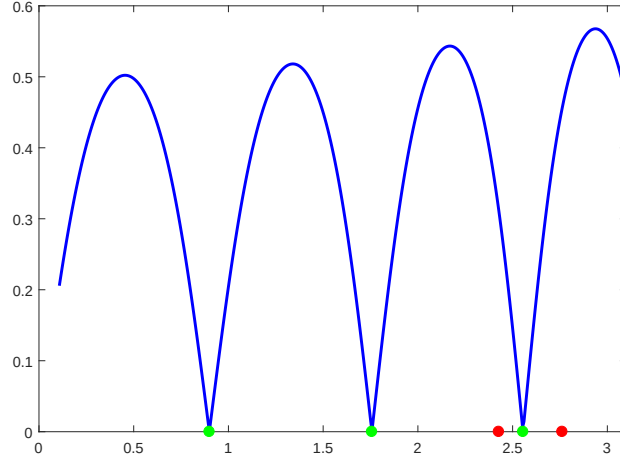


Figure 2.69: Curve $k \mapsto |R(k)|$ (modulus of the reflection coefficient) for $k \in (0.1; 3.1)$. The green and red dots represent respectively the reflectionless modes and the trapped modes computed in Figure 2.67. We indeed observe that $R(k)$ vanishes for reflectionless k .

In Figure 2.70, we represent the modulus of reflectionless mode eigenfunctions of B_θ associated with one real eigenvalue and two complex conjugated eigenvalues. We observe, and this is true in general, a symmetry with respect to the (Oy) axis for modes corresponding to real eigenvalues which disappears for complex ones. This is the so-called broken symmetry phenomenon which is well-known for $\mathcal{PT}$ -symmetric operators (see *e.g.* the review [47]).

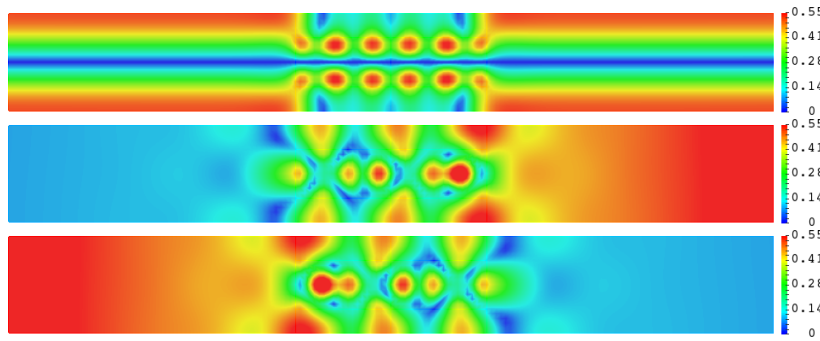


Figure 2.70: Modulus of eigenfunctions of B_θ associated to the eigenvalues $k \approx 5.31$ (top), $k \approx 5.29 - 0.13i$ (middle) and $k \approx 5.29 + 0.13i$ (bottom) obtained in Figure 2.67. The symmetry $x \rightarrow -x$ for real k (due to $\mathcal{PT}$ -symmetry) disappears for complex k .

Finally in Figure 2.71, we display the square root of the spectrum of B_θ for a coefficient γ which is not symmetric in x nor in y . More precisely, we take γ such that $\gamma = 5$ in $\mathcal{O} = (-1; 0] \times (0.25; 0.5) \cup [0; 1) \times (0.25; 0.75)$ and $\gamma = 1$ in $\Omega \setminus \overline{\mathcal{O}}$ (see Figure 2.64 (b)). We observe that the spectrum is no longer stable by conjugation ($\sigma(B_\theta) \neq \overline{\sigma(B_\theta)}$) since the operator B_θ is not $\mathcal{PT}$ -symmetric, and there is no “help” for the eigenvalues to be real. However, a closer look shows the presence of eigenvalues close to the real axis, in particular for $k \approx 1.0 + 0.13i$, $k \approx 1.9 + 0.005i$, $k \approx 2.5 + 0.02i$, $k \approx 2.8 + 0.08i$ and $k \approx 3.0 - 0.008i$. In Figure 2.72, we represent $k \mapsto |R(k)|$ for $k \in (0.1; 3.1)$, that is when there is only one propagating mode in the leads. It is interesting to note that the

above computed complex reflectionless modes (located close to the real axis) have an influence on this curve. More precisely, $k \mapsto |R(k)|$ attains minima for $k \in (0.1; 3.1)$ close to the real part of these complex reflectionless modes. Therefore complex reflectionless modes also have significance for scattering at real frequencies.

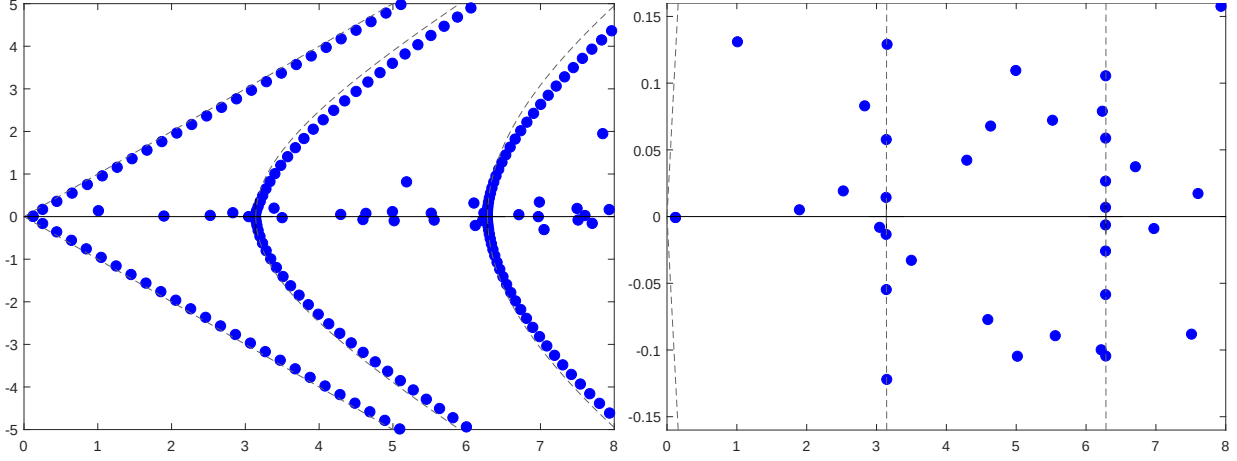


Figure 2.71: Spectrum of B_θ in the complex k plane for a non symmetric obstacle (Figure 2.64 (b)). The dashed lines represent the essential spectrum of B_θ (see (2.84)). The spectrum is not stable by conjugation. The picture on the right is a zoom-in of that on the left.

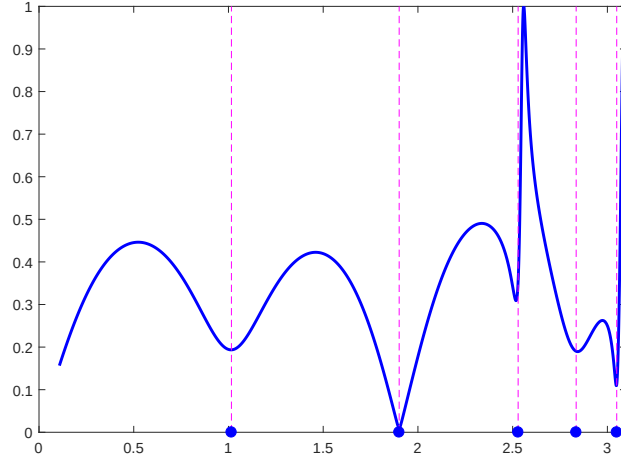


Figure 2.72: Curve $k \mapsto |R(k)|$ (modulus of the reflection coefficient) for $k \in (0.1; 3.1)$ and a non symmetric obstacle. The blue dots and the vertical dashed lines correspond to the real parts of the eigenvalues of B_θ located close to the real axis computed in Figure 2.71. We observe that $|R(k)|$ is minimal for these particular k .

2.9.5 Concluding remarks

It is often desirable to determine frequencies for which a wave can be completely transmitted through a structure, a task usually leading to the tedious work of evaluating the scattering coefficients for each frequency. The study of [74] shows that reflectionless frequencies can be directly computed as the eigenvalues of a non-selfadjoint operator B_θ (see (2.83)) with conjugated complex scalings enforcing ingoing behavior in the incident lead and outgoing behavior in the other lead. The reflectionless spectrum of this operator B_θ provides a complementary information to the one contained in the classical complex resonance spectrum associated with leaky modes which decom-

pose only on outgoing waves (see the operator A_θ in (2.79)). Note that eigenvalues corresponding to trapped modes belong to both the reflectionless spectrum and to the classical complex resonance spectrum because trapped modes do not excite propagating waves. Let us make a few additional comments and highlight future directions as well as open questions.

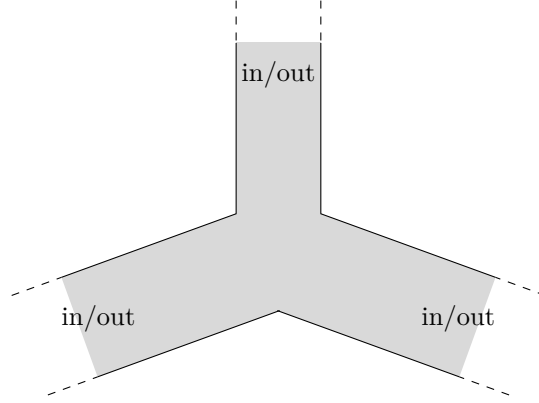


Figure 2.73: For this waveguide, one can impose ingoing or outgoing behaviors in each of the $N = 3$ leads so that there are $2^3 - 2 = 6$ reflectionless spectra.

i) In this work, we investigated scattering problems in waveguides with $N = 2$ leads for which two reflectionless spectra exist: one associated with right-going incident waves and another corresponding to left-going incident waves. The more general case with N ($N \geq 2$) leads can be considered as well (see Figure 2.73). Among the total of 2^N different spectra with an ingoing or an outgoing complex scaling in each lead, two spectra correspond to eigenmodes which decompose on waves which are all outgoing or all ingoing. As a consequence, there are $2^N - 2$ reflectionless spectra.

ii) We have seen that the non-selfadjoint operator B_θ is $\mathcal{PT}$ -symmetric when the structure has mirror symmetry. Interestingly, a direct calculus shows that in the very simple case of a 1D transmission problem through a slab of constant index, reflectionless frequencies are all real. This gives an example of a non-selfadjoint $\mathcal{PT}$ -symmetric operator with only real eigenvalues.

iii) Above we computed reflectionless modes in waveguides containing penetrable obstacles. We can work completely similarly with perturbations of the geometry. In Figure 2.74, we give two examples of reflectionless modes in such structures. Note that in each case, B_θ is $\mathcal{PT}$ -symmetric due to symmetry of the geometry.

There are many open questions with this work. Here are just a few of them (see also Section 2.12).

iv) First, it would be nice to obtain criteria on the index material/geometry ensuring that the spectrum of B_θ is discrete outside of $\sigma_{\text{ess}}(B_\theta)$. In 1D, we have been able to prove rather general results. In higher dimension this is quite open.

v) On the other hand, could we show that B_θ has always real eigenvalues corresponding to reflectionless modes, at least for $\mathcal{PT}$ -symmetric problems?

vi) The question of the approximation of the spectrum of B_θ is a field of research in itself, the reason being that B_θ is a non selfadjoint operator. Indeed, the theory of perturbations of non selfadjoint operators is not well developed and many phenomena can occur. Here it is easy to see in the situation where the waveguide contains only (Neumann) horizontal cracks. In that case we explained before Figure 2.65 that $\sigma(B_\theta)$ fills a whole sector. However when we truncate the domain at some distance L , we can show that the corresponding spectrum is discrete. This seems particularly pathological. But even when $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete, proving that the spectrum of the problem in the truncated waveguide converges to the one of B_θ when L tends to $+\infty$ is a challenging task. Additionally, it would be interesting to prove that spurious eigenvalues, which

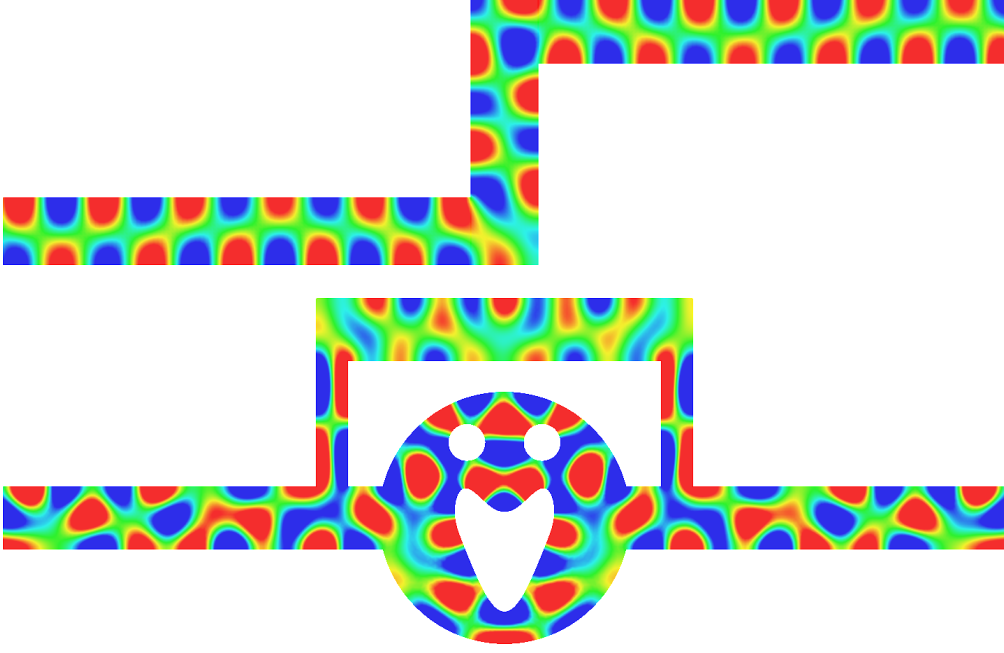


Figure 2.74: Examples of reflectionless modes in waveguides with sound hard walls (Neumann).

would converge to some non physical values, do not exist. In practice, we not only truncate the domain but also approximate the problem by working in finite dimension with finite elements. This is also an approximation which has to be studied. In this context, it seems that the notion of pseudo-spectrum, see *e.g.* [459], which has been developed to understand the properties of non normal operators, may provide useful information. This is a direction that we are currently investigating.

We end this chapter concerning wave propagation in waveguides by two studies which are not related to invisibility questions.

2.10 Wave propagation in unbounded strips for thin plate models

In the article [101], with L. Bourgeois and S. Fliss, we studied the propagation of elastic waves in time-harmonic regime in waveguides which are unbounded in one direction and bounded in the two other (transverse) directions. Assuming that the waveguide is thin in one of these transverse directions, this leads us to consider a Kirchhoff-Love plate model in a locally perturbed 2D strip. The corresponding problem writes

$$\left\{ \begin{array}{ll} \Delta^2 u - k^4 u = f & \text{in } \Omega \\ u = Cu = 0 & \text{on } \partial\Omega \setminus \partial\mathcal{O} \\ Mu = Nu = 0 & \text{on } \partial\mathcal{O} \\ u \text{ satisfies (RC)} & \end{array} \right. \quad (2.89)$$

Let us describe this system. Here $\Omega := \mathbb{R} \times (0; 1) \setminus \overline{\mathcal{O}}$ where $\mathcal{O}$ is a $\mathcal{C}^{1,1}$ bounded domain such that $\overline{\mathcal{O}} \subset \mathbb{R} \times (0; 1)$. From a physical point of view, the first line of (2.89) comes from the equation of motion of the strip

$$D\Delta^2 u + \rho h \frac{\partial^2 u}{\partial t^2} = p$$

in time-harmonic regime. Here, we have $D = Eh^3/12(1 - \nu^2)$, where E is the Young's modulus, $\nu \in [0; 1)$ is the Poisson's ratio and h is the thickness of the strip. Moreover, ρ refers to the density

per unit of volume and p corresponds to the pressure applied to the strip. Hence the wavenumber k is defined by $k^4 = \rho h \omega^2 / D$ and the volume source f by p/D .

In the third equation of (2.89), M and N are the boundary differential operators defined by

$$Mu = \nu \Delta u + (1 - \nu) M_0 u, \quad Nu = -\frac{\partial}{\partial n} \Delta u - (1 - \nu) \frac{\partial}{\partial s} N_0 u, \quad (2.90)$$

where $n = (n_x, n_y)$ is the outward unit normal to $\partial\Omega$ and $s = (-n_y, n_x)$. Above, we use the notation

$$\frac{\partial}{\partial n} := n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial s} := -n_y \frac{\partial}{\partial x} + n_x \frac{\partial}{\partial y}.$$

Moreover, in (2.90), the operators M_0 and N_0 are respectively defined by

$$M_0 u = \frac{\partial^2 u}{\partial x^2} n_x^2 + 2 \frac{\partial^2 u}{\partial x \partial y} n_x n_y + \frac{\partial^2 u}{\partial y^2} n_y^2, \quad N_0 u = \frac{\partial^2 u}{\partial x \partial y} (n_x^2 - n_y^2) - \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right) n_x n_y.$$

The boundary condition $Mu = 0$ and $Nu = 0$ on $\partial\mathcal{O}$ correspond to a free obstacle, that is a hole. Concerning the boundary conditions on $\partial\Omega$, *i.e.* the second equation of (2.89), we considered the following two cases:

- i) $C = M$, which leads to impose $u = Mu = 0$ on $\partial\Omega$. This models a simply supported strip.
- ii) $C = \partial_n$, which leads to impose $u = \partial_n u = 0$ on $\partial\Omega$. This models a clamped strip.

Finally in (2.89), (RC) stands for the radiation conditions which have to be specified and which differ whether we consider the simply supported or the clamped case. The goal of [101] was to study the well-posedness of Problem (2.89). We proposed two angles of attack, depending on the boundary condition. In the case of the simply supported strip, we benefit from the factorization of the transverse underlying differential operator to decompose any scattering solution in terms of the modes of the waveguide. Then we prescribe the radiation conditions with the help of these modes and introduce Dirichlet-to-Neumann operators – based on these radiation conditions – in order to reduce the analysis to the one of a problem set in a bounded domain. Such strategy also offers a method to compute the solution numerically. However this technique is not applicable to the case of a clamped strip because we are not able to prove that the eigenfunctions of the transverse operator form a Hilbert basis of the corresponding space (as in §1.3.2). For this problem, we obtain the result of modal decomposition needed to express the radiation conditions at infinity using the Kondratiev approach [312] (see also [353, 382, 314, 315]). As already said, it consists in applying the Fourier transform in the unbounded direction. Then working in weighted Sobolev spaces and using the residue theorem, we get our decomposition. In a second step, in order to impose radiation conditions, we integrate them to the functional space in which we look for the solution. To proceed, we work with spaces with detached asymptotics introduced in [381] (see also the reviews [374, 371]). Finally we show that the corresponding scattering problem with radiation conditions is well-posed in the Fredholm sense. We also prove that the solutions are the physical (outgoing) solutions in the sense of the limiting absorption principle. Let us emphasize that the methodology we follow to study the clamped problem can be used also to deal with the simply supported problem and provides completely similar results.

The goal of [101] was first to investigate problems of thin plates in unbounded strips but also to show that when the result of Hilbert basis in the transverse section is not available, we can still use an alternative route. By presenting successively the two methods, our hope was to help the reader to get familiar with the second approach which may be less known and which requires a slightly longer analysis. For application of the technique to other situations, one may consult [386, 69, 378, 65, 384]. In [368, 387], periodic problems are also considered. Let us mention that

some analysis of modal solutions in a strip for various boundary conditions have already been conducted (see for example [400, 279]). But a rigorous existence and uniqueness analysis of the scattering problem, whatever the boundary conditions, was missing.

We worked with a hole in the waveguide. We could also have considered other types of perturbations (change of material, local perturbation of the geometry,...). On the other hand, our article does not address the question of uniqueness of the solution. Can one find conditions on the geometry and on the wavenumber k so that trapped modes are absent? Can one show the existence of settings where trapped modes exist? These are interesting questions to study. A natural direction after this analysis would be to investigate how to approximate the solutions. For the simply supported case, the fact that we know explicitly the Dirichlet-to-Neumann operator allows one to adapt the methods used to deal with the Helmholtz problem. In the clamped case, another approach must be found. Could the technique of Perfectly Matched Layers be used and justified?

2.11 Spectrum of the Dirichlet Laplacian in quasi 1D periodic waveguides

Recently I have started to study periodic waveguides. In [171], with S.A. Nazarov, we were interested in the lower part of the spectrum $\sigma(A^\varepsilon)$ of the Dirichlet Laplacian A^ε in a thin waveguide Π^ε obtained by repeating periodically a pattern, itself constructed by scaling an inner field geometry Ω by a small factor $\varepsilon > 0$. The Floquet-Bloch theory ensures that $\sigma(A^\varepsilon)$ has a band-gap structure. Due to the Dirichlet boundary conditions, the bands all move to $+\infty$ as $O(\varepsilon^{-2})$ when $\varepsilon \rightarrow 0^+$. Concerning their widths, the results depend on the dimension of the so-called space of almost standing waves in Ω that we denote by $X_\dagger$. Generically, *i.e.* for most Ω , there holds $X_\dagger = \{0\}$ and the lower part of $\sigma(A^\varepsilon)$ is very sparse, made of bands of length at most $O(\varepsilon)$ as $\varepsilon \rightarrow 0^+$. For certain Ω however, we have $\dim X_\dagger = 1$ and then there are bands of length $O(1)$ which allow for wave propagation in Π^ε . In [171], we study the behavior of the spectral bands when perturbing Ω around a particular $\Omega_\star$ where $\dim X_\dagger = 1$. We show a breathing phenomenon of $\sigma(A^\varepsilon)$: when inflating Ω around $\Omega_\star$, the spectral bands rapidly expand before shrinking. In the process, a band dives below the normalized threshold, stops breathing and becomes extremely short as Ω continues to inflate.

2.11.1 Setting of the problem

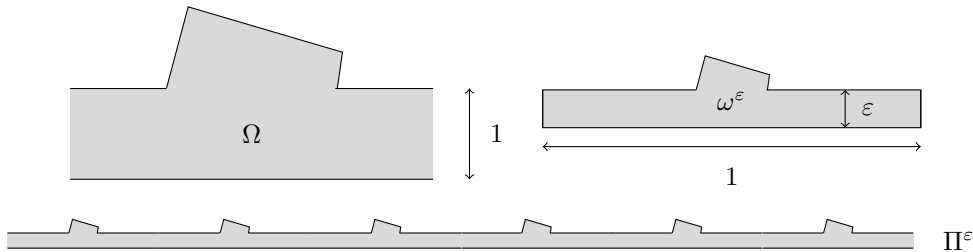


Figure 2.75: Geometries of Ω (top left), ω^ε (top right) and Π^ε (bottom).

Let $\Omega \subset \mathbb{R}^2$ be a waveguide which coincides with the strip $\mathbb{R} \times (-1/2; 1/2)$ outside of a bounded region (Figure 2.75 top left). For $\varepsilon > 0$, we consider the unit cell

$$\omega^\varepsilon := \{z = (x, y) \in \mathbb{R}^2 \mid z/\varepsilon \in \Omega \text{ and } |x| < 1/2\}$$

and set $\partial\omega_\pm^\varepsilon := \{\pm 1/2\} \times (-\varepsilon/2; \varepsilon/2)$. Finally we define the periodic waveguide

$$\Pi^\varepsilon := \{z \in \mathbb{R}^2 \mid (x - m, y) \in \omega^\varepsilon \cup \partial\omega_\pm^\varepsilon, m \in \mathbb{Z}\}.$$

We assume that Ω , ω^ε and Π^ε are connected with Lipschitz boundaries. In Π^ε , we consider the spectral problem for the Dirichlet Laplacian

$$\begin{cases} -\Delta u^\varepsilon &= \lambda^\varepsilon u^\varepsilon & \text{in } \Pi^\varepsilon \\ u^\varepsilon &= 0 & \text{on } \partial\Pi^\varepsilon. \end{cases} \quad (2.91)$$

We denote by A^ε the unbounded selfadjoint operator of $L^2(\Pi^\varepsilon)$, with domain $\mathcal{D}(A^\varepsilon) \subset H_0^1(\Pi^\varepsilon) := \{\varphi \in H^1(\Pi^\varepsilon) \mid \varphi = 0 \text{ on } \partial\Pi^\varepsilon\}$, associated with (2.91). Since the geometry is periodic, the Floquet-Bloch theory ensures that the spectrum of A^ε has a band/gap structure:

$$\sigma(A^\varepsilon) = \bigcup_{p \in \mathbb{N}^* := \{1, 2, \dots\}} \Upsilon_p^\varepsilon \quad (2.92)$$

where the Υ_p^ε are compact segments. Our goal is to study the behavior of $\sigma(A^\varepsilon)$ as $\varepsilon \rightarrow 0^+$.

2.11.2 Near field problem and first result

The analysis developed in particular in [243] shows that the asymptotic behavior of the Υ_p^ε with respect to ε depends on the features of the Dirichlet Laplacian A^Ω in Ω . Its continuous spectrum occupies the ray $[\pi^2; +\infty)$. To set ideas, we assume that A^Ω has exactly $N_\bullet \in \mathbb{N} := \{0, 1, 2, \dots\}$ eigenvalues (counted with multiplicity) in its discrete spectrum, that we denote by

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_{N_\bullet} < \pi^2. \quad (2.93)$$

Of particular importance in the study are the features of the inner field problem with a spectral parameter coinciding with the bottom of the continuous spectrum of A^Ω :

$$\begin{cases} \Delta W + \pi^2 W &= 0 & \text{in } \Omega \\ W &= 0 & \text{on } \partial\Omega. \end{cases} \quad (2.94)$$

To simplify the exposition, assume that the only solution of (2.94) in $L^2(\Omega)$, *i.e.* which decays at infinity, is the null function.

Definition 2.11.1 Let $X_\dagger$ denote the space of almost standing waves in Ω defined as the set of bounded solutions of (2.94).

Using techniques of dimension reduction on the spectral problem depending on the Floquet parameter obtained when applying the Floquet-Bloch transform to (2.91), we show the statement:

Theorem 2.11.2 For $p \in \mathbb{N}^*$, let $\Upsilon_p^\varepsilon = [a_{p-}^\varepsilon; a_{p+}^\varepsilon]$, with $a_{p-}^\varepsilon \leq a_{p+}^\varepsilon$, be the spectral band in (2.92). There are some constants $c_{p-} < c_{p+}$, C_p , β_p , $\varepsilon_p > 0$ and $\delta_p > 1$ such that we have

For $p = 1, \dots, N_\bullet$:

$$|a_{p\pm}^\varepsilon - (\varepsilon^{-2}\mu_p + \varepsilon^{-2}e^{-\beta_p/\varepsilon}c_{p\pm})| \leq C_p e^{-\delta_p\beta_p/\varepsilon};$$

For $p = N_\bullet + m$, $m \in \mathbb{N}^*$:

$$i) \text{ if } X_\dagger = \{0\}, \quad |a_{p\pm}^\varepsilon - (\varepsilon^{-2}\pi^2 + m^2\pi^2 + \varepsilon c_{p\pm})| \leq C_p \varepsilon^{\delta_p};$$

$$ii) \text{ if } \dim X_\dagger = 1, \quad |a_{p\pm}^\varepsilon - (\varepsilon^{-2}\pi^2 + c_{p\pm})| \leq C_p \varepsilon^{\delta_p};$$

$$iii) \text{ if } \dim X_\dagger = 2, \quad |a_{p\pm}^\varepsilon - (\varepsilon^{-2}\pi^2 + (m-1)^2\pi^2 + \varepsilon c_{p\pm})| \leq C_p \varepsilon^{\delta_p}.$$

Each estimate above is valid for all $\varepsilon \in (0; \varepsilon_p]$ and the μ_p are the ones introduced in (2.93).

Let us comment on these results. First, as already mentioned, when $\varepsilon \rightarrow 0^+$, the whole spectrum of A^ε goes to $+\infty$ as ε^{-2} . Besides, the first $N_\bullet$ spectral bands of A^ε become extremely short, in $O(e^{-c/\varepsilon})$ for some $c > 0$ which depends on the band. Concerning the next spectral bands Υ_p^ε , $p = N_\bullet + m$ with $m \in \mathbb{N}^*$, the behavior depends on the dimension of $X_\dagger$. When the latter is zero (the generic case) or two (cases *i*) or *iii*), the spectral bands are of length $O(\varepsilon)$. Moreover, between Υ_p^ε and $\Upsilon_{p+1}^\varepsilon$, there is a gap, that is, a segment of spectral parameters λ^ε such that waves cannot propagate, whose length tends to $(2m+1)\pi^2$ (resp. $(2m-1)\pi^2$) in case *i*) (resp. *iii*)). In other words, for these two cases, the propagation of waves in the thin lattice Π^ε is hampered and occurs only for very narrow (closed) intervals of frequencies. When the dimension of $X_\dagger$ is one (case *ii*)), the situation is very different. Indeed, asymptotically the spectral band Υ_p^ε is of length $c_{p+} - c_{p-}$, with in general $c_{p+} > c_{p-}$. As a consequence, waves can propagate in Π^ε for much larger intervals of frequencies than in cases *i*) and *iii*).

2.11.3 Breathing of the spectral bands

Now assume that the inner field geometry $\Omega = \Omega(H)$ depends smoothly on a parameter H . We denote by $X_\dagger(H)$ the corresponding space of almost standing waves. Consider some $H_\star$ such that $\dim X_\dagger(H_\star) = 1$ (see [171, Prop. 7.1] for the proof of existence of such geometries) and work with $H = H_\star + \varepsilon\rho$, $\rho \in \mathbb{R}$. Let $\Upsilon_p^{\rho,\varepsilon}$ stand for the spectral bands of the operator A^ε defined in the Π^ε constructed from $\Omega(H_\star + \varepsilon\rho)$. Computing an asymptotic expansion of the $\Upsilon_p^{\rho,\varepsilon}$ as $\varepsilon \rightarrow 0^+$, we get results depending on the parameter ρ . By varying $\rho \in \mathbb{R}$, this provides a model describing the transition of $\sigma(A^\varepsilon)$ when inflating the inner field geometry around $\Omega(H_\star)$. With this model, we proved that the spectral bands above the normalized threshold $\varepsilon^{-2}\pi^2$ first expand and then shrink (see [171, Thm. 6.1]). This is what we call the breathing phenomenon of the spectrum of A^ε . In the process, in $\sigma(A^\varepsilon)$ a band dives below $\varepsilon^{-2}\pi^2$, stops breathing and becomes extremely short as the inner field geometry continues to inflate.

In Figure 2.76 we illustrate this phenomenon with some numerics on a particular case. The inner field geometry $\Omega(H)$, with a step of height H , is the one appearing at the top right. We shrink $\Omega(H)$ by a factor $\varepsilon = 0.05$ to create a thin periodic waveguide. Then we compute the spectrum of (2.91) with a finite element method. To proceed, we exploit the Floquet-Bloch theory which ensures that

$$\sigma(A^\varepsilon) = \bigcup_{p \in \mathbb{N}^*} \bigcup_{\eta \in [0; 2\pi)} \Lambda_p^\varepsilon(\eta)$$

where the $\Lambda_p^\varepsilon(\eta)$ are the eigenvalues of a spectral problem set in the unit cell with quasi-periodic boundary conditions depending on the Floquet parameter $\eta \in [0; 2\pi)$. In Figure 2.76 we display the spectrum $\sigma(A^\varepsilon)$ for H varying in $[1.5; 3.5]$. On the other hand, by solving a problem in $\Omega(H)$ with adapted transparent conditions (see [170, §6.2] for more details), we can compute the so-called threshold scattering matrix and identify the values of $H = H_\star$ such that $\dim X_\dagger(H_\star) = 1$. The latter are marked by vertical dashed lines in Figure 2.76. In accordance with the above theory, we indeed observe a rapid expansion of the spectral bands around these particular $H_\star$. In the process, as expected, a very thin band (in magenta) dives below the normal threshold π^2/ε^2 (the horizontal red dashed line).

Finally, let us mention that in [170, 175], we showed results concerning the features of A^Ω , the Dirichlet Laplacian in Ω , for 3D geometries Ω such as the one of Figure 2.77 right. In particular, we identified the number of eigenvalues below the continuous spectrum and studied in detail the dimension of $X_\dagger$. As we have seen above, this is important to understand the properties of the spectrum of the Dirichlet Laplacian in the corresponding thin periodic waveguide (see Figure 2.77 left). The proofs in [170, 175] rely on classical arguments of spectral theory such as the max-min principle. The main originality lies rather in the delicate use of the features of the geometry.

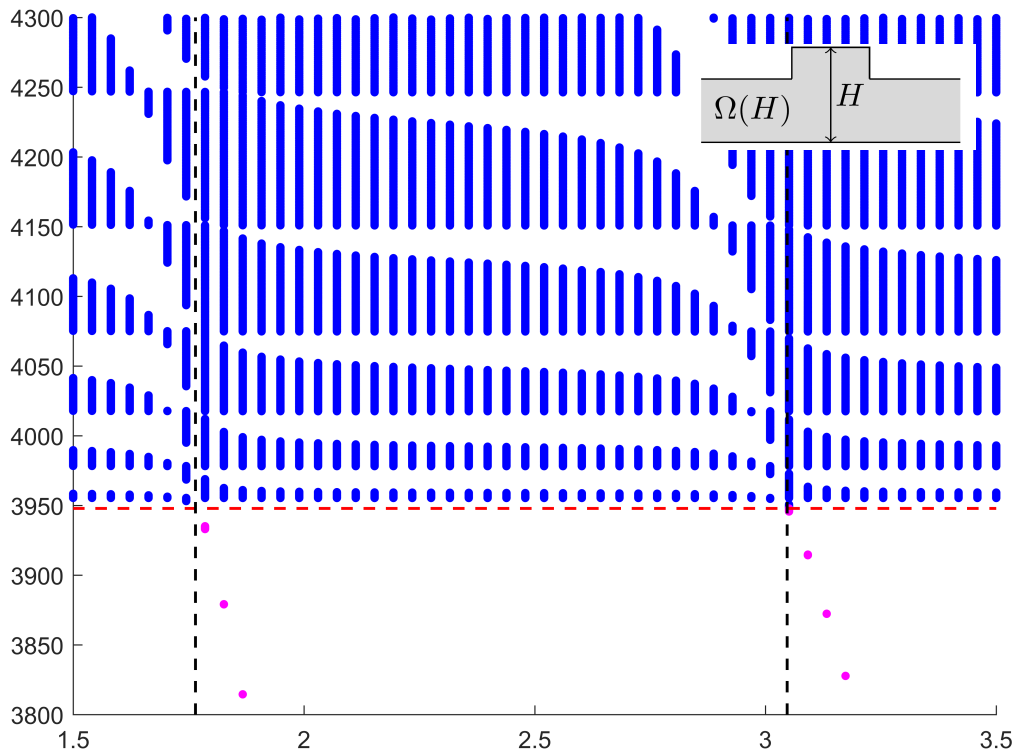


Figure 2.76: Spectrum of A^ε with respect to $H \in [1.5; 3.5]$. The horizontal red dashed line corresponds to $\varepsilon^{-2}\pi^2$. The vertical dashed lines mark the values of $H_\star$ such that $\dim X_\dagger(H_\star) = 1$.

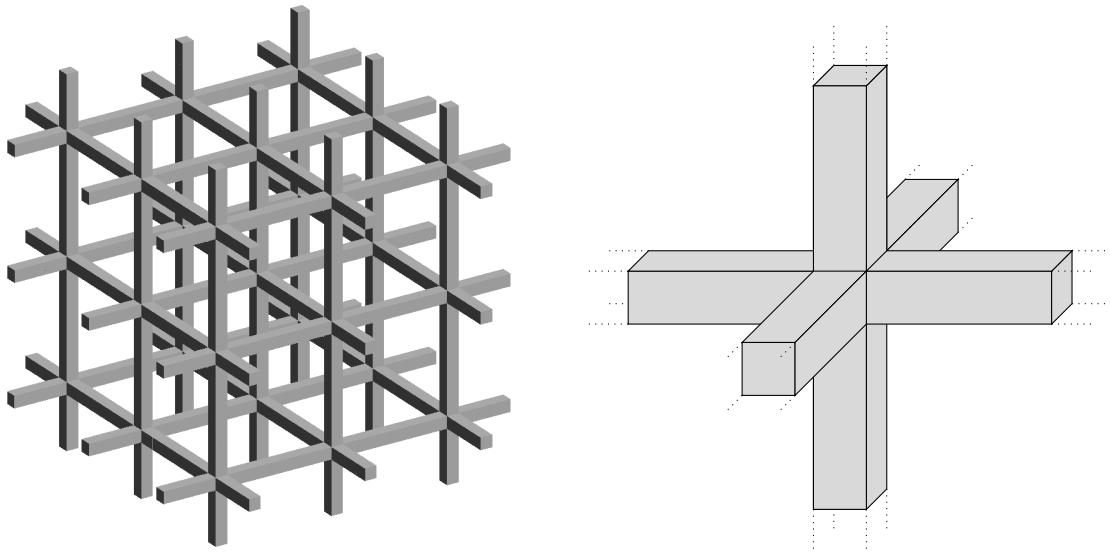


Figure 2.77: Left: unbounded periodic cubic lattice made of thin bars of width ε with square cross-sections. Right: geometry Ω of the near field problem at the junction regions.

2.12 Future directions and open questions

Over the preceding paragraphs, we already presented open problems and future possible directions. We discuss here additional questions which would be interesting to consider.

i) Invisibility has been obtained for well-chosen geometries/wavenumbers. In practice, it is important to have some robustness with respect to the parameters of the problem, in particular to k . As already mentioned, it is in general impossible to have invisibility for a continuum of k . But can we

get small reflection for an interval of k ? How to proceed?

ii) Invisibility has been studied in time-harmonic regime. What can be said for time dependent problems? How does it relate to questions of identifiability in control theory?

iii) The whole theme of invisibility above has been developed for simple scalar problems, mainly for the Helmholtz equation with Dirichlet/Neumann boundary conditions. In the PhD of A. Parigaux, we are currently investigating closed waveguide problems for Maxwell's equations, in particular how to write transparent conditions to truncate the domains for numerics. This is not a direct adaptation of the existing techniques to deal with the Helmholtz equation and the literature is little developed. But we are making progress and now we have a method to compute the corresponding fields, even in the difficult case where the permittivity and the permeability vary in the transverse section (while being constant in the longitudinal direction). From this work, it would be interesting to study if our techniques to exhibit situations of invisibility could be adapted to electromagnetism. It would also be worth investigating the plate model described in Section 2.10.

iv) Some of the techniques presented in the first sections rely on the perturbation of a setting where there exist embedded eigenvalues in the continuous spectrum. It is well-known how to create such geometries for the Helmholtz problem, in particular by using min-max arguments and by playing with symmetries. In electromagnetism or for plate problems, these ideas do not work directly and the theory must be developed.

v) In sections 2.7 and 2.8, we played with complex resonances to obtain interesting scattering properties. This has been done in closed waveguides here. It is not clear if this could be adapted in open waveguides or in free space where one can also create resonators giving birth to complex resonances located very close to the real axis.

vi) In Section 2.9, we defined a new spectrum of reflectionless modes. As we observed in 1D in some study not presented here, trying to obtain a spectrum of perfect invisibility, *i.e.*, without phase shift for the transmitted field, is not very relevant because this happens only in rare circumstances. However the zero transmission phenomenon, the fact that an incident field is completely backscattered in energy, seems to be much more common (see in particular the two results of robustness of §2.6.6 and Theorem 2.7.2). It is easy to define this spectrum in an abstract way but for the moment, we fail to find a method, for example using modified Perfectly Matched Layers, to compute efficiently the frequencies such that one has zero transmission for particular incident fields.

vii) In Section 2.11, we studied the spectrum of the Dirichlet Laplacian in quasi 1D periodic waveguides. For Neumann boundary conditions, it is well-known that the picture is completely different. In particular, the right transmission conditions to impose at the junction points of the 1D model are Kirchhoff conditions (continuity of the field and continuity of the flux). It is not clear what happens for Maxwell's equations which somehow mix problems with Dirichlet and Neumann boundary conditions.

viii) For frequencies in the spectral bands, waves can propagate in periodic waveguides. Therefore, we can study scattering problems in such structures. Can we create local defects of the periodicity such that we have zero reflection, perfect transmission or zero transmission? During the summer of 2025, with Z. Moitier, we started to consider these questions in 1D by supervising the Ensta 2A internship of T. Creuset. Besides, in the literature, some works appear (see *e.g.* [199]) concerning topologically invisible defects. It would be interesting to study in more detail the mathematical aspects of these techniques.

TOPICS RELATED TO INVERSE PROBLEMS

As mentioned in the introduction, I have started to get interested in inverse problems during my PhD when A.-S. Bonnet-Ben Dhia noticed that there is a close relation between the transmission problems with sign-changing coefficients presented in Chapter 1 and the so-called Interior Transmission Problem (ITP) which appears in the justification of certain imaging methods in inverse scattering. Then I continued to learn about this field of research during my post-doc and after. Retrospectively, I would not pretend to have worked directly on inverse problems. However I have been involved in topics appearing in the analysis or justification of methods developed in the theory of inverse problems.

3.1 Study of the Interior Transmission Problem

Let us start by presenting this ITP. Consider a reference medium, occupying say $\mathbb{R}^d$, $d = 2, 3$, and presenting a defect (penetrable inclusion) localized in some bounded domain Ω . This reference medium can be the human body, a cable, the sea, ... and the perturbation, a clot, a crack, a submarine. Generating incident plane waves and measuring the resulting far field pattern of the scattered fields (the only available information far from Ω) at a given frequency, one can try to reconstruct the features of the defect in the reference medium. In particular, the classical inverse scattering problem is to determine the support of the inclusion. In view of this aim, many methods have been developed [426, 119], as for example, the Linear Sampling Method (LSM) [188, 194]. Without modifications and *a priori* knowledge, the LSM only works at frequencies which are not Transmission Eigenvalues (TEs), the latter being defined, roughly speaking, as the frequencies such that there is an incident field whose scattered field by the defect vanishes outside of Ω . The Interior Transmission Problem (ITP) precisely characterizes these frequencies that one should avoid. This, and the fact that nice questions of spectral theory appear in its study, explains why the interior transmission eigenvalue problem has been so intensively investigated for 40 years now (see [303, 190, 437] and the review paper [123]).

The terminology “ITP” actually covers a family of spectral problems. Let us formulate the ITP in a particular situation which is relevant in acoustics or in electromagnetism (note that the ITP appears in other contexts, for example in elasticity, see [42, 252, 44]). Consider a penetrable inclusion located in some bounded domain Ω of $\mathbb{R}^d$, $d = 2, 3$. Introduce some coefficients A , n , valued respectively in $\mathbb{C}^{d \times d}$, $\mathbb{R}$, such that $A - \text{Id}$, $n - 1$ are supported in $\bar{\Omega}$. Assume that the propagation of waves is governed by the equation

$$\text{div}(A \nabla u) + k^2 n u = 0 \quad \text{in } \mathbb{R}^d$$

where $k > 0$ is the wavenumber. We consider the scattering of an incident field u_i solving

$$\Delta u_i + k^2 u_i = 0 \quad \text{in } \mathbb{R}^d$$

by the inclusion. Classically, we impose that the scattered field $u_s := u - u_i$ satisfies the radiation condition

$$\lim_{r \rightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_s}{\partial r} - i k u_s \right) = 0$$

where $r := |\mathbf{x}|$. Now, let us write what happens when u_i does not scatter. Assume that $\mathbb{R}^d \setminus \overline{\Omega}$ is connected and that its boundary $\partial\Omega$ is Lipschitz continuous. In Ω , we have $\operatorname{div}(A\nabla u) + k^2 n u = 0$ and $\Delta u_i + k^2 u_i = 0$. The total field satisfies the transmission conditions

$$[u]|_{\partial\Omega} = [\nu \cdot A\nabla u]|_{\partial\Omega} = 0$$

where $[\cdot]|_{\partial\Omega}$ stands for the jump on $\partial\Omega$. Since $u_s = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ (we look for an incident field which does not scatter), we deduce that we have

$$u = u_i \quad \text{and} \quad \nu \cdot A\nabla u = \nu \cdot \nabla u_i \quad \text{on } \partial\Omega.$$

Here ν stands for the outward unit normal vector to $\partial\Omega$. To avoid writing the index i , we denote by w the restriction of the incident field to Ω . The pair (u, w) then solves the ITP

$$\left| \begin{array}{ll} \text{Find } (u, w) \text{ such that} \\ \operatorname{div}(A\nabla u) + k^2 n u &= 0 \quad \text{in } \Omega \\ \Delta w + k^2 w &= 0 \quad \text{in } \Omega \\ u - w &= 0 \quad \text{on } \partial\Omega \\ \nu \cdot A\nabla u - \nu \cdot \nabla w &= 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (3.1)$$

Let us emphasize that in this problem the two functions u, w are defined in the same domain Ω while the transmission conditions are written on the whole boundary of the domain.

The first studies of (3.1) focused on the case where the inclusion is characterized by one single parameter: $A \neq \operatorname{Id}, n = 1$ or $A = \operatorname{Id}, n \neq 1$. The scalar problem has been studied in [190, 437, 192, 193] and its corresponding vectorial version in [254, 126, 121]. The situation with both $A \neq \operatorname{Id}$ and $n \neq 1$ has been investigated later in [120, 305, 128]. However we start by presenting this case because it is the one where the connection with sign-changing problems appears most clearly.

3.1.1 A first T-coercivity approach for the ITP

Assume additionally that $A \in L^\infty(\Omega, \mathbb{C}^{d \times d})$ is such that $A(\mathbf{x})$ is hermitian for almost all $\mathbf{x} \in \Omega$, that $n \in L^\infty(\Omega, \mathbb{R})$ and that these two parameters satisfy

$$\begin{aligned} A_- &:= \inf_{\mathbf{x} \in \Omega} \inf_{\xi \in \mathbb{C}^d, |\xi|=1} \xi \cdot A(\mathbf{x}) \bar{\xi} > 0; & A_+ &:= \sup_{\mathbf{x} \in \Omega} \sup_{\xi \in \mathbb{C}^d, |\xi|=1} \xi \cdot A(\mathbf{x}) \bar{\xi} < \infty; \\ n_- &:= \inf_{\mathbf{x} \in \Omega} n(\mathbf{x}) > 0; & n_+ &:= \sup_{\mathbf{x} \in \Omega} n(\mathbf{x}) < \infty. \end{aligned} \quad (3.2)$$

When $A \neq \operatorname{Id}$, as we will understand below, it is natural to look for solutions of (3.1) which are in $X := \{(u, w) \in H^1(\Omega) \times H^1(\Omega) \mid u - w \in H_0^1(\Omega)\}$, which leads to consider the problem

$$\left| \begin{array}{ll} \text{Find } (u, w) \in X \text{ such that} \\ \operatorname{div}(A\nabla u) + k^2 n u &= 0 \quad \text{in } \Omega \\ \Delta w + k^2 w &= 0 \quad \text{in } \Omega \\ \nu \cdot A\nabla u - \nu \cdot \nabla w &= 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (3.3)$$

Definition 3.1.1 Values of $k \in \mathbb{C}$ for which Problem (3.3) admits non zero solutions (u, w) are called transmission eigenvalues.

Classically, one shows that the pair (u, w) satisfies (3.3) if and only if (u, w) solves

$$\left| \begin{array}{l} \text{Find } (u, w) \in X \text{ such that, for all } (u', w') \in X, \\ a_k((u, w), (u', w')) := (A\nabla u, \nabla u')_\Omega - (\nabla w, \nabla w')_\Omega - k^2((nu, u')_\Omega - (w, w')_\Omega) = 0. \end{array} \right. \quad (3.4)$$

Here $(\cdot, \cdot)_\Omega$ stands for the inner product of $L^2(\Omega)$ or $(L^2(\Omega))^d$ according to the case. With the Riesz representation theorem, define the operator $\mathcal{A}_k : X \rightarrow X$ such that

$$(\mathcal{A}_k(u, w), (u', w'))_{H^1(\Omega) \times H^1(\Omega)} = a_k((u, w), (u', w')), \quad \forall (u, w), (u', w') \in X. \quad (3.5)$$

Note that $a_k(\cdot, \cdot)$ is not coercive in X nor of the form “coercive+compact”, like the form appearing in (1.3) in Chapter 1. In the article [71], with A.-S. Bonnet-Ben Dhia and H. Haddar, we explained how to work with the T-coercivity to study (3.4). In particular, we obtained criteria involving only the values of A , n in a neighborhood of $\partial\Omega$ to ensure that the set of transmission eigenvalues is discrete. Let us explain how we proceeded.

Again, the starting point of the T-coercivity is to observe that when T is an isomorphism of X , the pair $(u, w) \in X$ satisfies

$$a_k((u, w), (u', w')) = 0, \quad \forall (u', w') \in X$$

if and only if it solves

$$a_k^T((u, w), (u', w')) = 0, \quad \forall (u', w') \in X,$$

where

$$a_k^T((u, w), (u', w')) := a_k((u, w), T(u', w')).$$

Then we have to find some T such that $a_k^T(\cdot, \cdot)$ is coercive or of the form “coercive+compact”. The main difference with what was done in Chapter 1 is that here u and w are defined in the same domain. Therefore we do not need to work with the transfer operators based on geometrical constructions (see *e.g.* the $R_\pm$ in (1.15)).

Case $A \leq A^* \text{Id} < \text{Id}$ in Ω

To set ideas, consider the case where $A_+ < 1$ and $n_+ < 1$ in (3.2). Define T such that

$$T(u, w) := (u - 2w, -w). \quad (3.6)$$

Note that T is indeed an isomorphism because we have $T \circ T = \text{Id}$. Using Young’s inequality, one finds for $k = i\kappa$ with $\kappa > 0$, for $\alpha, \beta > 0$ and for all $(u, w) \in X$,

$$\begin{aligned} & |a_k^T((u, w), (u, w))| \\ &= |(A\nabla u, \nabla u)_\Omega + (\nabla w, \nabla w)_\Omega - 2(A\nabla u, \nabla w)_\Omega + \kappa^2((nu, u)_\Omega + (w, w)_\Omega - 2(nu, w)_\Omega)| \\ &\geq (A\nabla u, \nabla u)_\Omega + (\nabla w, \nabla w)_\Omega + \kappa^2((nu, u)_\Omega + (w, w)_\Omega) - 2|(A\nabla u, \nabla w)_\Omega| - 2\kappa^2|(nu, w)_\Omega| \\ &\geq (1 - \alpha)(A\nabla u, \nabla u)_\Omega + (1 - \alpha^{-1}A_+)(\nabla w, \nabla w)_\Omega + \kappa^2((1 - \beta)(nu, u)_\Omega + (1 - \beta^{-1}n_+)(w, w)_\Omega). \end{aligned}$$

By choosing α and β such that $A_+ < \alpha < 1$ and $n_+ < \beta < 1$, this estimate proves that a_k^T is coercive in X . From the Lax-Milgram theorem, and since T is an isomorphism of X , we deduce that $\mathcal{A}_k$ is an isomorphism of X for $k = i\kappa$ with $\kappa > 0$. But since the embedding of X in $L^2(\Omega) \times L^2(\Omega)$ is compact, for all $k \in \mathbb{C}$, $\mathcal{A}_k - \mathcal{A}_i$ is a compact operator. Consequently, the analytic Fredholm theorem guarantees that when $A_+ < 1$ and $n_+ < 1$, the set of transmission eigenvalues is discrete

and countable in $\mathbb{C}$.

From this first result, we explained in [71] how to weaken the assumptions made on A and n to ensure discreteness of transmission eigenvalues. In Chapter 1, we mentioned that the transmission problem between a positive material and a negative one is well-posed in the Fredholm sense as soon as the contrast is larger or smaller than -1 in a neighborhood of a $\mathcal{C}^1$ interface. For the ITP, the boundary $\partial\Omega$ plays the role of the interface. Quite naturally we established in [71] that the ITP is well-posed in the Fredholm sense when A is larger or smaller than the identity, in the sense of hermitian matrices, in a neighborhood of $\partial\Omega$. Moreover, since u, w are defined in the same domain, this is true for boundaries $\partial\Omega$ which have only Lipschitz regularity. Below we present the main results proved in [71].

Case $A \leq A^*\text{Id} < \text{Id}$ in a neighborhood of $\partial\Omega$

Assume that there is a neighborhood $\mathcal{V}$ of $\partial\Omega$, *i.e.* an open set of $\mathbb{R}^d$ such that $\partial\Omega \subset \mathcal{V}$, and a constant A^* such that, in the sense of hermitian matrices, $A \leq A^*\text{Id} < \text{Id}$ a.e. in $\Omega \cap \mathcal{V}$. By working with an *ad hoc* cut-off function in the definition of $\mathbf{T}$ in (3.6), we established in [71] the following result.

Lemma 3.1.2 *Assume that $A \leq A^*\text{Id} < \text{Id}$ and $n \leq n^* < 1$ a.e. in $\Omega \cap \mathcal{V}$. Then there exists $k = i\kappa$, with $\kappa \in \mathbb{R}$, such that the operator $\mathcal{A}_k : X \rightarrow X$ is an isomorphism.*

Remark 3.1.3 *In this lemma, we do not make any assumption concerning the sign of $A - \text{Id}$ and $n - 1$ in $\Omega \setminus \bar{\mathcal{V}}$.*

Consider $\kappa_0 \in \mathbb{R}$ such that $\mathcal{A}_{i\kappa_0} : X \rightarrow X$ is an isomorphism. Since $\mathcal{A}_k - \mathcal{A}_{i\kappa_0}$ is a compact operator of X for all $k \in \mathbb{C}$, by using the analytic Fredholm theorem, we obtain the following statement.

Theorem 3.1.4 *Assume that $A \leq A^*\text{Id} < \text{Id}$ and $n \leq n^* < 1$ a.e. in $\Omega \cap \mathcal{V}$. Then the set of transmission eigenvalues is discrete in $\mathbb{C}$.*

From Lemma 3.1.2, we also deduce in a straightforward way the following proposition.

Proposition 3.1.5 *Assume only that $A \leq A^*\text{Id} < \text{Id}$ a.e. in $\Omega \cap \mathcal{V}$. Then for all $k \in \mathbb{C}$, the operator $\mathcal{A}_k : X \rightarrow X$ is Fredholm of index zero.*

Now, let us state a result of localization of the transmission eigenvalues (see an illustration with Figure 3.1).

Theorem 3.1.6 *Assume that $A \leq A^*\text{Id} < \text{Id}$ and $n \leq n^* < 1$ a.e. in $\Omega \cap \mathcal{V}$. There exist two positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re k| < \delta |\Im k|$, then k is not a transmission eigenvalue.*

With a stronger assumption on A , we can weaken the condition on n . By taking $u' = w' = 1$ in (3.4), we remark first that transmission eigenfunction (u, w) satisfy $k^2 \int_{\Omega} nu - w \, d\mathbf{x} = 0$. This leads us to introduce the space $Y := \{(u, w) \in X \mid \int_{\Omega} nu - w \, d\mathbf{x} = 0\}$. Now assume that $\int_{\Omega} n - 1 \, d\mathbf{x} \neq 0$. By proceeding by contradiction, one proves the existence of a constant $C_P > 0$ (which depends on Ω and also on n through Y) such that

$$\|u\|_{\Omega}^2 + \|w\|_{\Omega}^2 \leq C_P (\|\nabla u\|_{\Omega}^2 + \|\nabla w\|_{\Omega}^2), \quad \forall (u, w) \in Y. \quad (3.7)$$

Moreover, one can verify that $k \neq 0$ is a transmission eigenvalue if and only if there exists a non zero pair $(u, w) \in Y$ such that $a_k((u, w), (u', w')) = 0$ for all $(u', w') \in Y$.

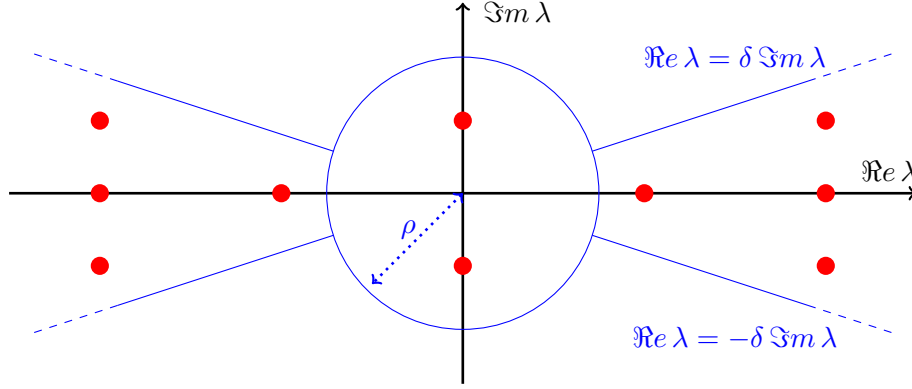


Figure 3.1: Localization of transmission eigenvalues in the complex plane when $A \leq A^* \text{Id} < \text{Id}$ and $n \leq n^* < 1$ a.e. in $\Omega \cap \mathcal{V}$.

Theorem 3.1.7 *Assume that $\int_{\Omega} n - 1 d\mathbf{x} \neq 0$ and $A_+ < 1$. Then the set of transmission eigenvalues is discrete and countable in $\mathbb{C}$. Besides, if $k \in \mathbb{C}^*$ satisfies the estimate $|k|^2 < (A_-(1 - \sqrt{A_+})) / (C_P \max(n_+, 1)(1 + \sqrt{n_+}))$, with C_P defined in (3.7), then k is not a transmission eigenvalue.*

Let us mention that one has similar results when A is larger than the identity in a neighborhood of $\partial\Omega$ (see [71]). Note that this tool of the T-coercivity was new in the study of the ITP and this result of discreteness of transmission eigenvalues under local assumption on A , n in a neighborhood of $\partial\Omega$ was not known. Since then, other approaches have been proposed in [321, 393] leading to refined criteria on A , n ensuring discreteness of transmission eigenvalues.

In a subsequent work [150], I adapted the technique to deal with Maxwell's equations. We will not present these results here.

3.1.2 Interior transmission problem with strongly oscillating singularities

In the article [65], with A.-S. Bonnet-Ben Dhia, we consider Problem (3.3) in situations where $A - \text{Id}$ changes sign on $\partial\Omega$, cases which had not been studied before. In a first step, under some assumptions on the parameters, we extended the variational approach of the T-coercivity to prove that the transmission eigenvalues form at most a discrete set. In the process, we also provide localization results similar to the ones of Theorem 3.1.6.

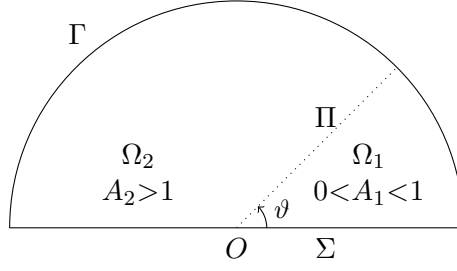
Then, we studied what happens when these assumptions are not satisfied. The main result is that oscillating singularities similar to the ones met in §1.3 can exist. In this case, the operator $\mathcal{A}_k$ defined in (3.5) in the H^1 framework is not of Fredholm type.

To simplify the presentation of the approach, let us work on a model problem.

Setting of the model problem

Denote by (r, θ) the polar coordinates centered at the origin O and for $\vartheta \in (0; \pi)$, define the sets

$$\begin{aligned} \Omega_1 &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, 0 < \theta < \vartheta\}; \\ \Omega_2 &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, \vartheta < \theta < \pi\}; \\ \Omega &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, 0 < \theta < \pi\}; \\ \Gamma &:= \{(r \cos \theta, r \sin \theta) \mid r = 1, 0 < \theta < \pi\}; \\ \Pi &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, \theta = \vartheta\}; \\ \Sigma &:= (-1; 1) \times \{0\}; \end{aligned}$$

Figure 3.2: Geometry of the domain Ω .

(see Figure 3.2). In particular, the boundary of Ω satisfies $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$. We introduce the function $A : \Omega \rightarrow \mathbb{R}$ such that $A = A_1$ in Ω_1 and $A = A_2$ in Ω_2 . Here $0 < A_1 < 1$ and $A_2 > 1$ are two constants. Our objective here is to understand the properties of the following problem

$$\left\{ \begin{array}{ll} \text{Find } (u, w) \in H^1(\Omega) \times H^1(\Omega) \text{ such that} \\ \operatorname{div}(A\nabla u) &= f \quad \text{in } \Omega \\ \Delta w &= g \quad \text{in } \Omega \\ u - w &= 0 \quad \text{on } \Sigma, & u = 0 \quad \text{on } \Gamma \\ \nu \cdot A\nabla u - \nu \cdot \nabla w &= 0 \quad \text{on } \Sigma, & w = 0 \quad \text{on } \Gamma. \end{array} \right. \quad (3.8)$$

In (3.8), ν denotes the unit outward normal vector to $\partial\Omega$. Moreover, f and g are two source terms which belong to some functional spaces which will be specified later. Compared to (3.3), in (3.8) the transmission conditions are written only on the part Σ of the boundary. Moreover we work at $k = 0$ to consider only the principal part of the associated operator.

Define the space $X_\Sigma := \{(u, w) \in H_\Gamma^1(\Omega) \times H_\Gamma^1(\Omega) \mid u - w = 0 \text{ on } \Sigma\}$ where $H_\Gamma^1(\Omega) := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma\}$. In the sequel, we will also use the space $H_\Gamma^1(\Omega_2) = \{\varphi|_{\Omega_2} \mid \varphi \in H_\Gamma^1(\Omega)\}$. Denote by $b(\cdot, \cdot)$ the sesquilinear form associated with (3.8) such that

$$b((u, w), (u', w')) := (A\nabla u, \nabla u')_\Omega - (\nabla w, \nabla w')_\Omega$$

and consider the variational problem

$$\left\{ \begin{array}{l} \text{Find } (u, w) \in X_\Sigma \text{ such that, for all } (u', w') \in X_\Sigma, \\ b((u, w), (u', w')) = F((u', w')), \end{array} \right.$$

where F is a continuous antilinear form in X_Σ . With the Riesz representation theorem, we define the operator $\mathcal{B} : X_\Sigma \rightarrow X_\Sigma$ such that, for all $(u, w), (u', w') \in X_\Sigma$,

$$(\mathcal{B}(u, w), (u', w'))_{H^1(\Omega) \times H^1(\Omega)} = b((u, w), (u', w')). \quad (3.9)$$

Again, the study of $\mathcal{B}$ is not simple because $b(\cdot, \cdot)$ is not coercive in X_Σ nor of the form “coercive+compact”.

Analysis via the T-coercivity approach

In this paragraph, we explain how to study the operator $\mathcal{B}$ using the variational approach of the T-coercivity. However, since we have neither $A > 1$ nor $A < 1$ on Σ , the computations presented in §3.1.1 fail. We must work with a different isomorphism $T : X_\Sigma \rightarrow X_\Sigma$. If φ is a measurable function on Ω , we denote $\varphi_1 := \varphi|_{\Omega_1}$, $\varphi_2 := \varphi|_{\Omega_2}$. Set

$$T(u, w) = (u', w'), \quad \text{with} \quad (u', w') = \begin{cases} (u_1 - 2w_1 + 2R_2w_2, -w_1 + 2R_2u_2) & \text{on } \Omega_1 \\ (u_2, -w_2 + 2u_2) & \text{on } \Omega_2. \end{cases} \quad (3.10)$$

In this definition, R_2 is the operator such that, on Ω_1 , $(R_2\varphi_2)(r, \theta) = \varphi_2(r, \frac{\vartheta-\pi}{\vartheta}\theta + \pi)$ for $\varphi_2 \in H^1_\Gamma(\Omega_2)$. First, notice that $R_2\varphi_2 = \varphi_2$ on Π for all $\varphi_2 \in H^1_\Gamma(\Omega_2)$. As a consequence, one can check that $u' \in H^1_\Gamma(\Omega)$ and $w' \in H^1_\Gamma(\Omega)$. Then, one can verify that $u' - w' \in H^1_0(\Omega)$ so that the operator $\mathbf{T}$ defined in (3.10) is indeed valued in X_Σ . Finally, noticing that $\mathbf{T}^2$ is equal to the identity of X_Σ , we deduce that $\mathbf{T}$ is an isomorphism. For all $(u, w) \in X_\Sigma$, we find

$$\begin{aligned} b((u, w), \mathbf{T}(u, w)) &= (A\nabla u, \nabla u)_\Omega + (\nabla w, \nabla w)_\Omega \\ &\quad + 2(A_1\nabla u_1, \nabla(-w_1 + R_2w_2))_{\Omega_1} - 2(\nabla w_1, \nabla(R_2u_2))_{\Omega_1} - 2(\nabla w_2, \nabla u_2)_{\Omega_2}. \end{aligned}$$

Let us introduce $\alpha, \beta, \gamma, \eta$ four strictly positive parameters. Using Young's inequality and setting

$$\|R_2\| := \sup_{\varphi_2 \in H^1_\Gamma(\Omega_2), \|\nabla\varphi_2\|_{\Omega_2}=1} \|\nabla(R_2\varphi_2)\|_{\Omega_1},$$

we can write

$$\begin{aligned} &|2(A_1\nabla u_1, \nabla(-w_1 + R_2w_2))_{\Omega_1}| \\ &\leq A_1(\alpha + \beta)(\nabla u_1, \nabla u_1)_{\Omega_1} + A_1\alpha^{-1}(\nabla w_1, \nabla w_1)_{\Omega_1} + A_1\beta^{-1}\|R_2\|^2(\nabla w_2, \nabla w_2)_{\Omega_2}, \\ &|2(\nabla w_1, \nabla(R_2u_2))_{\Omega_1}| \leq \gamma(\nabla w_1, \nabla w_1)_{\Omega_1} + \gamma^{-1}\|R_2\|^2(\nabla u_2, \nabla u_2)_{\Omega_2}, \\ &|2(\nabla w_2, \nabla u_2)_{\Omega_2}| \leq \eta(\nabla w_2, \nabla w_2)_{\Omega_2} + \eta^{-1}(\nabla u_2, \nabla u_2)_{\Omega_2}. \end{aligned}$$

We deduce

$$\begin{aligned} b((u, w), \mathbf{T}(u, w)) &\geq A_1(1 - \alpha - \beta)(\nabla u_1, \nabla u_1)_{\Omega_1} + (A_2 - \gamma^{-1}\|R_2\|^2 - \eta^{-1})(\nabla u_2, \nabla u_2)_{\Omega_2} \\ &\quad + (1 - \alpha^{-1}A_1 - \gamma)(\nabla w_1, \nabla w_1)_{\Omega_1} + (1 - A_1\beta^{-1}\|R_2\|^2 - \eta)(\nabla w_2, \nabla w_2)_{\Omega_2}. \end{aligned}$$

In [66], we obtained $\|R_2\|^2 = \Upsilon_\vartheta$ where

$$\Upsilon_\vartheta := \max\left(\frac{\pi - \vartheta}{\vartheta}, \frac{\vartheta}{\pi - \vartheta}\right). \quad (3.11)$$

Notice that there holds $\Upsilon_\vartheta \geq 1$ with $\Upsilon_\vartheta = 1$ if and only if $\vartheta = \pi/2$. Exploiting this result, we proved in [65] the following proposition.

Proposition 3.1.8 *Assume that A_1, A_2 satisfy*

$$A_1 < \frac{1}{1 + \Upsilon_\vartheta} \quad \text{and} \quad A_2 > \frac{1 + \Upsilon_\vartheta}{1 - A_1(1 + \Upsilon_\vartheta)}, \quad (3.12)$$

where Υ_ϑ appears in (3.11). Then the operator $\mathcal{B} : X_\Sigma \rightarrow X_\Sigma$ defined in (3.9) is an isomorphism.

Remark 3.1.9 *Thus, when $A_1 < \min(\vartheta/\pi, (\pi - \vartheta)/\pi)$, $\mathcal{B}$ is an isomorphism for A_2 large enough.*

Now, the question we propose to investigate is the following. Are the conditions of Proposition 3.1.8 necessary? In other words, what happens for the operator $\mathcal{B}$ when A does not meet the assumptions (3.12)? To answer this question, we have to compute the singularities of the problem. To facilitate the calculus, we will work not directly on the simplified interior transmission problem (3.8) but on an equivalent formulation. This equivalent problem is obtained from (3.8) by unfolding the domain Ω . First we explain this unfolding procedure. Then we compute the singularities in the unfolded domain.

The unfolding procedure

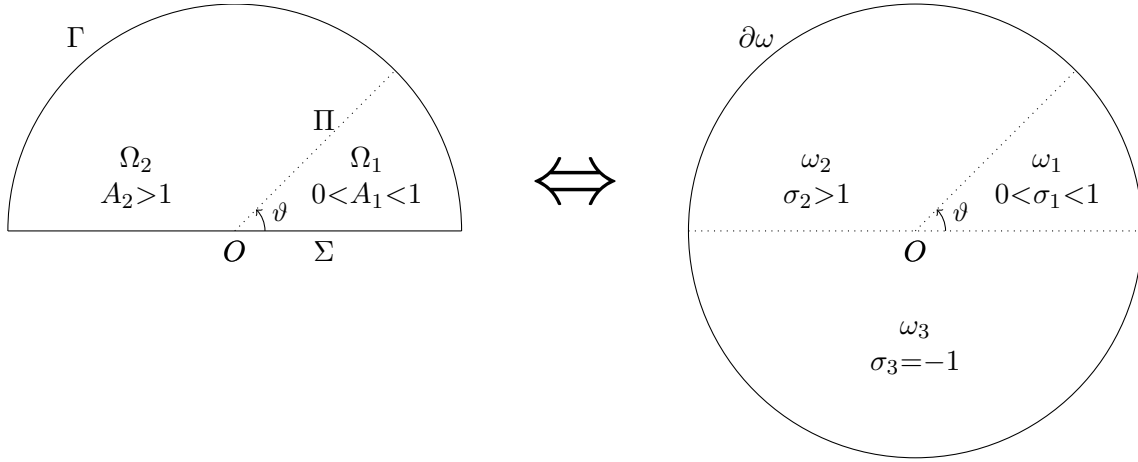


Figure 3.3: Initial domain Ω and unfolded domain ω .

Define the sets

$$\begin{aligned}\omega_1 &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, 0 < \theta < \vartheta\}; \\ \omega_2 &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, \vartheta < \theta < \pi\}; \\ \omega_3 &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, \pi < \theta < 2\pi\}; \\ \omega &:= \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, 0 \leq \theta < 2\pi\};\end{aligned}$$

(see Figure 3.3 right). Let σ denote the function such that $\sigma(\theta) = \sigma_1$ for $\theta \in (0; \vartheta)$, $\sigma(\theta) = \sigma_2$ for $\theta \in (\vartheta; \pi)$ and $\sigma(\theta) = \sigma_3 = -1$ for $\theta \in (\pi; 2\pi)$. Here $0 < \sigma_1 < 1$ and $\sigma_2 > 1$ are two constants. Now, we consider the problem

$$\left\{ \begin{array}{l} \text{Find } v \in H_0^1(\omega) \text{ such that} \\ \operatorname{div}(\sigma \nabla v) = h \quad \text{in } \omega, \end{array} \right. \quad (3.13)$$

where $H_0^1(\omega) := \{\varphi \in H^1(\omega) \mid \varphi = 0 \text{ on } \partial\omega\}$. Problem (3.13) can be seen as a transmission problem between a positive composite material filling the region $\omega_1 \cup \omega_2$ and a negative material in ω_3 . Let us assume that the source terms in (3.8) and (3.13) are such that $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ and $h \in L^2(\omega)$. Moreover, let us assume that h satisfies $h(x, y) = f(x, y)$ a.e. in $\omega_1 \cup \omega_2$ and $h(x, y) = -g(x, -y)$ a.e. in ω_3 .

★ Consider (u, w) a solution, if it exists, of the simplified interior transmission problem (3.8). Then define the function v such that $v = u$ on $\omega_1 \cup \omega_2$ and $v(x, y) = w(x, -y)$ a.e. on ω_3 . It is easy to check that v solves the transmission problem with a sign-changing coefficient (3.13) for $\sigma_1 = A_1$ and $\sigma_2 = A_2$.

★ Conversely, if v satisfies the transmission problem with a sign-changing coefficient (3.13), define the functions u and w such that $u = v$ in Ω , $w(x, y) = v(x, -y)$ a.e. in Ω . Then, one can check that the pair (u, w) is a solution of the simplified interior transmission problem (3.8) for $A_1 = \sigma_1$ and $A_2 = \sigma_2$.

The variational problem associated with (3.13) writes

$$\left\{ \begin{array}{l} \text{Find } v \in H_0^1(\omega) \text{ such that, for all } v' \in H_0^1(\omega), \\ (\sigma \nabla v, \nabla v')_\omega = \ell(v'), \end{array} \right. \quad (3.14)$$

where $\ell(\cdot)$ is an antilinear continuous form in $H_0^1(\omega)$. The main difficulty in the investigation of the properties of (3.14) comes from the triple point O : at this point, the solutions can be very singular.

Description of singularities

Let us look for functions of the form $s(r, \theta) = \kappa(r)\phi(\theta)$ (with separated variables in polar coordinates) which satisfy

$$\operatorname{div}(\sigma \nabla s) = r^{-2} \left(\sigma \phi (r \partial_r)^2 \kappa + \kappa (\partial_\theta \sigma \partial_\theta) \phi \right) = 0.$$

The above equation has been obtained by noticing that σ depends only on θ . Then we find that κ , ϕ must satisfy

$$(r \partial_r)^2 \kappa = \lambda^2 \kappa \quad \text{on } (0; +\infty) \quad \text{and} \quad (\partial_\theta \sigma \partial_\theta) \phi = -\lambda^2 \sigma \phi \quad \text{on } \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}),$$

for some constant $\lambda \in \mathbb{C}$. For $\lambda = 0$, one gets $\kappa(r) = A \ln r + B$ whereas for $\lambda \neq 0$, one obtains $\kappa(r) = A r^\lambda + B r^{-\lambda}$, A, B being two constants. On the other hand, one finds that (λ, ϕ) must be an eigenpair of the spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda, \phi) \in \mathbb{C} \times H^1(\mathbb{T}) \setminus \{0\} \text{ such that} \\ \frac{\partial}{\partial \theta} \left(\sigma \frac{\partial}{\partial \theta} \phi \right) = -\lambda^2 \sigma \phi \quad \text{on } \mathbb{T}. \end{array} \right. \quad (3.15)$$

The study of this problem leads us to consider the symbol $\mathcal{L}$ such that for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} \mathcal{L}(\lambda) : H^1(\mathbb{T}) &\longrightarrow H^1(\mathbb{T})^* \\ \phi &\longmapsto \partial_\theta(\sigma \partial_\theta \phi) + \lambda^2 \sigma \phi. \end{aligned}$$

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}$ when there exists a non trivial $\phi \in H^1(\mathbb{T})$ such that $\mathcal{L}(\lambda)\phi = 0$. We denote by Λ the set of the eigenvalues of $\mathcal{L}$. This set is also called the set of singular exponents associated with O . As for the interior transmission eigenvalue problem, the study of (3.15) is not standard because the sign-changing parameter σ appears both in the principal and in the compact part of the equation. However, now it is only a 1D problem. Proceeding as in [153, Lem. 4.9], we can prove the

Proposition 3.1.10 *Assume that $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$. Then, the set Λ is discrete. Moreover, there exist two positive constants ρ and δ such that if $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \rho$ and $|\operatorname{Re} \lambda| < \delta |\operatorname{Im} \lambda|$, then λ is not a singular exponent.*

Generally speaking, we can decompose the solutions of the source term problem (3.14) as the sum of a finite number of singularities and a regular term. The computations of the beginning of this paragraph show that for $\lambda \neq 0$, the singularities are of the form

$$(r, \theta) \mapsto r^\lambda \phi(\theta), \quad (3.16)$$

where (λ, ϕ) corresponds to an eigenpair of problem (3.15). Therefore, the regularity of the solutions of problem (3.14) only depends on the set Λ . For our study, we are particularly interested in the set $\Lambda \cap \mathbb{R}i \setminus \{0\}$, the reason being that for $\lambda \in \mathbb{R}i \setminus \{0\}$, as seen in §1.3, the singularity (3.16) is strongly oscillating and responsible for the loss of Fredholmness in the H^1 framework.

Explicit computation of the singularities in the case $\vartheta = \pi/2$

In [65], we were able to compute explicitly the set Λ in the particular case

$$\vartheta = \pi/2.$$

We simply present the results here. Define the region

$$\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4,$$

$$\begin{aligned}
\text{with } \mathcal{R}_1 &= \{(\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \sigma_1 > 1 \text{ and } \sigma_2 < 2 - \sigma_1\} \\
\mathcal{R}_2 &= \{(\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \sigma_1 > 1 \text{ and } \frac{\sigma_1}{2\sigma_1 - 1} < \sigma_2 < 1\} \\
\mathcal{R}_3 &= \{(\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \sigma_2 > 1 \text{ and } \sigma_1 < 2 - \sigma_2\} \\
\mathcal{R}_4 &= \{(\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \sigma_2 > 1 \text{ and } \frac{\sigma_2}{2\sigma_2 - 1} < \sigma_1 < 1\}
\end{aligned}$$

(see the representation of Figure 3.4).

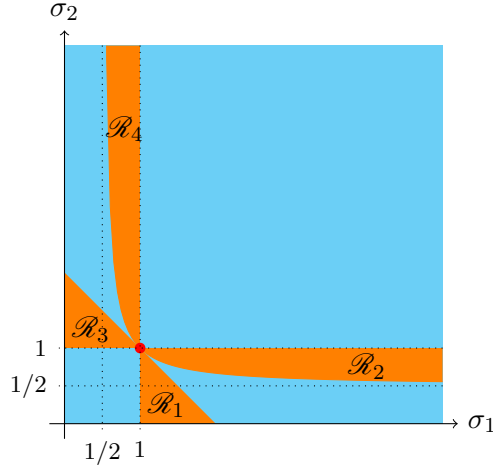


Figure 3.4: Representation of the set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$ in orange. For $(\sigma_1, \sigma_2) \in \mathcal{R}$, strongly oscillating singularities exist for Problem (3.13).

We established the following statement.

Proposition 3.1.11

For $(\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \setminus \overline{\mathcal{R}}$, one has $\Lambda \cap \mathbb{R}i = \{0\}$.

For $(\sigma_1, \sigma_2) \in \mathcal{R}$, one has $\Lambda \cap \mathbb{R}i = \{0, \pm i\eta\}$ for some real number $\eta > 0$.

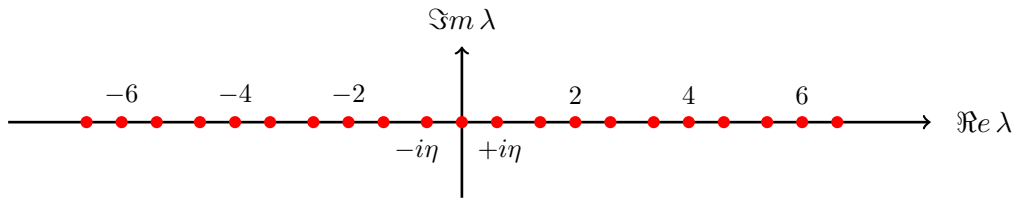


Figure 3.5: Set Λ for $(\sigma_1, \sigma_2) = (1/4, 2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \setminus \overline{\mathcal{R}}$.

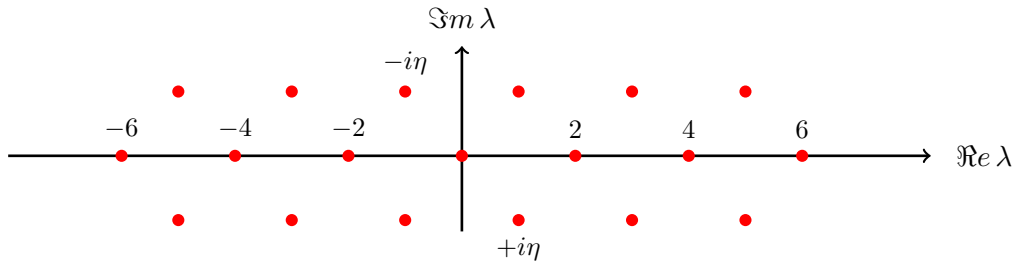


Figure 3.6: Set Λ for $(\sigma_1, \sigma_2) = (4, 2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \setminus \overline{\mathcal{R}}$.

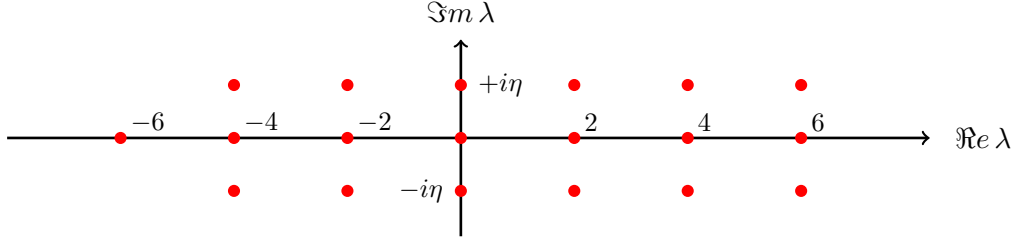


Figure 3.7: Set Λ for $(\sigma_1, \sigma_2) = (3/4, 2) \in \mathcal{R}$. Notice the two non trivial singular exponents on $\mathbb{R}i$.

Next steps of the analysis

When $(\sigma_1, \sigma_2) \in \mathcal{R}$, strongly oscillating singularities exist for Problem (3.13). As a consequence, the associated operator is not Fredholm in $H_0^1(\omega)$. Using the equivalence schematized in Figure 3.3, we deduce that the operator $\mathcal{B} : X_\Sigma \rightarrow X_\Sigma$ defined in (3.9) is not of Fredholm type when $(A_1, A_2) \in \mathcal{R}$ (and $\vartheta = \pi/2$). Above we focused our attention on Problem (3.8) to underline the connection existing with Problem (3.13) with a sign-changing coefficient. The results are similar for the true ITP (3.3). In the article [65], we then explained how to take into account the strongly oscillating singularities in the functional framework to obtain a Fredholm operator. The approach is the same as the one presented in §1.3.2 and we will not detail it. Let us emphasize that exhibiting a framework where Fredholmness holds is the first step to apply the analytic Fredholm theorem and prove that the set of TEs is discrete. To complete the proof, it is necessary to exhibit at least one k which is not a TE. However for now, we have not been able to obtain this result.

With A.-S. Bonnet-Ben Dhia and F. Monteghetti, we are currently continuing the study. Working with Perfectly Matched Layers (PMLs) as in §1.3.3, Florian has written a code based on a finite element method to compute in a robust way the spectrum of TEs. It is important to emphasize that if we do not use the PMLs, the computed eigenvalues do not converge as the mesh is refined. Therefore it is crucial to take into account the oscillating singularities to avoid spurious eigenvalues. The eigenfunctions we obtain in this setting may have singular behaviors which are not expected for the solutions of the direct problem (note that the composite material filling Ω is a classical positive material). We are investigating what happens for the LSM, the initial reconstruction technique, at these frequencies.

3.1.3 A bilaplacian with a sign changing coefficient

In §3.1.1, we considered cases where $A - \text{Id}$ is positive or negative in a neighborhood of $\partial\Omega$. Then in §3.1.2, we investigated situations where $A - \text{Id}$ changes sign on $\partial\Omega$. In the article [151], I studied the properties of (3.3) when $A = \text{Id}$. Then the ITP writes

$$\begin{cases} \Delta u + k^2 n u & = 0 & \text{in } \Omega \\ \Delta w + k^2 w & = 0 & \text{in } \Omega \\ u - w & = 0 & \text{on } \partial\Omega \\ \nu \cdot \nabla u - \nu \cdot \nabla w & = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.17)$$

According to the analysis presented in §3.1.2, this situation is similar to the case of a contrast equal to -1 for the transmission problem with a sign-changing coefficient. In that case, the corresponding operator is not of Fredholm type in the H^1 framework (see Theorem 1.2.4). This is the reason why people have considered a different setting. For Problem (3.17), one usually adopts the following definition for the transmission eigenvalues.

Definition 3.1.12 *The $k \in \mathbb{C}$ for which (3.17) admits non zero solutions $(u, w) \in L^2(\Omega) \times L^2(\Omega)$ such that $u - w \in H_0^2(\Omega)$ are called interior transmission eigenvalues.*

Here and below $H_0^2(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ for the H^2 -norm. Note that in Definition 3.1.12, the fields are only in $L^2(\Omega)$ but their difference is smooth. Now, following [437], we rewrite (3.17) as a fourth order equation. Consider (u, w) an eigenpair of (3.17). Define $v := u - w$. Notice that v is nothing else than the scattered field inside the inclusion. It satisfies the relation

$$\Delta v + k^2 n v = -k^2(n-1)w \quad \text{in } \Omega. \quad (3.18)$$

Assume that $n : \Omega \rightarrow \mathbb{R}$ is an element of $L^\infty(\Omega)$ such that $n > 0$ and $n \neq 1$ in Ω . Assume also that $(n-1)^{-1}$ belongs to $L^\infty(\Omega)$ ¹. Dividing on each side of (3.18) by $n-1$ and using the equation $\Delta w + k^2 w = 0$, we obtain, in the sense of distributions,

$$(\Delta + k^2) \left(\frac{1}{n-1} (\Delta v + k^2 n v) \right) = 0 \quad \text{in } \Omega.$$

We deduce that if the pair (u, w) satisfies (3.17) then $v = u - w$ solves the problem

$$\left| \begin{array}{l} \text{Find } v \in H_0^2(\Omega) \text{ such that} \\ \int_{\Omega} \frac{1}{n-1} (\Delta v + k^2 n v) (\Delta v' + k^2 v') d\mathbf{x} = 0, \quad \forall v' \in H_0^2(\Omega). \end{array} \right. \quad (3.19)$$

Conversely, one shows (see [437, Lem. 3.1] for the details) that if v is a solution of (3.19) then the pair $(u, w) := ((n-1)^{-1}(\Delta v + k^2 n v) - k^2 v, (n-1)^{-1}(\Delta v + k^2 n v))$ satisfies Problem (3.17). In $H_0^2(\Omega)$, for $k \in \mathbb{C}$, we define the sesquilinear form $a_k(\cdot, \cdot)$ such that

$$a_k(v, v') = ((n-1)^{-1}(\Delta v + k^2 n v), (\Delta v' + k^2 v'))_{\Omega}, \quad \forall v, v' \in H_0^2(\Omega).$$

With the Riesz representation theorem, let us introduce the bounded operator $\mathcal{A}_k : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ associated with a_k such that

$$(\mathcal{A}_k v, v')_{H_0^2(\Omega)} = a_k(v, v'), \quad \forall v, v' \in H_0^2(\Omega). \quad (3.20)$$

Above $(\cdot, \cdot)_{H_0^2(\Omega)}$ stands for the inner product of $H_0^2(\Omega)$ such that

$$(\varphi, \varphi')_{H_0^2(\Omega)} = (\Delta \varphi, \Delta \varphi')_{\Omega}, \quad \forall \varphi, \varphi' \in H_0^2(\Omega).$$

Observe that contrary to (3.4), (3.19) is not linear in k^2 but quadratic. Assume that there exists $k_0 \in \mathbb{C}$ such that $\mathcal{A}_{k_0}$ is an isomorphism of $H_0^2(\Omega)$. For all $k \in \mathbb{C}$, it is easy to see that $\mathcal{A}_k - \mathcal{A}_{k_0}$ is a compact operator of $H_0^2(\Omega)$. With the analytic Fredholm theorem, we deduce, since $\mathcal{A}_{k_0}$ is injective, that $\mathcal{A}_k$ is an isomorphism of $H_0^2(\Omega)$ for all $k \in \mathbb{C} \setminus \mathcal{S}$, where $\mathcal{S}$ is a discrete or empty set of $\mathbb{C}$. In other words, to prove that the set of transmission eigenvalues is discrete, it is sufficient to establish that there exists $k_0 \in \mathbb{C}$ such that $\mathcal{A}_{k_0}$ is an isomorphism of $H_0^2(\Omega)$.

When $n \geq C > 1$ in Ω for some constant C , the bilinear form $a_0(\cdot, \cdot)$ is coercive in $H_0^2(\Omega)$. This allows one to prove that $\mathcal{A}_0$ is an isomorphism of $H_0^2(\Omega)$, and as a consequence, that the set of transmission eigenvalues is discrete or empty. The case $0 < C_1 \leq n \leq C_2 < 1$, where C_1, C_2 are some constants, is similar because $-a_0(\cdot, \cdot)$ is coercive in $H_0^2(\Omega)$. These results have been known since [437]. When $n-1$ changes sign in Ω , the form $a_k(\cdot, \cdot)$ is no longer coercive nor “coercive+compact” in $H_0^2(\Omega)$. Is the set of transmission eigenvalues still discrete in this case? And before that, what are the properties of $\mathcal{A}_k$ in such configurations? This was the motivation of the study [151] that we summarize now. Let us emphasize that this does not correspond to a particular exotic configuration. Indeed, one meets this problem simply when considering the scattering of waves by an inclusion made of two different materials: one with an index $n > 1$, the other such that $n < 1$.

¹For the study of the Interior Transmission Eigenvalue Problem when n is smooth, one may consult [322, 435].

Let us focus on the principal part of $\mathcal{A}_k$, the part which contains the derivatives of highest order. To simplify the notation, set $\sigma := (n-1)^{-1} \in L^\infty(\Omega)$. We assume that σ is real valued. Our goal is to investigate the features of the following problem

$$(\mathcal{P}) \left| \begin{array}{l} \text{Find } v \in H_0^2(\Omega) \text{ such that} \\ (\sigma \Delta v, \Delta v')_\Omega = \ell(v'), \quad \forall v' \in H_0^2(\Omega), \end{array} \right.$$

where $\ell(\cdot)$ is a continuous antilinear form in $H_0^2(\Omega)$. Let us introduce the sesquilinear form $b(\cdot, \cdot)$ such that

$$b(v, v') = (\sigma \Delta v, \Delta v')_\Omega, \quad \forall v, v' \in H_0^2(\Omega)$$

and the continuous operator $B : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ defined by

$$(Bv, v')_{H_0^2(\Omega)} = b(v, v'), \quad \forall v, v' \in H_0^2(\Omega). \quad (3.21)$$

The goal of [151] was also to compare the features of the operators $\operatorname{div}(\sigma \nabla \cdot) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ and $\Delta(\sigma \Delta \cdot) : H_0^2(\Omega) \rightarrow H^{-2}(\Omega)$. Has the change of sign of σ the same consequences in each case?

Bilaplacian with mixed boundary conditions

Before investigating the properties of the operator B , let us study the problem obtained by replacing in $(\mathcal{P})$ the boundary condition “ $\nu \cdot \nabla v = 0$ on $\partial\Omega$ ” by the condition “ $\sigma \Delta v = 0$ on $\partial\Omega$ ”. Thus, consider the problem with mixed boundary conditions

$$\left| \begin{array}{l} \text{Find } v \in H_0^1(\Omega) \cap H^2(\Omega) \text{ such that} \\ \Delta(\sigma \Delta v) = f \quad \text{in } \Omega \\ \sigma \Delta v = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (3.22)$$

Problem (3.22) is equivalent to the following variational formulation

$$(\tilde{\mathcal{P}}) \left| \begin{array}{l} \text{Find } v \in H_0^1(\Omega) \cap H^2(\Omega) \text{ such that} \\ (\sigma \Delta v, \Delta v')_\Omega = \langle f, v' \rangle_\Omega, \quad \forall v' \in H_0^1(\Omega) \cap H^2(\Omega). \end{array} \right.$$

With these mixed boundary conditions, one can solve $(\tilde{\mathcal{P}})$ in two steps. Assume that f belongs to $H^{-1}(\Omega)$. There exists a unique $p \in H_0^1(\Omega)$ satisfying $-(\nabla p, \nabla p')_\Omega = \langle f, p' \rangle_\Omega$ for all $p' \in H_0^1(\Omega)$. Let $v \in H_0^1(\Omega)$ be the function such that $\Delta v = \sigma^{-1}p \in L^2(\Omega)$. If the domain Ω is of class $\mathcal{C}^2$ ([236, Thm. 8.12]) or convex ([249, Thm. 3.2.1.2]), we know that v belongs to $H^2(\Omega)$. Moreover, there holds, for all $v' \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$(\sigma \Delta v, \Delta v')_\Omega = (p, \Delta v')_\Omega = -(\nabla p, \nabla v')_\Omega = \langle f, v' \rangle_\Omega.$$

We deduce that v is a solution of $(\tilde{\mathcal{P}})$. Notice that, to obtain this result, the only assumptions for σ are $\sigma \in L^\infty(\Omega)$ and $\sigma^{-1} \in L^\infty(\Omega)$. Thus, σ can change sign. Let us refine the study of $(\tilde{\mathcal{P}})$.

With the Lax-Milgram theorem, we can show that the sesquilinear form $(v, v') \mapsto (v, v')_{H_0^2(\Omega)} = (\Delta v, \Delta v')_\Omega$ is an inner product in $H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, if Ω is of class $\mathcal{C}^2$ or convex, in $H_0^1(\Omega) \cap H^2(\Omega)$, the map $v \mapsto \|\Delta v\|_{L^2(\Omega)}$ defines a norm which is equivalent to the H^2 -norm. Therefore, $H_0^1(\Omega) \cap H^2(\Omega)$ endowed with the inner product $(\cdot, \cdot)_{H_0^2(\Omega)}$ is a Hilbert space. Let us introduce the sesquilinear form $\tilde{b}(\cdot, \cdot)$ such that

$$\tilde{b}(v, v') = (\sigma \Delta v, \Delta v')_\Omega, \quad \forall v, v' \in H_0^1(\Omega) \cap H^2(\Omega),$$

and the continuous operator $\tilde{B} : H_0^1(\Omega) \cap H^2(\Omega) \rightarrow H_0^1(\Omega) \cap H^2(\Omega)$, defined by

$$(\tilde{B}v, v')_{H_0^2(\Omega)} = \tilde{b}(v, v'), \quad \forall v, v' \in H_0^1(\Omega) \cap H^2(\Omega). \quad (3.23)$$

Theorem 3.1.13 Assume that the domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is of class $\mathcal{C}^2$ or convex. For all $\sigma \in L^\infty(\Omega)$ such that $\sigma^{-1} \in L^\infty(\Omega)$, the operator $\tilde{B} : H_0^1(\Omega) \cap H^2(\Omega) \rightarrow H_0^1(\Omega) \cap H^2(\Omega)$ defined in (3.23) is an isomorphism.

Proof. To establish this result, let us work again with the T-coercivity. Introduce the operator $T : H_0^1(\Omega) \cap H^2(\Omega) \rightarrow H_0^1(\Omega) \cap H^2(\Omega)$ such that, for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$, Tv is defined as the unique solution of the problem “find $Tv \in H_0^1(\Omega)$ satisfying $\Delta(Tv) = \sigma^{-1}\Delta v$ ”. Notice that since Ω is assumed to be of class $\mathcal{C}^2$ or convex, Tv is indeed an element of $H^2(\Omega)$. For all $v, v' \in H_0^1(\Omega) \cap H^2(\Omega)$, we can write

$$(\tilde{B}(Tv), v')_{H_0^1(\Omega)} = (\sigma \Delta(Tv), \Delta v')_\Omega = (\Delta v, \Delta v')_\Omega.$$

Therefore, the operator $\tilde{B} \circ T$ is equal to the identity of $H_0^1(\Omega) \cap H^2(\Omega)$. Since $\tilde{B}$ is selfadjoint, we deduce that $\tilde{B}$ is an isomorphism with $\tilde{B}^{-1} = T$. ■

The operator $\tilde{B}$ defined in (3.23) is an isomorphism of $H_0^1(\Omega) \cap H^2(\Omega)$ when $\sigma \in L^\infty(\Omega)$ satisfies $\sigma^{-1} \in L^\infty(\Omega)$ and when Ω is such that $\Delta : H_0^1(\Omega) \cap H^2(\Omega) \rightarrow L^2(\Omega)$ is an isomorphism. But what if this last assumption on Ω is not met? What happens for example if Ω is a 2D domain with reentrant corners? It is proved in [151, §2.2] that in this case, $\tilde{B}$ is not always an isomorphism. According to the values of the parameter σ , a kernel and a cokernel, both of finite dimension, can appear.

Bilaplacian with Dirichlet boundary condition

Let us come back to the study of the operator $B : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ introduced in (3.21).

Configurations where σ has a constant sign on the boundary

In this paragraph, Ω is a domain of $\mathbb{R}^d$, with $d \geq 1$. We prove that B is Fredholm of index zero when σ has a constant sign in a neighborhood of $\partial\Omega$. More precisely, consider the following assumption:

$$(\mathcal{H}_\sigma) \quad \left| \begin{array}{l} \sigma \in L^\infty(\Omega) \text{ is such that } \sigma^{-1} \in L^\infty(\Omega). \text{ Moreover, there holds } \sigma \geq C_1 > 0 \text{ a.e. in } \Omega \setminus \overline{\mathcal{O}} \\ \text{or } \sigma \leq C_2 < 0 \text{ a.e. in } \Omega \setminus \overline{\mathcal{O}}, \text{ where } C_1, C_2 \text{ are constants and } \mathcal{O} \text{ is an open set such that } \\ \overline{\mathcal{O}} \subset \Omega. \end{array} \right.$$

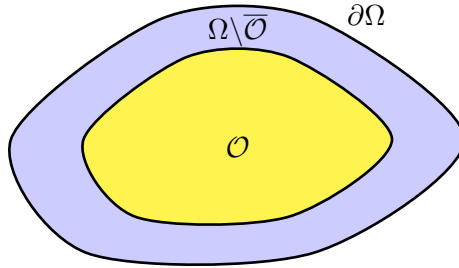


Figure 3.8: σ is assumed to be uniformly positive or uniformly negative in the region $\Omega \setminus \overline{\mathcal{O}}$.

Let us emphasize that in $(\mathcal{H}_\sigma)$, there is no assumption concerning the sign of σ in $\mathcal{O}$. To prove that B is a Fredholm operator of index zero, we build a right parametrix for B , i.e. we build a bounded operator T such that $B \circ T = \mathcal{I} + \mathcal{K}$ where $\mathcal{I} : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ is an isomorphism and $\mathcal{K} : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ is a compact operator.

Theorem 3.1.14 Assume that σ satisfies condition $(\mathcal{H}_\sigma)$. Then the operator $B : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ such that $(Bv, v')_{H_0^2(\Omega)} = (\sigma \Delta v, \Delta v')_\Omega$ for all $v, v' \in H_0^2(\Omega)$, is Fredholm of index zero.

Remark 3.1.15 Adapting the approach presented below, we can prove that for all $m \in \mathbb{N}^* := \{1, 2, 3, \dots\}$ the operator $\Delta^m(\sigma \Delta^m \cdot) : H_0^{2m}(\Omega) \rightarrow H^{-2m}(\Omega)$ is Fredholm of index zero when σ satisfies condition $(\mathcal{H}_\sigma)$. Here, $H_0^{2m}(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ for the H^{2m} -norm.

Proof. Let us give the proof in the case where $\sigma \geq C_1 > 0$ in a neighborhood of the boundary $\partial\Omega$. The configuration where $\sigma \leq C_2 < 0$ in a neighborhood of $\partial\Omega$ can be deduced easily by working with the operator $-B$. Let us introduce $\zeta \in \mathcal{C}_0^\infty(\Omega, [0; 1])$ a cut-off function equal to 1 in $\mathcal{O}$. Notice that $1 - \zeta$ is an element of $\mathcal{C}^\infty(\overline{\Omega}, [0; 1])$ which is equal to 1 in a neighborhood of $\partial\Omega$. Now, let us consider v an element of $H_0^2(\Omega)$. The function $(1 - \zeta)v$ belongs to $H_0^2(\Omega)$ and, by definition, we have, for all $v' \in H_0^2(\Omega)$,

$$(\sigma\Delta((1 - \zeta)v), \Delta v')_\Omega = b((1 - \zeta)v, v') = (B((1 - \zeta)v), v')_{H_0^2(\Omega)}.$$

This allows us to write, expanding $\Delta((1 - \zeta)v)$,

$$((1 - \zeta)\sigma\Delta v, \Delta v')_\Omega = (B((1 - \zeta)v), v')_{H_0^2(\Omega)} + (\sigma(2\nabla v \cdot \nabla \zeta + v\Delta\zeta), \Delta v')_\Omega.$$

On the support of ζ , we must proceed differently because we allow σ to change sign. Let us denote by ψ the unique element of $H_0^1(\Omega)$ such that $\Delta\psi = \sigma^{-1}\Delta v \in L^2(\Omega)$. Classical results of interior regularity indicate that, for all $\chi \in \mathcal{C}_0^\infty(\Omega)$, $\chi\psi \in H_0^2(\Omega)$ with the estimate $\|\chi\psi\|_{H_0^2(\Omega)} \leq C\|\sigma^{-1}\Delta v\|_{L^2(\Omega)} \leq C\|v\|_{H_0^2(\Omega)}$. In particular, the function $\zeta\psi$ belongs to $H_0^2(\Omega)$ and depends continuously on v . Since $(\sigma\Delta(\zeta\psi), \Delta v')_\Omega = (B(\zeta\psi), v')_{H_0^2(\Omega)}$, we obtain, expanding $\Delta(\zeta\psi)$,

$$(\zeta\Delta v, \Delta v')_\Omega = (\sigma\zeta\Delta\psi, \Delta v')_\Omega = (B(\zeta\psi), v')_{H_0^2(\Omega)} - (\sigma(2\nabla\psi \cdot \nabla\zeta + \psi\Delta\zeta), \Delta v')_\Omega.$$

Let us define the operator $T : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ such that $Tv = \zeta\psi + (1 - \zeta)v$ for all $v \in H_0^2(\Omega)$. With the Riesz representation, we introduce the operators $\mathcal{I} : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ and $\mathcal{K} : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ such that, for all $(v, v') \in H_0^2(\Omega) \times H_0^2(\Omega)$,

$$\begin{aligned} (\mathcal{I}v, v')_{H_0^2(\Omega)} &= ((\zeta + (1 - \zeta)\sigma)\Delta v, \Delta v')_\Omega \\ (\mathcal{K}v, v')_{H_0^2(\Omega)} &= (\sigma(2\nabla(\psi - v) \cdot \nabla\zeta + (\psi - v)\Delta\zeta), \Delta v')_\Omega. \end{aligned}$$

With these definitions, we have the relation $B \circ T = \mathcal{I} + \mathcal{K}$. One checks easily that $\mathcal{I} : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ is an isomorphism because the sesquilinear form $(v, v') \mapsto ((\zeta + (1 - \zeta)\sigma)\Delta v, \Delta v')_\Omega$ is coercive in $H_0^2(\Omega)$. On the other hand, classical arguments guarantee that $\mathcal{K} : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ is a compact operator. Therefore, the operator T constitutes a right parametrix for B . Since B is selfadjoint, we deduce that it is a Fredholm operator of index zero. $\blacksquare$

Recall that we denoted $\mathcal{A}_k$ the operator associated with the original interior transmission problem (see the definition in (3.20)). For all $k \in \mathbb{C}$, $\mathcal{A}_k - B$ is a compact operator. Since the index of an operator is stable under compact perturbations, we have the

Corollary 3.1.16 *Assume that σ satisfies condition $(\mathcal{H}_\sigma)$. Then, for all $k \in \mathbb{C}$, the operator $\mathcal{A}_k$ defined in (3.20) is Fredholm of index zero.*

In a quite surprising way, this result indicates that Fredholmness for the operator $\Delta(\sigma\Delta\cdot) : H_0^2(\Omega) \rightarrow H^{-2}(\Omega)$ does not depend on the changes of sign of σ which occur inside the domain Ω . As we have seen in §1.2, this property is not true for the operator $\operatorname{div}(\sigma\nabla\cdot) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$. On the other hand, Corollary 3.1.16 is not sufficient to guarantee that the set of transmission eigenvalues is discrete when σ satisfies $(\mathcal{H}_\sigma)$. To obtain such a result with the analytic Fredholm theorem, one has to exhibit one k such that $\mathcal{A}_k$ is injective. This is not simple due to the expression of $\mathcal{A}_k$ but this has been established in [457, 306].

Fredholm property can be lost when σ changes sign on the boundary

Finally in [151], I studied configurations as described in Figure 3.9 where $\sigma = n - 1$ changes sign on $\partial\Omega$ and proved that Fredholm property can be lost in certain situations. More precisely, during the Ensta 2A internship of J. Firozaly, we proved that singularities of the form

$$s(r, \theta) = r^{1+i\eta}\varphi(\theta), \quad \text{with } \eta \in \mathbb{R}, \quad (3.24)$$

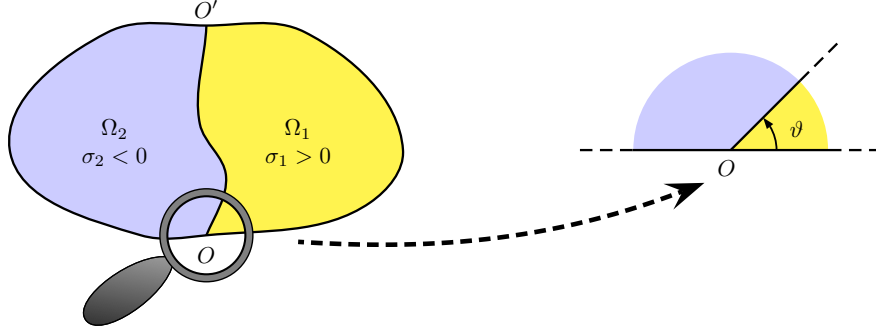


Figure 3.9: Example of configuration where σ changes sign on $\partial\Omega$.

can exist for the operator B in configurations as described in Figure 3.9, depending on σ_1 , σ_2 and the angle ϑ . Here (r, θ) are the polar coordinates associated with O . In that case, similarly to Theorem 1.2.6 for the operator $\operatorname{div}(\sigma \nabla \cdot) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ in §1.2, we established that the range of $B : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$ is not closed which shows that B is not of Fredholm type. By working as in §1.3, we could take into account certain of the singularities (3.24) to obtain an operator which is of Fredholm type. This has not been done in [151].

3.1.4 Concluding remarks concerning the analysis of the ITP

Above we studied the ITP in various frameworks. A noteworthy point of that problem is that depending on the assumptions made on the physical coefficients of the inclusion that one wishes to identify, different tools and approaches must be used. Let us mention that we have this flexibility in the framework because Herglotz waves, *i.e.* generalized combinations of plane waves, which are the waves we assume that the observer can produce, are dense in different functional frameworks. In the setting involving strongly oscillating singularities, we explained how to obtain Fredholm operators. However, to prove discreteness of TEs, one must find one k where the operator is injective. This is not simple and for the moment this remains an open question when $A - \operatorname{Id}$ changes sign on $\partial\Omega$ as well as in the case where $A = \operatorname{Id}$ and $n - 1$ changes sign on $\partial\Omega$. Besides, we focused our attention on results of discreteness of TEs. As we will see below, TEs can also be used to obtain information on the material index of the inclusion. For this reason, people have proved results of existence of real TEs in certain situations (see *e.g.* [412, 127, 125]). However the cases where $A - \operatorname{Id}$ or $n - 1$ change sign on the boundary have not been studied.

Let us come back to an inaccuracy we made in order to simplify the presentation. At the beginning of this section, we first introduced TEs as frequencies $k > 0$ such that there is an incident field whose scattered field vanishes outside of the inclusion. Then we explained that they correspond to values of $k > 0$ such that there is a non zero pair (u, w) which solves the ITP (3.1). In general these two notions do not coincide. More precisely, in the literature, the $k > 0$ such that there is an incident field, defined in $\mathbb{R}^d$, which does not scatter, are usually called non-scattering frequencies. The $k > 0$ such that the ITP admits a non zero solution are the TEs. A non-scattering frequency is always a TE but the converse is often false. Actually, it is easy to see that a TE is a non-scattering frequency if and only if the corresponding w can be extended as a solution to the homogeneous Helmholtz equation in $\mathbb{R}^d$. It has been established that certain scatterers with non-smooth geometries (with corners in 2D or conical tips/edges in 3D) have an infinite number of TEs while their spectrum of non-scattering frequencies is empty [56, 411, 221, 222]. In the recent article [160] with H. Haddar, H. Li and J. Xiao, we gave examples of defects, including some domains with corners or cusps, for which there exist non-scattering frequencies, which finally seems a rather rare phenomenon. Adapting an example proposed by J. Eckmann and C. Pillet in [216],

we also gave situations where the incident non-scattering fields can be extended in a neighborhood of the inhomogeneity but not in the whole space.

3.2 Invisibility in free space with a finite number of measurements

We mentioned that the Linear Sampling Method (LSM) can be used to recover the support of a defect embedded in a homogeneous background if the frequency is not a Transmission Eigenvalue (TE), *i.e.* if the only solution of the ITP (3.1) is zero. This theory supposes that we can produce incident plane waves in all directions $\theta_i \in \mathbb{S}^{d-1}$ and measure the far field pattern of the associated scattered fields for all $\theta_s \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. However, this is not quite realistic. Some works have been devoted to the study of the LSM in partial aperture (see [117] and [186, 187, 191] for the corresponding numerical experiments), that is when the observer knows only the restriction of the far field operator on a non empty open set of $\mathbb{S}^{d-1}$. Thanks to the analytic dependence with respect to the wavenumber, the theory can be developed as in the case of full aperture. But again, this is not completely satisfactory for applications because in practice, one cannot access this continuum of information. Very often, for example in the numerical implementation of the LSM, one can produce incident plane waves and measure the far field pattern of the associated scattered fields only in a finite set of directions. It is this setting that we decided to investigate in [72] with A.-S. Bonnet-Ben Dhia and S.A. Nazarov.

We assume that the propagation of acoustic waves in time harmonic regime in the reference medium $\mathbb{R}^d$, $d = 2, 3$, is governed by the Helmholtz equation $\Delta u + k^2 u = 0$. Again, the localized perturbation in the reference medium occupies the bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. To represent the physical properties of the penetrable inclusion Ω , we introduce $A \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d})$ and $n \in L^\infty(\mathbb{R}^d, \mathbb{R})$, two parameters such that $A = \text{Id}$ and $n = 1$ in $\mathbb{R}^d \setminus \bar{\Omega}$. We assume that $A(\mathbf{x})$ is symmetric for all $\mathbf{x} \in \Omega$, such that $A(\mathbf{x})\zeta \cdot \zeta \geq C|\zeta|^2$ for all $\mathbf{x} \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$ and that n satisfies $n(\mathbf{x}) \geq C$ for all $\mathbf{x} \in \mathbb{R}^d$, for some constant $C > 0$. The scattering of the incident plane wave $u_i := e^{ik\theta_i \cdot \mathbf{x}}$, of direction of propagation $\theta_i \in \mathbb{S}^{d-1}$, by Ω , is described by the problem

$$\left\{ \begin{array}{l} \text{Find } u \text{ such that} \\ -\text{div}(A\nabla u) = k^2 n u \quad \text{in } \mathbb{R}^d, \\ u = u_i + u_s \quad \text{in } \mathbb{R}^d, \\ \lim_{r \rightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_s}{\partial r} - ik u_s \right) = 0. \end{array} \right. \quad (3.25)$$

It is known that for all $k > 0$, Problem (3.25) has a unique solution in $H_{\text{loc}}^1(\mathbb{R}^d)$, the Sobolev space of measurable functions whose H^1 -norm is finite on each bounded subset of $\mathbb{R}^d$. We denote by $u_s(\cdot, \theta_i)$ the associated scattered field. Using Green's representation formula and computing explicitly the asymptotic behavior of the Green's function for the Helmholtz equation far from Ω , one proves (see [189, Thm. 2.6]) the expansion

$$u_s(\mathbf{x}, \theta_i) = e^{ikr} r^{-\frac{d-1}{2}} \left(u_s^\infty(\theta_s, \theta_i) + O(1/r) \right), \quad (3.26)$$

as $r \rightarrow +\infty$, uniformly in $\theta_s \in \mathbb{S}^{d-1}$. Here θ_s is the direction of observation. The function $u_s^\infty(\cdot, \theta_i) : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$, is called the far field pattern associated with $u_i := e^{ik\theta_i \cdot \mathbf{x}}$. In other words, at infinity, $u_s(\cdot, \theta_i)$ behaves at the first order like a cylindrical wave in 2D or like a spherical wave in 3D. The far field pattern is given by the following integral representation

$$u_s^\infty(\theta_s, \theta_i) := c_d \langle \partial_\nu u_s^+(\cdot, \theta_i), e^{-ik\theta_s \cdot \mathbf{x}} \rangle_{\partial\Omega} - c_d \langle \partial_\nu (e^{-ik\theta_s \cdot \mathbf{x}}), u_s(\cdot, \theta_i) \rangle_{\partial\Omega}. \quad (3.27)$$

In this expression, $\langle \cdot, \cdot \rangle_{\partial\Omega}$ stands for the duality pairing (without complex conjugation) between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. On the other hand, ν denotes the unit normal vector to $\partial\Omega$ oriented

toward the interior of Ω and $\partial_\nu u_s^+(\cdot, \boldsymbol{\theta}_i)$ refers to the normal trace of $u_s(\cdot, \boldsymbol{\theta}_i)|_{\mathbb{R}^d \setminus \overline{\Omega}}$. Finally, the constant c_d is such that $c_2 = e^{i\pi/4}/\sqrt{8\pi k}$ and $c_3 = 1/(4\pi)$. In particular when $A = \text{Id}$ in $\mathbb{R}^d$, we deduce from (3.27) that

$$u_s^\infty(\boldsymbol{\theta}_s, \boldsymbol{\theta}_i) = c_d k^2 \int_{\Omega} (n-1)(u_s(\cdot, \boldsymbol{\theta}_i) + e^{ik\boldsymbol{\theta}_i \cdot \mathbf{x}}) e^{-ik\boldsymbol{\theta}_s \cdot \mathbf{x}} d\mathbf{x}. \quad (3.28)$$

We assume that one has a finite set of emitters and receivers located at the same positions so that we can produce incident plane waves in some given directions $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N \in \mathbb{S}^{d-1}$ and measure the far field pattern of the resulting scattered field only in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$ (backscattering directions). This corresponds to knowing all elements of the relative scattering matrix $\mathcal{M}(k) \in \mathbb{C}^{N \times N}$ such that

$$\mathcal{M}_{mn}(k) = u_s^\infty(-\boldsymbol{\theta}_m, \boldsymbol{\theta}_n).$$

In this context, we shall say that $k > 0$ is a non-scattering wavenumber if $\mathcal{M}(k)$ is not injective, equivalently, if there is a non trivial incident field, combination of the plane waves of directions of propagation $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$, such that the far field pattern of the associated scattered field vanishes in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$. Let us emphasize that in this case, unlike in the continuous setting, the scattered field has no reason to be identically null outside of the defect.

3.2.1 Discreteness of non-scattering wavenumbers

In the first part of [72], by working with energy identities and by exploiting the meromorphic dependence of $\mathcal{M}(\cdot)$ with respect to k , we provided criteria on the physical parameters of the inclusion to guarantee that the non-scattering wavenumbers, defined by means of the relative scattering matrix, form a discrete set. One of the results we proved is as follows.

Assumption 3.2.1 *There is a neighborhood of $\partial\Omega$, denoted $\mathcal{V}$, such that $\text{Id} - A$ is nonnegative definite in $\Omega \cap \mathcal{V}$. Moreover, there exists a constant C such that there holds $0 < C \leq 1 - n$ in $\Omega \cap \mathcal{V}$.*

Assumption 3.2.2 *There is a neighborhood of $\partial\Omega$, denoted $\mathcal{V}$, such that $A - \text{Id}$ is nonnegative definite in $\Omega \cap \mathcal{V}$. Moreover, there exists a constant C such that there holds $0 < C \leq n - 1$ in $\Omega \cap \mathcal{V}$.*

Proposition 3.2.3 *Assume that A and n satisfy Assumption 3.2.1 or Assumption 3.2.2. Then the set of non-scattering wavenumbers for Problem (3.25) is either empty or discrete.*

3.2.2 Construction of non-scattering inclusions

In the second part of [72], imposing $A = \text{Id}$ in $\mathbb{R}^d$, for a given $k > 0$ and a given domain Ω , we adapted the method presented in §2.2 to construct, both theoretically and numerically, inclusions supported in $\overline{\Omega}$ for which the corresponding relative scattering matrix $\mathcal{M}(k)$ vanishes. The main result is as follows.

Proposition 3.2.4 *Let $k > 0$ be a given wavenumber and Ω refer to a given Lipschitz domain. Consider $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$, N vectors of $\mathbb{S}^{d-1}$ such that the $N(N+1)/2$ elements of the family $\{\boldsymbol{\theta}_{n'} + \boldsymbol{\theta}_{m'}\}_{1 \leq m' \leq n' \leq N}$ are all non null and all different. There exists an infinite family of parameters n , with $n-1$ supported in $\overline{\Omega}$, such that the relative scattering matrix $\mathcal{M}(k)$ associated with (3.25) (for $A = \text{Id}$) satisfies $\mathcal{M}(k) = 0_{N \times N}$. As a consequence, for such inclusions, for any incident field combination of the plane waves $e^{ik\boldsymbol{\theta}_1 \cdot \mathbf{x}}, \dots, e^{ik\boldsymbol{\theta}_N \cdot \mathbf{x}}$, the far field pattern of the scattered field vanishes in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$.*

Remark 3.2.5 *In dimension two, the assumptions of this proposition can be slightly simplified. Indeed, we can see that if $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$ denote N vectors of $\mathbb{S}^1$, then the elements of the family $\{\boldsymbol{\theta}_{n'} + \boldsymbol{\theta}_{m'}\}_{1 \leq m' \leq n' \leq N}$ are all different as soon as they are all non null.*

In Figures 3.10–3.12, we give an example of defect which produces zero far field patterns in three directions when enlightened from the left.

Let us underline that here we consider only non dissipative isotropic inclusions, the result being simpler to establish for dissipative or anisotropic inclusions. This indicates in particular that the data of the relative scattering matrix does not uniquely determine the defect. Note that in the continuous setting, this construction is impossible. Indeed, the knowledge of the far field operator on $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ uniquely determines the parameter of the inclusion (see [189, Thm. 10.5] and the references therein: [363, 458] in 3D, [107, 285] in 2D).

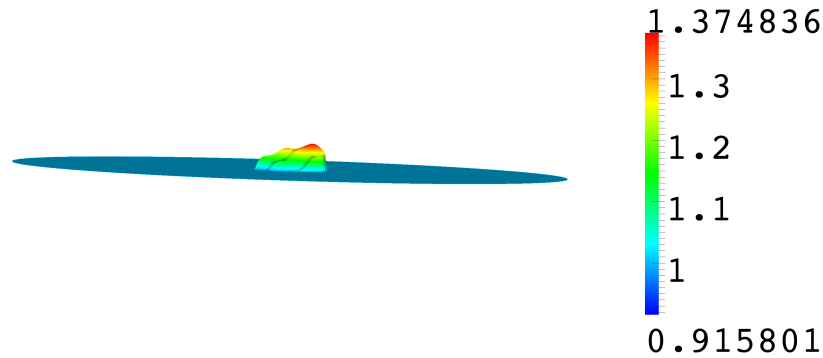


Figure 3.10: Example of parameter n which produces zero far field patterns in the directions corresponding to the angles 90° , 180° , 225° and for an incident direction at 0° . The domain represented here is B_8 , the ball of radius 8, and the defect Ω is located in B_1 .

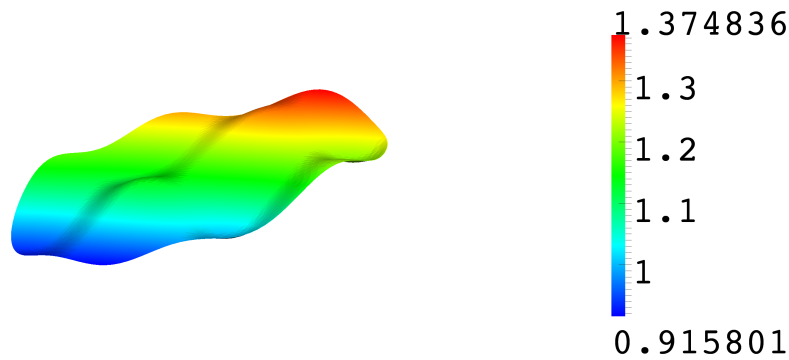


Figure 3.11: Restriction of the coefficient n of Figure 3.10 to Ω . Here we have been able to construct n such that $n|_{\overline{\Omega}} \in \mathcal{C}^\infty(\overline{\Omega})$. The domain represented here is B_1 .

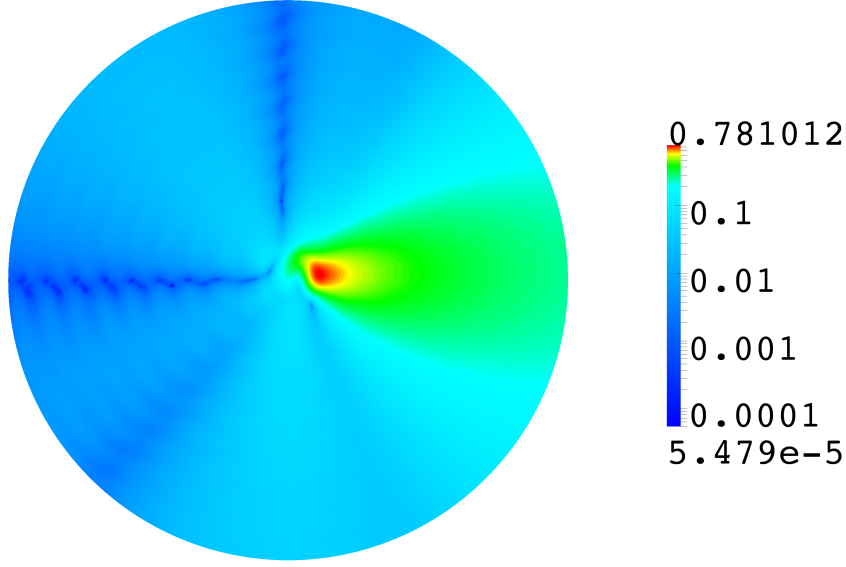


Figure 3.12: Absolute value of the scattered field in logarithmic scale associated with an incident plane wave of incident angle 0° . As desired, we see that it is very small far from Ω in the directions corresponding to the angles 90° , 180° and 225° . The domain represented here is equal to B_8 .

3.2.3 The case of the scattering direction coinciding with the incident direction

In the statement of Proposition 3.2.4, we assume that $\theta_s \neq \theta_i$. When $\theta_s = \theta_i$ (see Figure 3.13), the approach we proposed cannot be implemented. To cope with this problem, one could consider a complex valued parameter n or an anisotropic material characterized by some matrix valued coefficient A . But imagine that we want to use non dissipative isotropic materials with $A = \text{Id}$ only. For a given wavenumber $k > 0$, can we find a real valued parameter n for the inclusion Ω such that the far field pattern associated with the incident plane wave $u_i := e^{ik\theta_i \cdot x}$, vanishes in the direction θ_i ? In the sequel, we shall assume there is only one scattering direction ($N = 1$).

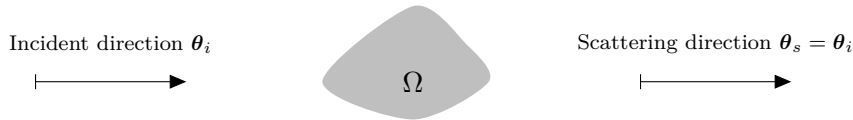


Figure 3.13: Scattering problem with $\theta_s = \theta_i$.

According to (3.28), we know that the far field pattern in the direction θ_s of the scattered field solution of the problem

$$\begin{cases} -\Delta u = k^2 n u & \text{in } \mathbb{R}^d, \\ u = u_i + u_s & \text{in } \mathbb{R}^d, \\ \lim_{r \rightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_s}{\partial r} - i k u_s \right) = 0, \end{cases} \quad (3.29)$$

is given by

$$u_s^\infty(\theta_s, \theta_i) = c_d k^2 \int_{\Omega} (n - 1) u e^{-ik\theta_s \cdot x} dx.$$

When $\theta_s = \theta_i$, we find, using the relation $-\Delta u_s - k^2 n u_s = k^2 (n-1) u_i$,

$$\begin{aligned} u_s^\infty(\theta_i, \theta_i) &= c_d k^2 \int_{\Omega} (n-1) (u_i + u_s) \overline{u_i} d\mathbf{x} \\ &= c_d k^2 \int_{\Omega} (n-1) |u_i|^2 d\mathbf{x} - c_d \int_{\Omega} u_s \overline{\Delta u_s + k^2 n u_s} d\mathbf{x} \\ &= c_d k^2 \int_{\Omega} (n-1) |u_i|^2 d\mathbf{x} + c_d \int_{\Omega} |\nabla u_s|^2 - k^2 n |u_s|^2 d\mathbf{x} + c_d \langle \overline{\partial_\nu u_s}, u_s \rangle_\Gamma. \end{aligned} \quad (3.30)$$

Let us work on the last term of the right-hand side of the previous equation as in the proof of [189, Thm. 2.5]. We take R large enough such that $\overline{\Omega}$ is contained in the ball $B_R := \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < R\}$ and apply Green formula in $B_R \setminus \overline{\Omega}$ to obtain

$$\langle \overline{\partial_\nu u_s}, u_s \rangle_\Gamma = -\langle \overline{\partial_\nu u_s}, u_s \rangle_{\partial B_R} + \int_{B_R \setminus \overline{\Omega}} |\nabla u_s|^2 - k^2 |u_s|^2 d\mathbf{x}.$$

On ∂B_R , ν refers to the unit outward normal to B_R . But u_s satisfies the radiation condition of (3.29) and admits the decomposition (3.26). Therefore, there holds

$$\Im m \langle \overline{\partial_\nu u_s}, u_s \rangle_\Gamma = k \int_{\mathbb{S}^{d-1}} |u_s^\infty(\theta, \theta_i)|^2 d\theta. \quad (3.31)$$

Using (3.31) in (3.30), we conclude

$$\Im m (c_d^{-1} u_s^\infty(\theta_i, \theta_i)) = k \int_{\mathbb{S}^{d-1}} |u_s^\infty(\theta, \theta_i)|^2 d\theta.$$

According to Rellich's lemma, this proves what is called the optical theorem in the literature and which was originally discovered independently by W. Sellmeier and Lord Rayleigh in 1871.

Proposition 3.2.6 *Let u_s refer to the scattered field, defined by Problem (3.29), associated with the incident plane wave $u_i = e^{ik\theta_i \cdot \mathbf{x}}$. Then the far field pattern of u_s vanishes in the direction θ_i if and only if there holds $u_s = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$.*

This shows that it is much more complicated to impose far field invisibility in the direction θ_i than in other directions. In particular, if $k > 0$ and n are such that the far field pattern of u_s vanishes in the direction θ_i , according to Proposition 3.2.6, we must have $u_s = u - u_i = 0$ and $\partial_\nu u_s = \partial_\nu(u - u_i) = 0$ on Γ . This implies that the pair $(u, u_i) = (u_i + u_s, u_i)$ satisfies

$$\begin{cases} \Delta u + k^2 n u &= 0 & \text{in } \Omega, \\ \Delta u_i + k^2 u_i &= 0 & \text{in } \Omega, \\ u - u_i &= 0 & \text{on } \Gamma, \\ \partial_\nu(u - u_i) &= 0 & \text{on } \Gamma. \end{cases} \quad (3.32)$$

This is nothing but a transmission eigenvalue problem with a strong hypothesis on u_i . Compared to the classical transmission eigenvalue problem where the restriction of the incident field to Ω belongs to $L^2(\Omega)$, here we additionally impose $u_i = e^{ik\theta_i \cdot \mathbf{x}}$. As a consequence, for a given n which satisfies $0 < c \leq n \leq C < 1$ in a neighborhood of Γ or $1 < c \leq n$ in a neighborhood of Γ , we know according to [457] that the set of wavenumbers for which (3.32) has a non trivial solution is either empty or discrete. However, we cannot use the results of existence of transmission eigenvalues (see [412, 125]) to conclude the existence of non trivial solutions to (3.32) because here, we impose a very restrictive condition for u_i . More interesting for our configuration are the results of [56, 411, 221, 222]. In these articles, situations where the support of $n-1$ has corners or edges are considered. In this context, the authors provide sufficient criteria on n which ensure that all non trivial incident Herglotz functions (*i.e.* all non trivial generalized combinations of plane waves), for all wavenumbers $k > 0$, produce

non trivial scattered field. For such coefficients n , the only solution of (3.32) is zero. For more general n , for example such that the support of $n - 1$ is smooth, even assuming $u_i = e^{ik\theta_i \cdot x}$, we do not know any result showing that (3.32) has only the trivial solution. Nevertheless, in any case, and this is the important message of this discussion, it seems particularly delicate to impose far field invisibility in the direction θ_i .

3.3 ITP and artificial background

We mentioned that the Linear Sampling Method (LSM) fails at frequencies coinciding with Transmission Eigenvalues (TEs) which, we remind the reader, are defined as the eigenvalues of the Interior Transmission Problem (ITP). This is the reason why substantial efforts have been made to prove discreteness of the set of TEs. On the other hand, from the work [122], it has been proved that TEs can be determined from measured data and then can be exploited to get additional properties of the defect. Roughly speaking, the idea is to work in two steps. First, one recovers the support of the defect by applying the LSM at a frequency which is not a TE. Then one uses the spectrum of TEs to obtain information on the material constituting the inclusion (below we shall assume that this material is characterized by an index n). Since [122], the determination of TEs from measured data has been improved using the framework of the Generalized Linear Sampling Method (GLSM) in which exact knowledge of the support is no longer needed [29, 123] (see also [307] for a different approach).

Although mathematically appealing, relying on transmission eigenvalues to determine quantitative features on n is difficult. The reason is twofold. First, information is lost in complex eigenvalues, which generically exist and cannot be measured in practice. Second, it is difficult to establish sharp estimates for real TEs with respect to n due to the complexity of the ITP.

In [24], with L. Audibert and H. Haddar, we explained how to work with another far field operator F^{ab} corresponding to an artificial background (reference medium) for which the associated TEs have a more direct connection with n . Put differently, working with F^{ab} , our goal is to simplify the solution of the inverse spectral problem consisting in determining n from the knowledge of real positive TEs. Important in the analysis is the fact that F^{ab} is given by the formula $F^{\text{ab}} = F - \tilde{F}$ where $\tilde{F}$ can be obtained via a rather direct numerical computation. Therefore, in practice F^{ab} can also be considered as a data. We will see that our approach does not require a priori knowledge of the exact support of the inhomogeneity. It is sufficient to know that the defect in the reference medium is located in a given bounded region.

Close to our study are the articles [237, 124, 23]. In the first one, the authors reformulate the ITP as an eigenvalue problem for the material coefficient. In the second and third ones, it is explained how to identify n from the knowledge of $F(k) - \tilde{F}(k, \gamma)$ at a single wavenumber k and for a range of γ . Here $F(k) - \tilde{F}(k, \gamma)$ can be seen as the far field operator corresponding to a background depending on an artificial parameter γ . In comparison with our approach, this method is interesting because it requires knowing F at a single wavenumber ($\tilde{F}(k, \gamma)$ can be computed numerically). However, the relation between associated TEs and n is a bit more complex than in our case.

3.3.1 Setting

Again, the localized perturbation in the reference medium is modeled by some bounded domain $\Omega \subset \mathbb{R}^d$ and a real valued refractive index $n \in L^\infty(\mathbb{R}^d)$. We assume that $n = 1$ in $\mathbb{R}^d \setminus \Omega$. The scattering of the incident plane wave $u_i(\cdot, \theta_i) := e^{ik\theta_i \cdot x}$ of direction of propagation $\theta_i \in \mathbb{S}^{d-1}$ by Ω

is described by the problem

$$\left\{ \begin{array}{l} \text{Find } u = u_i + u_s \text{ such that} \\ \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d, \\ \lim_{r \rightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_s}{\partial r} - i k u_s \right) = 0, \end{array} \right. \quad (3.33)$$

with $u_i = u_i(\cdot, \boldsymbol{\theta}_i)$. As in (3.26), the scattered field $u_s(\cdot, \boldsymbol{\theta}_i)$ has the expansion

$$u_s(\mathbf{x}, \boldsymbol{\theta}_i) = e^{ikr} r^{-\frac{d-1}{2}} \left(u_s^\infty(\boldsymbol{\theta}_s, \boldsymbol{\theta}_i) + O(1/r) \right),$$

as $r \rightarrow +\infty$, uniformly in $\boldsymbol{\theta}_s \in \mathbb{S}^{d-1}$. The function $u_s^\infty(\cdot, \boldsymbol{\theta}_i) : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$, is called the far field pattern associated with $u_i(\cdot, \boldsymbol{\theta}_i)$. From the far field pattern, we define the far field operator $F : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ such that

$$(Fg)(\boldsymbol{\theta}_s) = \int_{\mathbb{S}^{d-1}} g(\boldsymbol{\theta}_i) u_s^\infty(\boldsymbol{\theta}_s, \boldsymbol{\theta}_i) d\boldsymbol{\theta}_i. \quad (3.34)$$

The function Fg corresponds to the far field pattern for the scattered field in (3.33) with $u_i = u_i(g) := \int_{\mathbb{S}^{d-1}} g(\boldsymbol{\theta}_i) e^{ik\boldsymbol{\theta}_i \cdot \mathbf{x}} d\boldsymbol{\theta}_i$ (Herglotz wave function). Define the operator $\mathcal{H} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\Omega)$ such that $\mathcal{H}g = u_i(g)|_\Omega$ and the space $H_{\text{inc}}(\Omega) := \{v \in L^2(\Omega); \Delta v + k^2 v = 0 \text{ in } \Omega\}$. It is known that $H_{\text{inc}}(\Omega)$ is nothing but the closure of the range of the operator $\mathcal{H}$ in $L^2(\Omega)$. Observing that $\Delta u_s + k^2 n u_s = k^2(1-n)u_i(g)$ (in particular u_s depends only on the values of $u_i(g)|_\Omega$), we can factorize F as $F = G\mathcal{H}$ where the operator $G : H_{\text{inc}}(\Omega) \rightarrow L^2(\mathbb{S}^{d-1})$ is the extension by continuity of the mapping $u_i(g)|_\Omega \mapsto u_s^\infty$. The real transmission eigenvalues coincide with the values of $k \in \mathbb{R}$ for which G is not injective. In such a case, there is a (generalized) incident wave $v \in H_{\text{inc}}(\Omega)$ such that the associated far field is zero and therefore, by Rellich's Lemma, the scattered field u_s is zero outside of Ω . As already seen, this leads to the equivalent definition of TEs as the values of $k \in \mathbb{R}$ for which the problem

$$\left\{ \begin{array}{l} \Delta u_s + k^2 n u_s = k^2(1-n)v \quad \text{in } \Omega \\ \Delta v + k^2 v = 0 \quad \text{in } \Omega \end{array} \right. \quad (3.35)$$

admits a non trivial solution $(u_s, v) \in H_0^2(\Omega) \times L^2(\Omega)$. One of the main troubles with TEs is that their link with the index of refraction n is not explicit nor easily accessible. Some monotonicity results have been obtained in [125] but only for some of the TEs. We hereafter explain how by modifying the far field operator, one can obtain new transmission eigenvalue problems for which it is easier to get information on n .

3.3.2 Transmission eigenvalues with a Zero-Index Material background

Assume that one has a priori knowledge of a Lipschitz domain Ω_b such that $\Omega \subset \Omega_b$. We emphasize that we do not require to know exactly the support $\overline{\Omega}$ of the defect in the reference medium. Consider the scattering problem

$$\left\{ \begin{array}{l} \text{Find } \tilde{u} = u_i + \tilde{u}_s \text{ such that} \\ \Delta \tilde{u} + k^2 \rho \tilde{u} = 0 \quad \text{in } \mathbb{R}^d, \\ \lim_{r \rightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial \tilde{u}_s}{\partial r} - i k \tilde{u}_s \right) = 0 \end{array} \right. \quad (3.36)$$

where ρ is the function such that $\rho = 0$ in Ω_b and $\rho = 1$ in $\mathbb{R}^d \setminus \overline{\Omega_b}$. This artificial medium is referred to as Zero-Index Material (ZIM) because we choose $\rho = 0$ inside Ω_b . This choice greatly simplifies the structure of the associated interior transmission problem as we shall see. For $u_i = u_i(\cdot, \boldsymbol{\theta}_i) = e^{ik\boldsymbol{\theta}_i \cdot \mathbf{x}}$ with $\boldsymbol{\theta}_i \in \mathbb{S}^{d-1}$, as for (3.33), this problem admits a unique solution in $H_{\text{loc}}^1(\mathbb{R}^d)$. We denote by $\tilde{u}_s(\cdot, \boldsymbol{\theta}_i)$ the associated scattered field and $\tilde{u}_s^\infty(\cdot, \boldsymbol{\theta}_i) : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$ the corresponding far field

pattern. From the far field pattern, we define the far field operator $\tilde{F} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ such that $(\tilde{F}g)(\boldsymbol{\theta}_s) = \int_{\mathbb{S}^{d-1}} g(\boldsymbol{\theta}_i) \tilde{u}_s^\infty(\boldsymbol{\theta}_s, \boldsymbol{\theta}_i) ds(\boldsymbol{\theta}_i)$. Finally we define the artificial far field operator $F^{\text{ab}} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ as

$$F^{\text{ab}} := F - \tilde{F}. \quad (3.37)$$

From a practical point of view, notice that F is given by the measurements while $\tilde{F}$ has to be computed, which is achievable because Problem (3.36) does not involve n (and Ω_b is known). Therefore, we can consider F^{ab} as a data for the inverse problem of determining n . Let us denote

$$u_i^{\text{ab}}(g) := \int_{\mathbb{S}^{d-1}} g(\boldsymbol{\theta}_i) \tilde{u}(\cdot, \boldsymbol{\theta}_i) ds(\boldsymbol{\theta}_i) \quad (3.38)$$

for given $g \in L^2(\mathbb{S}^{d-1})$ where $\tilde{u}(\cdot, \boldsymbol{\theta}_i)$ is the solution of (3.36) for $u_i = u_i(\cdot, \boldsymbol{\theta}_i) := e^{ik\boldsymbol{\theta}_i \cdot \mathbf{x}}$. Define the operator $\mathcal{H}^{\text{ab}} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\Omega_b)$ such that $\mathcal{H}^{\text{ab}}g = u_i^{\text{ab}}(g)|_{\Omega_b}$. Set also $\tilde{H}_{\text{inc}}(\Omega_b) := \{v \in L^2(\Omega_b); \Delta v = 0 \text{ in } \Omega_b\}$. Observing that $\Delta(u_s - \tilde{u}_s) + k^2 n(u_s - \tilde{u}_s) = k^2(\rho - n)u_i^{\text{ab}}(g)$, we can factorize F^{ab} as $F^{\text{ab}} = G^{\text{ab}}\mathcal{H}^{\text{ab}}$ where the operator $G^{\text{ab}} : \tilde{H}_{\text{inc}}(\Omega_b) \rightarrow L^2(\mathbb{S}^{d-1})$ is the extension by continuity of the mapping $u_i^{\text{ab}}(g)|_{\Omega_b} \mapsto u_s^\infty - \tilde{u}_s^\infty$. Similarly as above, we now define the transmission eigenvalues as the values of k for which the operator G^{ab} is not injective. Denoting $w := u_s - \tilde{u}_s$, this is now equivalent, due to Rellich's Lemma, to define the transmission eigenvalues as the values of k for which the problem

$$\begin{cases} \Delta w + k^2 n w &= -k^2 n v & \text{in } \Omega_b \\ \Delta v &= 0 & \text{in } \Omega_b \end{cases} \quad (3.39)$$

admits a non trivial solution $(w, v) \in H_0^2(\Omega_b) \times L^2(\Omega_b)$. We observe that $k = 0$ is an eigenvalue of infinite multiplicity of (3.39). Now we consider the case $k \neq 0$. Then we find that (3.39) admits a non trivial solution if and only if there is $w \neq 0$ such that $(\Delta + k^2 n)(n^{-1}\Delta w) = 0$ in Ω_b , that is, if and only if (w, k) is a solution of the problem

$$\begin{cases} \text{Find } (w, k) \in H_0^2(\Omega_b) \setminus \{0\} \times \mathbb{R} \text{ such that} \\ \Delta(n^{-1}\Delta w) = -k^2 \Delta w \text{ in } \Omega_b. \end{cases} \quad (3.40)$$

In contrast to Problem (3.35) which is quadratic in k^2 , one ends up here with a well-known linear eigenvalue problem similar to the so-called plate buckling eigenvalue problem. We can use classical results concerning linear self-adjoint compact operators to show that the spectrum of (3.40) is made of real positive isolated eigenvalues (no more complex eigenvalues) of finite multiplicity $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p \leq \dots$ such that $\lim_{p \rightarrow +\infty} \lambda_p = +\infty$. Moreover, we have the simple *min-max* formulas

$$\lambda_p = \min_{E_p \in \mathcal{E}_p} \max_{w \in E_p \setminus \{0\}} \frac{(n^{-1}\Delta w, \Delta w)_{L^2(\Omega_b)}}{\|\nabla w\|_{L^2(\Omega_b)}^2}. \quad (3.41)$$

Here $\mathcal{E}_p$ denotes the sets of subspaces E_p of $H_0^2(\Omega_b)$ of dimension p . Observe that the characterisation of the spectrum of Problem (3.39) is much simpler than the one of Problem (3.35). Moreover, it holds under very general assumptions for n : we just require that $n|_{\Omega_b} \in L^\infty(\Omega_b)$ with $\text{ess inf}_\Omega n > 0$. In particular, n can be equal to one inside Ω_b (thus, as already mentioned, we do not need to know exactly the support $\bar{\Omega}$ of the defect in the reference medium) and $n - 1$ can change sign on the boundary. In comparison, the analysis of the spectrum of (3.35) in such situations is much more complex and the functional framework must be adapted according to the values of n as seen in §3.1.3. The second advantage of considering Problem (3.39) instead of Problem (3.35) is that the spectrum of (3.39) is entirely real.

Now we provide a Theorem which justifies a method to identify eigenvalues of the modified ITP from the knowledge of $\tilde{F}$. To state the result, we need to introduce some notation. Set $\tilde{F}_\# := |\tilde{F} + \tilde{F}^*| + |\tilde{F} - \tilde{F}^*|$, where $\tilde{F}^*$ is the adjoint of $\tilde{F}$, and define for $\alpha > 0$, $g, \phi \in L^2(\mathbb{S}^{d-1})$, the functional

$$J_\alpha(g, \phi) := \alpha(\tilde{F}_\# g, g)_{L^2(\mathbb{S}^{d-1})} + \|F^{\text{ab}}g - \phi\|_{L^2(\mathbb{S}^{d-1})}^2.$$

Notice that in the penalty term, we use the operator $\tilde{F}_\#$ and not $F_\#^{\text{ab}}$ defined similarly as $\tilde{F}_\#$. The reason is that we do not know if $g \mapsto (F_\#^{\text{ab}}g, g)_{L^2(\mathbb{S}^{d-1})}$ is equivalent to $\|u_i(g)\|_{L^2(\Omega_b)}^2$. We then consider for $\mathbf{z} \in \mathbb{R}^d$ a function $g_{\mathbf{z}}^\alpha \in L^2(\mathbb{S}^{d-1})$ such that

$$J_\alpha(g_{\mathbf{z}}^\alpha, \phi_{\mathbf{z}}^\infty) \leq \alpha + \inf_{g \in L^2(\mathbb{S}^{d-1})} J_\alpha(g, \phi_{\mathbf{z}}^\infty)$$

where $\phi_{\mathbf{z}}^\infty(\boldsymbol{\theta}_s) := e^{ik\boldsymbol{\theta}_s \cdot \mathbf{z}}$ is the far field associated with a point source at the point $\mathbf{z}$.

Theorem 3.3.1 *Assume that the far field operator F^{ab} has dense range. Then k^2 is an eigenvalue of (3.40) if and only if the set of points $\mathbf{z}$ for which $(\tilde{F}_\#g_{\mathbf{z}}^\alpha, g_{\mathbf{z}}^\alpha)_{L^2(\mathbb{S}^{d-1})}$ is bounded as $\alpha \rightarrow 0$ is nowhere dense in Ω_b .*

This theorem guarantees that peaks in the curve

$$k \mapsto I(k, n) := \int_{\Omega_0 \subset \Omega_b} (\tilde{F}_\#g_{\mathbf{z}}^\alpha, g_{\mathbf{z}}^\alpha)_{L^2(\mathbb{S}^{d-1})} d\mathbf{z} \quad (3.42)$$

for small values of α correspond to k^2 which are eigenvalues of (3.40). Therefore, from the knowledge of F^{ab} for real $k \in (0; +\infty)$, we can identify all the spectrum of (3.40).

Formula (3.41) is interesting because it guarantees that the λ_p have monotone dependence with respect to n . If we denote by $\hat{\lambda}_p(\hat{n})$ the p -th eigenvalue of Problem (3.40) with n replaced by $\hat{n}$, then we have $\lambda_p(n) \leq \hat{\lambda}_p(\hat{n})$ if $n \geq \hat{n}$. This estimate can be useful to derive qualitative information from a reference situation: for a given setting where the $\lambda_p(n)$ are known, if n is changed into $\hat{n}$, one can have an idea of the nature of the perturbation. From (3.41), we can also write

$$\text{ess inf}_{\Omega_b} n \leq \lambda_p(1)/\lambda_p(n) \quad \text{and} \quad \text{ess sup}_{\Omega_b} n \geq \lambda_p(1)/\lambda_p(n).$$

Note that the $\lambda_p(1)$ can be explicitly computed with a numerical code because they do not depend on n and Ω_b is known. In particular, when n is constant and $\Omega_b = \Omega$, we have the formula

$$n = \lambda_p(1)/\lambda_p(n). \quad (3.43)$$

Thus, in this case, from the knowledge of the far field operator F^{ab} on an interval of wavenumbers k containing an eigenvalue of Problem (3.40), we can identify the value of n .

Let us discuss other possibilities to determine simple spectral signatures associated with the index of refraction by modifying the background.

3.3.3 A modified background with a non penetrable obstacle

First, we consider a background with a non penetrable obstacle Ω_b such that $\Omega \subset \Omega_b$. On $\partial\Omega_b$, we prescribe the boundary condition $B(\tilde{u}) = 0$ where B is a given boundary operator. The construction is similar to above where (3.36) is replaced by

$$\begin{cases} \Delta \tilde{u} + k^2 \tilde{u} = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega_b} \\ B(\tilde{u}) = 0 & \text{on } \partial\Omega_b. \end{cases} \quad (3.44)$$

We investigate here only the cases $B(\tilde{u}) = \tilde{u}$ and $B(\tilde{u}) = \partial\tilde{u}/\partial\nu + \gamma\tilde{u}$ which respectively correspond to Dirichlet and Robin scattering problems, with $\gamma \in \mathbb{R}$ being a fixed impedance parameter. The normal unit vector ν to $\partial\Omega_b$ is chosen directed to the exterior of Ω_b . We redefine respectively F^{ab} , $u_i^{\text{ab}}(g)$ as in (3.37), (3.38) from $\tilde{u}$ introduced in (3.44). Then by adapting the definition of $\tilde{F}_\#$ in

(3.42), we showed that by modifying the data F by some computed quantities, one can access the eigenvalues of the cavity problem

$$\begin{cases} \Delta w + k^2 n w &= 0 & \text{in } \Omega_b \\ B(w) &= 0 & \text{on } \partial\Omega_b. \end{cases} \quad (3.45)$$

For this problem, we obtain similar conclusions as in the previous section on the determination of n from TEs. Let us emphasize that one can combine the information provided by the different spectra to determine n .

3.3.4 Other choices of artificial backgrounds

For the artificial background in (3.36), we are not obliged to choose ρ such that $\rho = 0$ in Ω_b and $\rho = 1$ in $\mathbb{R}^d \setminus \overline{\Omega_b}$. Consider an arbitrary real valued $\rho \in L^\infty(\mathbb{R}^d)$ such that $\rho - 1$ is compactly supported. Let Ω_b be a domain such that $\text{supp}(n - \rho) \subset \overline{\Omega_b}$. For such a ρ , setting $w := u - v$, the corresponding transmission eigenvalue problem reads

$$\begin{cases} \Delta w + k^2 n w &= k^2(\rho - n)v & \text{in } \Omega_b \\ \Delta v + k^2 \rho v &= 0 & \text{in } \Omega_b \end{cases}$$

with $(w, v) \in H_0^2(\Omega_b) \times L^2(\Omega_b)$. And if ρ is such that $n - \rho \neq 0$ in Ω_b , this leads to the problem

$$\begin{cases} \text{Find } (w, k^2) \in H_0^2(\Omega_b) \setminus \{0\} \times \mathbb{C} \text{ such that} \\ (\Delta + k^2 \rho) \left(\frac{1}{n - \rho} (\Delta w + k^2 n w) \right) = 0 & \text{in } \Omega_b. \end{cases} \quad (3.46)$$

All the game with these artificial backgrounds consists in choosing ρ such that TEs of (3.46) give interesting or simply usable information on n . One can also vary the boundary condition in the case discussed in Section 3.3.3 by varying the Robin parameter γ . A natural question then would be to ask whether these spectra provide unique determination of the refractive index.

When n is complex valued, the spectra of Problems (3.40) and (3.45) are complex. In particular when n is constant with a non-zero imaginary part, real eigenvalues do not exist. In that case, we cannot identify transmission eigenvalues from the knowledge of F^{ab} for real k and therefore we cannot retrieve information on n . We do not know if one can find a well-chosen artificial background such that the knowledge of the corresponding F^{ab} for $k \in (0; +\infty)$ would allow one to get information on complex valued n .

3.3.5 Numerical illustrations

We provide here a simple numerical example showing how the algorithm of determining constant n using the ZIM background would work. The inclusion Ω is a kite shape as depicted in Figure 3.14. First, we generate the far field matrices for 200 incident directions θ_i and 200 observation directions θ_s by solving Problem (3.33). This is done for a range of wavenumbers k . Thus we obtain a discretization of the far field operator F defined in (3.34) which in practice would be given by measurements. We consider two cases: $n = n_1 = 2$ in Ω and $n = n_2 = 4$ in Ω . Then we choose $\Omega_b = \Omega$ and we build a discretization of the artificial far field operator F^{ab} introduced in (3.37). This allows us to compute the indicator function $I(k, n)$ defined in (3.42) with $\Omega_0 = \Omega$. In Figure 3.14, we display the curve $k \mapsto I(k, n_1)$ for $k \in (3.5; 7)$. On the other hand, using the **FreeFem++** software, we solve the eigenvalue Problem (3.40) for $n = n_1$ in Ω . We observe that the square roots of the eigenvalues of (3.40), marked by dashed lines on Figure 3.14, correspond to the k for which $k \mapsto I(k, n_1)$ has some peaks. This is coherent with the result of Theorem 3.3.1. Therefore, from the measurements, we can assess the $\lambda_p(n)$ in (3.43). Computing $\lambda_p(1)$ (solving (3.40) with $n = 1$),

we can recover n when n is constant using Formula (3.43). In Figure 3.14, we also display the curve $k \mapsto I(k/\sqrt{2}, n_2)$ for $k \in (3.5; 7)$. We see that the peaks of this curve coincide with the ones of $k \mapsto I(k, n_1)$. This is in accordance with the theory which guarantees that $n_2/n_1 = \lambda_p(n_1)/\lambda_p(n_2)$ for all $p \geq 0$. Numerically $|n_2/n_1 - \lambda_p(n_1)/\lambda_p(n_2)|$ is of order 10^{-2} .

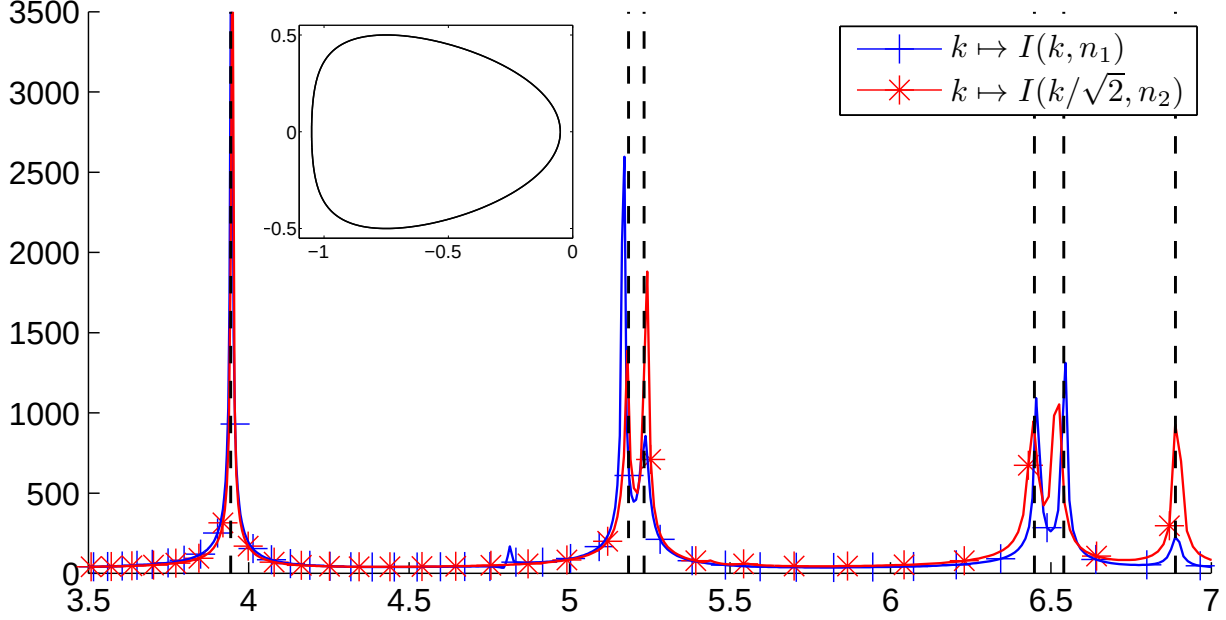


Figure 3.14: Curves $k \mapsto I(k, n_1)$ and $k \mapsto I(k/\sqrt{2}, n_2)$ for $k \in (3.5; 7)$. The vertical dashed lines correspond to the eigenvalues of Problem (3.40) computed with $n = n_1$ in Ω (the quantities we want to retrieve in practice). The kite shaped domain represents the inclusion Ω .

3.4 Inside-outside duality with artificial background

We continue our study of the problem consisting in retrieving information on the material index n of a penetrable inclusion embedded in a reference medium from far field data associated with incident plane waves. As mentioned above, (classical) Transmission Eigenvalues (TEs) can be used as qualitative spectral signatures of n . The first step consists in identifying TEs from far field data known for a range of frequencies. To proceed, two main approaches exist. The first class of methods exploit the failure of sampling methods at these special frequencies to reconstruct them [122, 22, 29]. They require some *a priori* knowledge on the inclusion location and size. Another class of methods exploit the inside-outside duality revealed in [212, 216, 211, 217] that relates TEs to spectral properties of the far field operator. These methods have been introduced to identify TEs in [307] and further developed in many subsequent works [323, 332, 330, 287, 331, 422, 421, 309] both theoretically and numerically. These approaches only use the far field data and do not need *a priori* knowledge on the shape or location of the inclusion. Let us describe the main idea.

Consider the same scattering problem as in (3.33), say in $\mathbb{R}^3$ ($d = 3$). Let $F : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)$ denote the far field operator introduced in (3.34). From F , define the scattering operator $S : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)$ such that

$$S := \text{Id} + \frac{ik}{2\pi} F.$$

It is known that S is unitary ($SS^* = S^*S = \text{Id}$) and that F is compact. Therefore, for all $k > 0$, the eigenvalues of S are located on the unit circle and only accumulate at 1. Now, if the material index n in (3.33) satisfies $n - 1 \geq c > 0$ in Ω , one can show that the eigenvalues of S accumulate

at 1 only in the upper half complex plane $\{z \in \mathbb{C} \mid \Im m z > 0\}$ (see the illustration of Figure 3.15). As a consequence, if we denote by $e^{i\delta_j}$, with

$$2\pi \geq \delta_1 \geq \delta_2 \geq \dots \geq \delta_j \geq \dots > 0,$$

the eigenvalues of S , then there holds $\lim_{j \rightarrow +\infty} \delta_j = 0^+$.

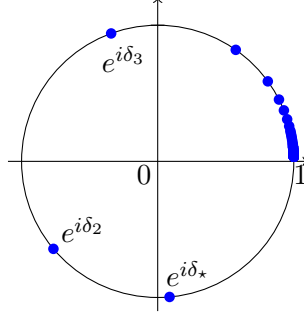


Figure 3.15: Eigenvalues of S in a situation where $n - 1 \geq c > 0$ in Ω .

The phase δ_1 plays a particular role in the analysis and we set $\delta_\star := \delta_1$. Moreover, note that the phase of the eigenvalues of S depends on k . The main result of [307] can be summarized as follows: under certain additional assumptions on n , the number $k_0 > 0$ is a TE if and only if there holds

$$\lim_{k \nearrow k_0} \delta_\star(k) = 2\pi,$$

see Figure 3.16 (the notation $k \nearrow k_0$ means that k tends to k_0 by increasing values).

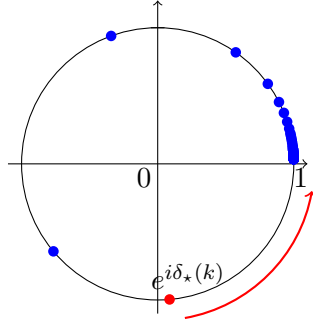


Figure 3.16: Illustration of the behavior of $k \mapsto e^{i\delta_\star(k)}$ when $k \nearrow k_0$ where k_0 is a TE. Here we consider a situation where $n - 1 \geq c > 0$ in Ω .

There exists a similar statement in the case where $1 - n \geq c > 0$ in Ω . In that case, eigenvalues of S accumulate at 1 in the lower half complex plane and under additional assumption on n , one shows that $k_0 > 0$ is a TE if and only if an eigenvalue of S converges to 1 with a positive phase as $k \searrow k_0$. Let us emphasize that the mathematical justification of these results only applies to some simplified asymptotic configurations of the material properties, namely small perturbation of large enough or small enough constants.

On the other hand, as said above, one of the main troubles with classical TEs is that their link with the index of refraction is not explicit nor easily accessible. This is why in the previous section, we explained how to work with artificial backgrounds for which the associated TEs have a more direct connection with the material index of the inclusion (see [124, 23, 185, 24]).

In the article [25], with L. Audibert and H. Haddar, our goal was to show how to develop and justify the inside-outside duality for identifying TEs associated with artificial backgrounds.

We investigated cases where the modification of the background consists in introducing an artificial inhomogeneity with index n_b that contains the targeted inclusion. As in (3.37), we first defined a relative far field operator as the difference between the far field operator from data and the far field operator associated with the background. Scaling this difference with the adjoint of the scattering matrix for the artificial background, we obtained an operator that admits a symmetric factorisation in terms of Herglotz wave operators (see [365, 248, 320, 247]). The spectrum of this operator was then exploited to identify TEs. In the case when n_b is constant, one is faced with the same difficulties as for classical TEs ($n_b = 1$, see [307]) for the theoretical justification. To overcome this problem, we exploited that the artificial background is a numerical tool that can be adapted at wish and proposed to use some n_b scaling like ρ/k^2 where ρ is an arbitrary constant and where k is the wavenumber. For this particular choice, we have been able to obtain a necessary and sufficient condition for the identification of TEs from the behavior of the phases of the eigenvalues of the modified far field operator. The main property that made this result possible to prove is that with this scaling, the space of generalized incident waves is independent from the wavenumber k . In addition to this characterization, we also showed that one can recover the generalized incident wave which does not scatter. This result holds for any continuous dependence of n_b with respect to k and may be useful for the inverse spectral problem consisting in determining n from the TEs.

3.5 Detection of cracks via the ITP with artificial background

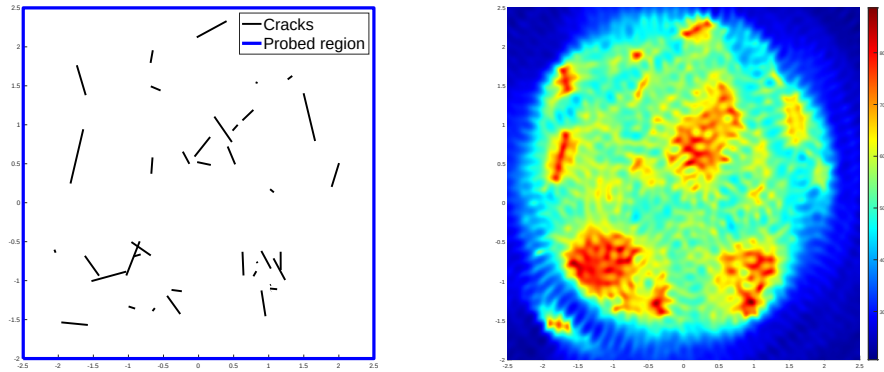


Figure 3.17: Imaging of a simulated damaged background with the indicator of the so-called Factorization Method. The reconstruction uses far field data generated at the wavelength $\lambda := 2\pi/k = 0.3$. The resolution of the image is set by the sampling points step equal to 0.01 here.

During the PhD of Kevish Napal [366] that I have supervised with L. Audibert and H. Haddar, we considered the problem of identifying a set of cracks embedded in some homogeneous background from measured far field data generated by acoustic plane waves. This type of problem has been extensively studied in the literature and we refer to [308, 118, 316, 7, 45, 289, 286, 478, 10, 342, 61, 102, 43, 46, 250] for an overview of some non-iterative methods that have been employed in this context. In the PhD of Kevish, we were interested in cases where the cracks form a relatively dense network that causes failure of most of these methods (due to important multiple scattering effects). In general, the above techniques provide an indicator function that only reveals a volumetric domain encompassing the cracks network as illustrated in Figure 3.17. In our work, we proposed and studied two approaches to remove this limitation and provide indicator functions that are sensitive to local densities of the cracks inside the network.

These two methods belong to the family of so-called sampling methods (see *e.g.* [190, 304]) and make use in particular of the Generalized Linear Sampling Method (GLSM) [29, 123]. In the case of inclusions with non empty interior, sampling methods fail if the interrogating frequency

coincides with so-called Transmission Eigenvalues (TEs) (that coincide with resonant frequencies in the case of non penetrable obstacles). As we have seen in the previous two sections, exploiting the failure of these methods at TEs, it has been shown that TEs can be computed from far field data, and moreover that the knowledge of these quantities can be used to infer qualitative information on the material. However, recovering sharp information from TEs is not an easy task for penetrable inclusions because they cannot be viewed as the spectrum of a selfadjoint operator. To overcome this difficulty it has been suggested to consider a modified spectrum that we refer to as Relative Transmission Eigenvalues (RTEs) and that can still be computed from far field data (see §3.3). The original proposed idea consists in the introduction of an artificial background that can be chosen by the observer. Then RTEs correspond to wavenumbers such that there exists an incident field for which the far field resulting from the effective background and the one resulting from the artificial background are arbitrarily close. The advantage of using RTEs is that they depend on the artificial background which can be fit as desired, in accordance with the problem under consideration. Hence choosing the appropriate setting for the artificial obstacle, such as the position, the geometry, whether the obstacle is penetrable or not and the corresponding refractive index or boundary conditions, can greatly simplify the link between the RTEs and the parameters of interest.

In the PhD of Kevish, that led to the articles [26, 27], we adapted the idea of RTEs for crack monitoring perspectives. Note that cracks have an empty interior and therefore TEs do not exist (according to their definition for volumetric inclusions). However, one can still define RTEs as long as the artificially included background contains inclusions with a non empty interior. This is what we exploited to build indicator functions that are sensitive to the density of cracks inside the artificial inclusion. Considering the latter to be a non-penetrable obstacle Ω (with some prescribed boundary conditions), the RTEs then coincide with the eigenvalues of the Laplace operator with the boundary conditions on $\partial\Omega$ and the boundary conditions on the cracks intersecting Ω . This indicates that these RTEs are sensitive to the number of cracks contained in Ω . We have shown for instance that RTEs are monotone with respect to the crack (for the inclusion order). As for TEs, we proved that RTEs can be determined from the multi-frequency data using an adaptation of the GLSM [29]. This led to a first algorithm where the domain Ω is swept over the interrogated region and for each position, the computed RTEs are compared with RTEs associated with a crack free inclusion Ω . The desired resolution of the method is determined by the size of Ω which is in turn limited by the frequency band of the data (since it should contain at least the first RTE associated with Ω).

A weak point of this first method is the relatively high numerical cost of the computations of RTEs associated to one artificial background. Since increasing the resolution, or equivalently, reducing the size of Ω , requires increasing the number of artificial backgrounds considered, obtaining high resolution images may be prohibitive. To bypass this drawback, we suggested a second method which uses far field data at only one fixed wavenumber k .

This alternative approach mixes the notion of artificial background with ideas from the Differential Linear Sampling Method [28]. We consider the same Ω as previously but instead of comparing RTEs, we compare the solutions of the corresponding transmission problems. We showed that a measure of the difference between these solutions can be obtained using the GLSM. This measure is indeed sensitive to the presence of cracks but we were unable to prove monotonicity results as for the first indicator function. The numerical results however suggest that the obtained indicator function depends on the density of cracks inside Ω . Then, varying Ω over the probed domain provides an image that reflects how badly the material is damaged.

We illustrate the principle of the first method in Figures 3.18–3.20 in a simplified 2D setting. In Figure 3.21, we compare the performances of the two approaches with the ones obtained with

the classical Factorization Method. Numerically we observe the superiority of the second method for relatively sparse networks. When the crack network becomes denser, only the first method provides an indicator function that shows variations with respect to local crack densities.

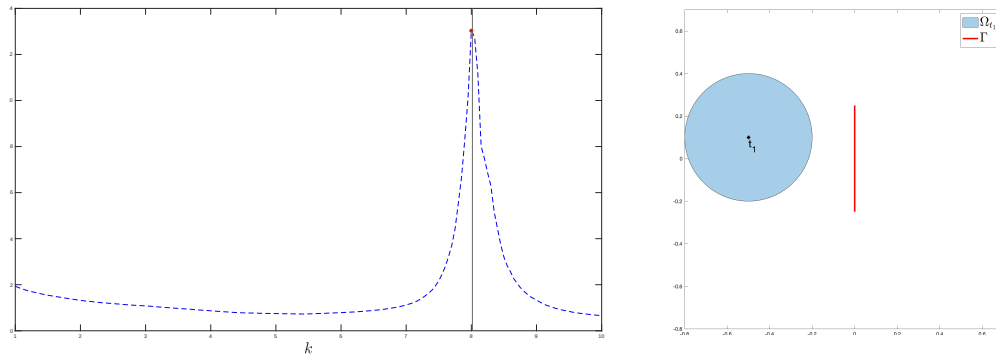


Figure 3.18: Plot of the indicator function (left) computed with the artificial obstacle Ω represented on right. The peak of the curve coincides with the known eigenvalue of the problem in Ω without crack (indicated by the vertical line): we deduce that there is no crack inside Ω .

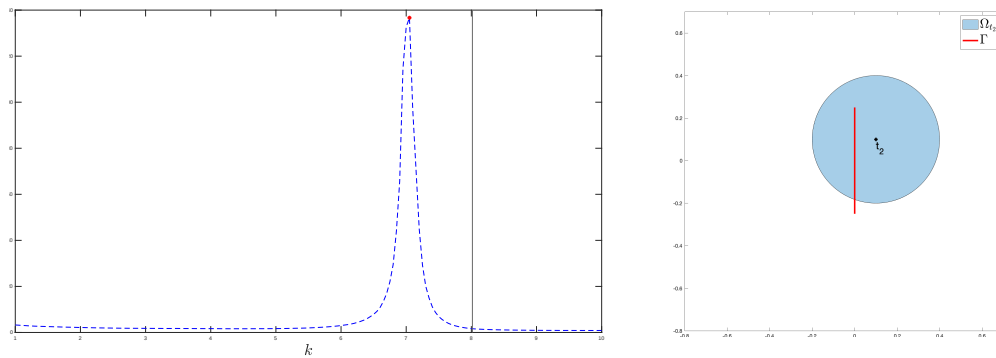


Figure 3.19: Plot of the indicator function (left) computed with the artificial obstacle Ω represented on right. The peak of the curve does not coincide with the known eigenvalue of the problem in Ω without crack (indicated by the vertical line): we deduce that there is some crack inside Ω .

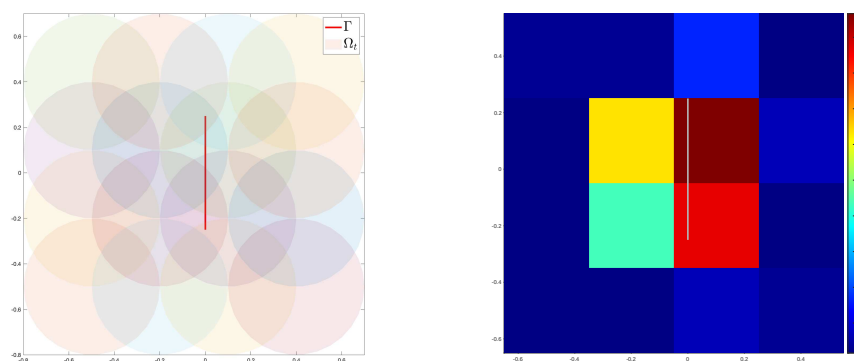


Figure 3.20: Left: collection of artificial disks used to identify the crack. Right: reconstruction provided by the first method. The amplitude of the indicator function gives an idea of the density of cracks inside the corresponding square.

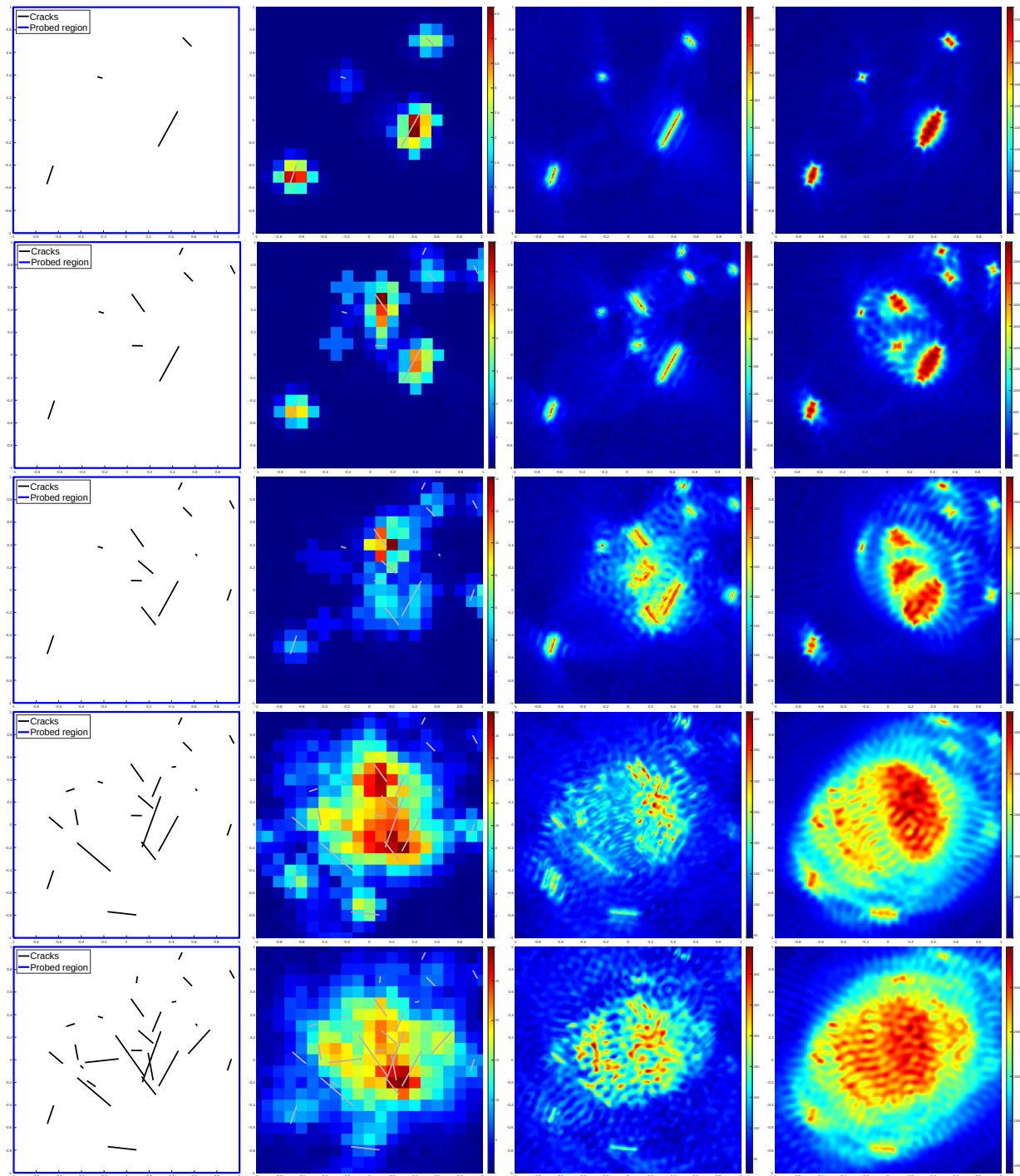


Figure 3.21: From left to right: exact configuration of the cracks, first method, second method and classical Factorization Method indicator functions. From top to bottom, additional cracks are added to the medium. The synthetic data is corrupted with 1% random noise.

In my work, I have been mainly interested in topics related to inverse wave scattering theory. But I have also considered a question appearing in Electrical Impedance Tomography (EIT).

3.6 Construction of indistinguishable conductivity perturbations in EIT

EIT is a noninvasive imaging technique with applications *e.g.* in medical imaging, process tomography, and nondestructive testing of materials [90, 148, 460]. The aim of EIT is to reconstruct the conductivity distribution inside the examined bounded physical body $\Omega \subset \mathbb{R}^d$, $d \geq 2$, from boundary measurements of current and potential. From the purely theoretical standpoint, the inverse problem of EIT, also known as the inverse conductivity problem, corresponds to determining the strictly positive conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ in the elliptic equation

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega \quad (3.47)$$

from knowledge of the corresponding Neumann-to-Dirichlet (or Dirichlet-to-Neumann) map at the object boundary $\partial\Omega$. This formulation corresponds to the idealized continuum model (CM), which is mathematically attractive in its simplicity but lacks a straightforward connection to practical EIT measurements that are always performed with a finite number of contact electrodes. On the other hand, it is widely acknowledged that the most accurate model for real-life EIT is the complete electrode model (CEM), which takes into account electrode shapes and contact resistances at electrode-object interfaces [149, 452].

Instead of the CM or the CEM, in [164], with N. Hyvönen and S. Staboulis during my post-doc at Aalto University in Helsinki, we decided to work with the point electrode model (PEM). The latter treats the current-feeding electrodes of EIT as delta-like boundary current sources and models the potential measurements as pointwise evaluations of the corresponding solution to (3.47) at the electrode locations [262]. Note that to make such a model well defined, relative potential measurements need to be considered due to the singularities induced by the localized current inputs (see below). The PEM is a reasonably good model for practical EIT measurements - in particular, superior to the CM, see [280] - if the electrodes are small compared to the size of the imaged body. In [164], we adapted the method presented in §2.2 to construct invisible perturbations of a given reference conductivity for an arbitrary but fixed electrode configuration modeled by the PEM.

To put this work into perspective, let us briefly review the main results on the unique solvability of the inverse conductivity problem, which was proposed to the mathematical community by Calderón in [129]. For $d \geq 3$, the global uniqueness for $\mathcal{C}^2$ -conductivities was proven in [458]; an extension of this result to the case of Lipschitz conductivities can be found in [253]. The first global uniqueness proof in two spatial dimensions was given by [364] for $\mathcal{C}^2$ -conductivities; subsequently, [21] proved uniqueness for general L^∞ -conductivities. All these articles assume the Cauchy data for (3.47) are known on all of $\partial\Omega$, but the partial data problem of having access only to some subset(s) of $\partial\Omega$ has also been tackled by many mathematicians; see *e.g.* [284, 296] and the references therein. From our view point, the most important uniqueness results are presented in [281, 444] where it is shown that any two-dimensional conductivity that equals a known constant in some interior neighborhood of $\partial\Omega$ is uniquely determined by the PEM measurements with a countably infinite number of electrodes. Our article [164] complements [281, 444] by constructively showing that for an arbitrary, but fixed and finite, electrode configuration there exists a perturbation of the unit conductivity that is invisible for the EIT measurements in the framework of the PEM. For completeness, it should also be mentioned that the nonuniqueness of the inverse conductivity problem has been studied for finite element method (FEM) discretizations with piecewise linear basis functions in [324]. Take note that we restrict our attention exclusively to isotropic conductivities because it is well-known that the inverse conductivity problem is not uniquely solvable for

anisotropic conductivities; see *e.g.* [20, 456] and the references therein.

Following what has been presented in §2.2, our constructive proof for the existence of “invisible conductivities” is based on introducing a small perturbation of a given reference conductivity and applying a suitable fixed point iteration. Although the technique is, in principle, applicable in any spatial dimension $d \geq 2$ and for an arbitrary smooth enough reference conductivity, a necessary intermediate result about linear independence of certain auxiliary functions is proved in [164] only for $d = 2$ and the unit reference conductivity, making the main theoretical result of this work two-dimensional. In [164], we implemented the constructive proof as a numerical algorithm that is capable of producing conductivities that are indistinguishable from the unit conductivity for a given set of point electrodes. We also numerically demonstrated that such conductivities are almost invisible for small electrodes within the CEM as well.

It should be emphasized that the existence of invisible conductivity perturbations for a finite number of electrodes is not very surprising: the space of admissible conductivities is infinite dimensional whereas EIT measurements with $N + 1$ electrodes include only $N(N + 1)/2$ degrees of freedom. Be that as it may, we were not aware of any previous works providing examples on this nonuniqueness that is of importance in practical EIT.

Setting

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a simply connected and bounded domain with a $\mathcal{C}^\infty$ -boundary. Introduce a real valued reference conductivity $\sigma^0 \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\sigma^0 \geq c > 0$ in Ω . Here and below $C, c > 0$ denote generic constants that may change from one occurrence to the next. Consider the Neumann boundary value problem

$$\operatorname{div}(\sigma^0 \nabla u^0) = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma^0 \nabla u^0 = f \quad \text{on } \partial\Omega \quad (3.48)$$

for a current density f in

$$H_\diamond^s(\partial\Omega) := \{g \in H^s(\partial\Omega) \mid \langle g, 1 \rangle_{\partial\Omega} = 0\}, \quad (3.49)$$

with some $s \in \mathbb{R}$. Here and in what follows, ν denotes the unit normal vector of $\partial\Omega$ oriented toward the exterior of Ω . We observe that the dual of $H_\diamond^s(\partial\Omega)$ is realized by

$$H^{-s}(\partial\Omega)/\mathbb{R} := H^{-s}(\partial\Omega)/\operatorname{span}\{1\}, \quad s \in \mathbb{R}. \quad (3.50)$$

It follows from standard theory of elliptic boundary value problems that (3.48) has a unique solution $u^0 \in H^{s+3/2}(\Omega)/\mathbb{R}$, which is smooth away from $\partial\Omega$ due to the interior regularity for elliptic equations. In particular, for any domain $\omega \subset \Omega$ such that $\overline{\omega} \subset \Omega$, it holds that

$$c \|u^0\|_{H^1(\omega)/\mathbb{R}} \leq \|u^0\|_{H^{s+3/2}(\Omega)/\mathbb{R}} \leq C \|f\|_{H_\diamond^s(\partial\Omega)},$$

where $c = c(s, \omega)$ and $C = C(s)$ [341]. Let us next consider a perturbed conductivity

$$\sigma^\varepsilon := \sigma^0 + \varepsilon \kappa, \quad (3.51)$$

where $\kappa \in L^\infty(\Omega)$ is compactly supported in Ω and $\varepsilon > 0$ is such that $\sigma^\varepsilon \geq c > 0$ in Ω . The corresponding perturbed Neumann boundary value problem

$$\operatorname{div}(\sigma^\varepsilon \nabla u^\varepsilon) = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma^\varepsilon \nabla u^\varepsilon = f \quad \text{on } \partial\Omega, \quad (3.52)$$

with $f \in H_\diamond^s(\partial\Omega)$ for some $s \in \mathbb{R}$, has a unique solution in $(H^{s+3/2}(\Omega) \cap H_{\operatorname{loc}}^1(\Omega))/\mathbb{R}$ (cf. *e.g.* [341, 263]). We note that the relative Neumann-to-Dirichlet map

$$\Lambda^\varepsilon - \Lambda^0 : f \mapsto (u^\varepsilon - u^0)|_{\partial\Omega}, \quad \mathcal{D}'_\diamond(\partial\Omega) \rightarrow \mathcal{D}(\partial\Omega)/\mathbb{R}$$

is well-defined. Here, the mean-free distributions $\mathcal{D}'_\diamond(\partial\Omega)$ and the quotient space of smooth boundary potentials $\mathcal{D}(\partial\Omega)/\mathbb{R}$ are defined in accordance with (3.49) and (3.50). This regularity result can be deduced from standard elliptic theory (cf. e.g. [341, 263]) and follows from the fact that Λ^ε and Λ^0 are pseudodifferential operators with the same symbol on $\partial\Omega$ because $\sigma^\varepsilon - \sigma^0$ vanishes in some interior neighborhood of $\partial\Omega$. One can also prove the following symmetry property for the relative Neumann-to-Dirichlet map (see e.g. [444, Thm. 2.1]).

Proposition 3.6.1 *For all $\varphi, \varphi' \in \mathcal{D}'_\diamond(\partial\Omega)$, we have*

$$\langle \varphi, (\Lambda^\varepsilon - \Lambda^0)\varphi' \rangle_{\partial\Omega} = \langle \varphi', (\Lambda^\varepsilon - \Lambda^0)\varphi \rangle_{\partial\Omega} = \int_{\Omega} (\sigma^0 - \sigma^\varepsilon) \nabla u_\varphi^\varepsilon \cdot \nabla u_{\varphi'}^0 \, d\mathbf{x},$$

where $u_\varphi^\varepsilon, u_{\varphi'}^0$ denote respectively the solutions of (3.52), (3.48) with f equal to φ, φ' .

Now, consider an observer who can impose currents between pairs of (small) electrodes located at $\mathbf{x}_0, \dots, \mathbf{x}_N \in \partial\Omega$ and measure the resulting voltages for both conductivities σ^0 and σ^ε . According to the PEM [262], this corresponds to knowing all elements in the matrix of relative measurements $\mathcal{M}(\sigma^\varepsilon) \in \mathbb{R}^{N \times N}$ defined via

$$\mathcal{M}_{ij}(\sigma^\varepsilon) = \langle \delta_i - \delta_0, (\Lambda^\varepsilon - \Lambda^0)(\delta_j - \delta_0) \rangle_{\partial\Omega}, \quad i, j = 1, \dots, N,$$

where $\delta_n \in H^{-(d-1)/2-\eta}(\partial\Omega)$, $\eta > 0$, stands for the Dirac delta distribution supported at $\mathbf{x}_n$. It follows from Proposition 3.6.1 that $\mathcal{M}(\sigma^\varepsilon)$ is symmetric and, furthermore,

$$\mathcal{M}_{ij}(\sigma^\varepsilon) = \int_{\Omega} (\sigma^0 - \sigma^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla u_j^0 \, d\mathbf{x}, \quad (3.53)$$

where $u_n^0, u_n^\varepsilon \in (H^{-(d-4)/2-\eta}(\Omega) \cap H_{\text{loc}}^1(\Omega))/\mathbb{R}$, $\eta > 0$, are the solutions of the Neumann problems

$$\operatorname{div}(\sigma^0 \nabla u^0) = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma^0 \nabla u^0 = \delta_n - \delta_0 \quad \text{on } \partial\Omega \quad (3.54)$$

and

$$\operatorname{div}(\sigma^\varepsilon \nabla u^\varepsilon) = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma^\varepsilon \nabla u^\varepsilon = \delta_n - \delta_0 \quad \text{on } \partial\Omega, \quad (3.55)$$

respectively. The goal of [164] was to find $\sigma^\varepsilon \not\equiv \sigma^0$ such that $\mathcal{M}(\sigma^\varepsilon)$ vanishes.

Main result

Let $\omega \subset \Omega \subset \mathbb{R}^d$ be a nonempty Lipschitz domain such that $\bar{\omega} \subset \Omega$. We recall the decomposition $\sigma^\varepsilon = \sigma^0 + \varepsilon\kappa$ and require that $\kappa \in L^\infty(\Omega)$ satisfies $\operatorname{supp}(\kappa) \subset \bar{\omega}$. Let us compute an asymptotic expansion for $\mathcal{M}(\sigma^\varepsilon)$. To this end, we write

$$u_n^\varepsilon = u_n^0 + \varepsilon \tilde{u}_n^\varepsilon \quad (3.56)$$

where $u_n^\varepsilon, u_n^0 \in (H^{-(d-4)/2-\eta}(\Omega) \cap H_{\text{loc}}^1(\Omega))/\mathbb{R}$ are defined by (3.55) and (3.54), respectively. Plugging (3.56) in (3.55), we see that $\tilde{u}_n^\varepsilon$ must satisfy the Neumann problem

$$\operatorname{div}(\sigma^\varepsilon \nabla \tilde{u}_n^\varepsilon) = -\operatorname{div}(\kappa \nabla u_n^0) \quad \text{in } \Omega, \quad \nu \cdot \sigma^0 \nabla \tilde{u}_n^\varepsilon = 0 \quad \text{on } \partial\Omega. \quad (3.57)$$

Since $\operatorname{div}(\kappa \nabla u_n^0)$ defines a compactly supported source in $H^{-1}(\Omega)$, by applying the Lax-Milgram theorem to the variational formulation of (3.57), we see that (3.57) has a unique solution $\tilde{u}_n^\varepsilon \in H^1(\Omega)/\mathbb{R}$. Inserting (3.56) in (3.53), we deduce that

$$\mathcal{M}_{ij}(\sigma^\varepsilon) = -\varepsilon \int_{\Omega} \kappa \nabla u_i^0 \cdot \nabla u_j^0 \, d\mathbf{x} - \varepsilon^2 \int_{\Omega} \kappa \nabla \tilde{u}_i^\varepsilon \cdot \nabla u_j^0 \, d\mathbf{x}. \quad (3.58)$$

The representation (3.58) demonstrates that $\mathcal{M}(\sigma^\varepsilon)$ is of the order ε , which was to be expected as $\mathcal{M}(\sigma^0)$ is the null matrix and $\sigma^\varepsilon - \sigma^0 = \varepsilon\kappa$. It is relatively easy to find a nontrivial κ such that $\mathcal{M}(\sigma^\varepsilon)$ is of the order ε^2 . To cancel the higher order terms in ε which depend less explicitly on κ , we introduce a suitable fixed point problem. To proceed, we redecompose κ in the form

$$\kappa = \kappa_0 + \sum_{j=1}^N \sum_{i=1}^j \tau_{ij} \kappa_{ij} \quad (3.59)$$

where $\tau_{ij} \in \mathbb{R}$ are free parameters. Moreover, $\kappa_0, \kappa_{ij} \in L^\infty(\Omega)$ are fixed functions that are supported in $\bar{\omega}$ and assumed to satisfy the conditions

$$\int_{\Omega} \kappa_0 \nabla u_{i'}^0 \cdot \nabla u_{j'}^0 d\mathbf{x} = 0 \quad \text{for } 1 \leq i' \leq j' \leq N \quad (3.60)$$

and

$$\int_{\Omega} \kappa_{ij} \nabla u_{i'}^0 \cdot \nabla u_{j'}^0 d\mathbf{x} = \begin{cases} 1 & \text{if } (i, j) = (i', j') \text{ or } (j, i) = (i', j'), \\ 0 & \text{else.} \end{cases} \quad (3.61)$$

Establishing that such κ_{ij} exist is the main difficulty in the analysis. In [164] we showed that this is the case when $\sigma^0 \equiv 1$ and when Ω is a simply connected domain of $\mathbb{R}^2$ with a $\mathcal{C}^\infty$ boundary. Our proof exploits conformal maps and we have not been able to extend it to 3D. Note that it is reasonable to assume that the $N(N+1)/2$ free parameters τ_{ij} in the perturbation κ are enough to cancel out the symmetric matrix $\mathcal{M}(\sigma^\varepsilon) \in \mathbb{R}^{N \times N}$ (cf. Proposition 3.6.1). Substituting (3.59) in (3.58) and using (3.60)–(3.61), we obtain the expansion

$$\mathcal{M}(\sigma^\varepsilon) = -\varepsilon \tau - \varepsilon^2 \tilde{\mathcal{M}}^\varepsilon(\tau)$$

where

$$\tilde{\mathcal{M}}_{ij}^\varepsilon(\tau) = \tilde{\mathcal{M}}_{ji}^\varepsilon(\tau) = \int_{\Omega} \kappa \nabla \tilde{u}_i^\varepsilon \cdot \nabla u_j^0 d\mathbf{x}, \quad 1 \leq i \leq j \leq N,$$

and $\tau \in \mathbb{R}^{N \times N}$ is the symmetric matrix defined by the parameters τ_{ij} . Thus imposing $\mathcal{M}(\sigma^\varepsilon) = 0$ is equivalent to solving the following fixed point problem:

$$\left| \text{Find } \tau \in S^N \text{ such that } \tau = F^\varepsilon(\tau), \right. \quad (3.62)$$

where S^N denotes the space of symmetric $N \times N$ matrices and $F^\varepsilon : S^N \rightarrow S^N$ is defined by

$$F^\varepsilon(\tau) = -\varepsilon \tilde{\mathcal{M}}^\varepsilon(\tau).$$

In [164] we showed that for any fixed parameter $\gamma > 0$, there is $\varepsilon > 0$ small enough such that the map F^ε has the invariant set

$$\mathbb{B}_\gamma := \{\tau \in S^N \mid |\tau| \leq \gamma\} \quad (3.63)$$

on which it is a contraction. In (3.63), $|\cdot|$ denotes an arbitrary norm of S^N . As a consequence, the Banach fixed point theorem guarantees the existence of $\varepsilon_0 = \varepsilon_0(\gamma) > 0$ such that for all $\varepsilon \in (0; \varepsilon_0]$, the fixed point problem (3.62) has a unique solution in $\mathbb{B}_\gamma$, enabling the construction of σ^ε for which $\mathcal{M}(\sigma^\varepsilon) = 0$ via (3.59) and (3.51).

Theorem 3.6.2 *Assume that $\sigma^0 \equiv 1$. Let $\Omega \subset \mathbb{R}^2$ be a simply connected and bounded domain with a $\mathcal{C}^\infty$ -boundary. For any Lipschitz domain $\omega \subset \Omega$ such that $\bar{\omega} \subset \Omega$, there exists a conductivity $\sigma^\varepsilon \in L^\infty(\Omega)$, with $\sigma^\varepsilon \geq c > 0$, such that $\sigma^\varepsilon - \sigma^0 \not\equiv 0$, $\text{supp}(\sigma^\varepsilon - \sigma^0) \subset \bar{\omega}$ and*

$$\mathcal{M}_{i,j}(\sigma^\varepsilon) = \langle \delta_i - \delta_0, (\Lambda^\varepsilon - \Lambda^0)(\delta_j - \delta_0) \rangle_{\partial\Omega} = 0$$

for all $i, j = 1, \dots, N$.

It is important to notice that the constructed κ defined by (3.59) satisfies $\kappa \neq 0$ whenever $\kappa_0 \neq 0$ by virtue of the orthogonality conditions (3.60)–(3.61). In other words, if $\kappa_0 \neq 0$, then also $\sigma^\varepsilon \neq \sigma^0$ as required. Moreover, for given σ^0 , κ_0 , κ_{ij} and $\gamma > 0$, the upper bound $\varepsilon_0 > 0$ can be tuned to ensure that the perturbed conductivity corresponding to κ of (3.59) satisfies $\sigma^\varepsilon \geq c > 0$ for all $\varepsilon \in [0; \varepsilon_0]$ and $\tau \in \mathbb{B}_\gamma$, which guarantees that all conductivities involved in the fixed point iteration are admissible.

In Figures 3.22, 3.23, we give examples of conductivities which provide the same measurements as the reference conductivity $\sigma^0 \equiv 1$ for different numbers N and positions of electrodes. As N increases, the values of ε such that we have convergence for the iterative procedure used to solve the fixed point problem (3.62), become smaller and smaller. As a consequence, the amplitude of the indistinguishable conductivities is small for large N .

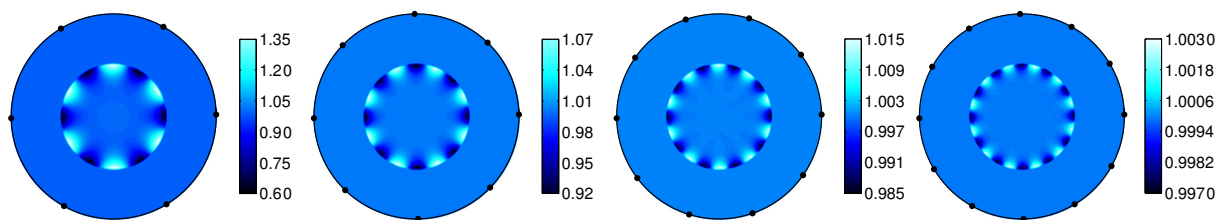


Figure 3.22: Examples of invisible conductivities for $N = 5, 7, 9, 11$ (recall that $N + 1$ is the number of electrodes). The black dots represent the electrodes.

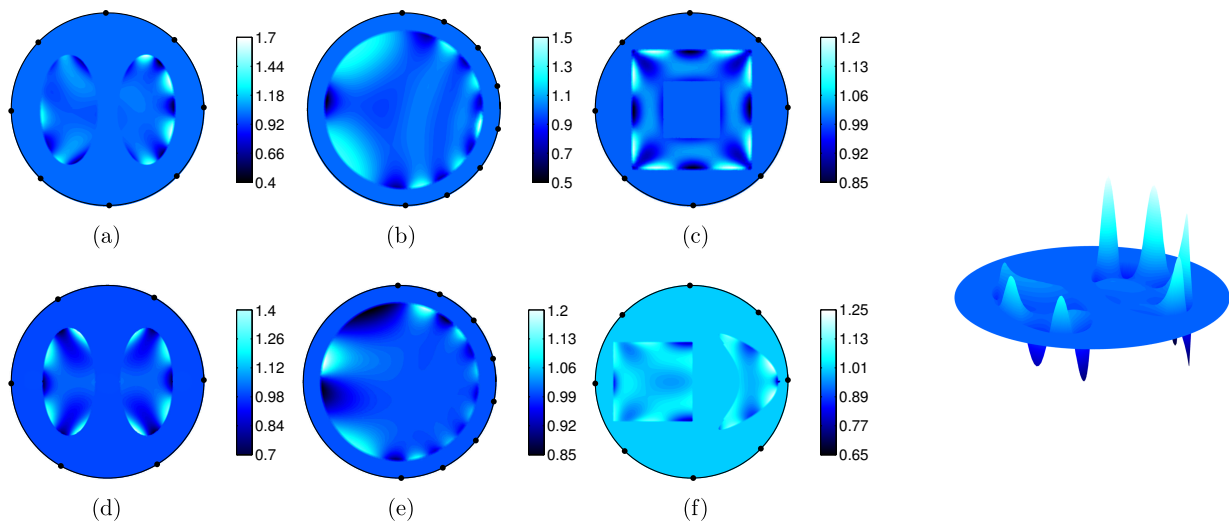


Figure 3.23: Other examples of invisible conductivities. The black dots represent the electrodes. The right picture is a surface plot of the conductivity (a).

3.7 Future directions and open questions

In this section, we gather a few open problems and possible directions of research related to the topics of that chapter.

i) The Interior Transmission Problem (ITP) introduced in (3.1) is a fertile ground of open questions and we already mentioned several of them in §3.1.4. In addition, it would be interesting to study more TEs which are complex but not real. Numerically they seem rather common. Some proofs

of existence have been obtained in simple configurations (in particular for 1D problems). How to show their existence in 2D/3D? for Maxwell's problem? And what can one say concerning their influence on reconstruction methods? In practice, one works with real $k > 0$ but what happens for example to the LSM at a real $k > 0$ such that there is a complex TE close to it?

ii) In §§3.2 and 3.6, we explained how to construct invisible inclusions for two inverse problems with a finite number of measurements. The consequence of the existence of these indistinguishable defects for the corresponding reconstruction methods is not clear and would deserve to be studied further.

iii) The idea of working with artificial backgrounds to improve reconstruction methods as studied in §3.3 looks very promising. The main point here consists in exploiting more cleverly the given data by comparing/modifying them with computed quantities. For the artificial background, there are many possibilities and finding the right choice for a given problem is a vast field of research. In his PhD [427], Fabien Pourre considered this question in the acoustic case for media with many small scatterers.

iv) The inside-outside duality considered in §3.4 allows one to characterize TEs via the behavior of the eigenvalues of the scattering matrix. This has been done in $\mathbb{R}^d$. Is there some similar property for the scattering matrix defined in waveguides as in Chapter 2?

v) In the previous chapter, we saw that in waveguides, complex resonances located close to the real axis have a strong influence on scattering properties for real frequencies in a neighborhood of the resonance. We can imagine that this is true in $\mathbb{R}^d$ as well (think of an almost trapping domain). How do imaging techniques behave for these types of objects? Does the presence of the complex resonance destroy or improve the reconstruction?

vi) We mentioned in §3.6 that the most accurate model for real-life EIT is the complete electrode model (CEM) which takes into account electrode shapes and contact resistances at electrode-object interfaces. We tried to construct indistinguishable conductivities for this model, with again a finite number of electrodes, but we were unable to justify the method. This deserves to be studied.

MISCELLANEOUS PROBLEMS

In this chapter, we present briefly four works which are not directly connected to the themes developed in the previous chapters. The first two ones are related to problems of asymptotic analysis. The other two concern questions of singular behaviors for solutions of boundary value problems.

4.1 A numerical approach for a problem with a small inclusion

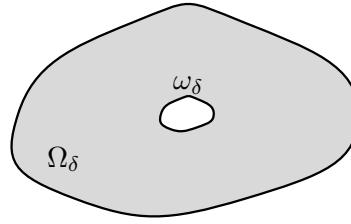


Figure 4.1: Domain Ω_δ with a small obstacle.

In the article [155], with X. Claeys, we considered the Poisson equation in a domain with a small hole of size δ . We presented a simple numerical method, based on asymptotic analysis, which allows one to approximate robustly the far field of the solution as δ goes to zero without meshing the small hole.

Let $\omega, \Omega \subset \mathbb{R}^2$ be two bounded Lipschitz domains such that $\bar{\omega} \subset \Omega$. For $\delta > 0$, define $\omega_\delta := \{\mathbf{x} \in \mathbb{R}^2, \mathbf{x}/\delta \in \omega\}$ and $\Omega_\delta := \Omega \setminus \bar{\omega}_\delta$ (see Figure 4.1). Given a data $f \in L^2(\Omega)$, we are interested in devising a robust and accurate numerical method for approximating, for small values of δ , the far field of the function satisfying

$$u_\delta \in H_0^1(\Omega_\delta) \quad \text{and} \quad -\Delta u_\delta = f \quad \text{in } \Omega_\delta. \quad (4.1)$$

In (4.1), $H_0^1(\Omega_\delta)$ denotes the subspace of the elements of the Sobolev space $H^1(\Omega_\delta)$ vanishing on $\partial\Omega_\delta$. On the other hand, we call far field of u_δ the restriction of u_δ to $\Omega \setminus \bar{B}_r$, where $B_r := B(0, r)$ is the disk with fixed arbitrary radius $r > 0$. Problem (4.1), or variants of it, arises as a simple but relevant model in many applications ranging from electrical engineering [52, 431] to flow transport around wells [147, 414]. This kind of problem also appears when considering wave scattering by small impenetrable inclusions [136].

In order to solve numerically Problem (4.1), a crude but rather natural idea would consist in neglecting the influence of the small inclusion on the total field u_δ . Indeed (see for example [352]), as $\delta \rightarrow 0$ the function u_δ converges toward u_0 the solution to the limit problem where the inclusion has disappeared

$$u_0 \in H_0^1(\Omega) \quad \text{and} \quad -\Delta u_0 = f \quad \text{in } \Omega.$$

However, in general (more precisely when $u_0(0) \neq 0$), the convergence turns out to be very slow: for any arbitrary radius $r > 0$ such that $\overline{B_r} \subset \Omega$, we have $\|u_\delta - u_0\|_{H^1(\Omega \setminus \overline{B_r})} \geq C |\ln \delta|^{-1} \|f\|_{L^2(\Omega)}$, for some constant $C > 0$ independent of δ . To give an idea $|\ln \delta|^{-1} \approx 0.0434$ for $\delta = 10^{-10}$. Thus, neglecting the presence of the small inclusion is not satisfactory and a reasonable approach for approximating the solution of (4.1) should reproduce accurately the perturbation induced by the presence of the small inclusion ω_δ .

Most of the numerical approaches that could be considered for dealing with this problem suffer from a numerical locking effect [31]: performances deteriorate as $\delta \rightarrow 0$. Admittedly, robust methods already exist in the literature, like the multi-scale finite element method coupled with some mesh refinement strategy [219], or the boundary element method (see *e.g.* [14]). These techniques provide very good results in many cases, but they require careful and thorough implementation efforts, and/or rely on strong assumptions such as homogeneity of the coefficients of the equation under consideration. Other methods are based on an approximation of u_δ of the form “ $u_0 + \text{corrector}$ ”, where both terms of this sum are computed separately (see for example [203, 60]). These approaches may induce substantial additional computational cost in a real life simulation as we explain below. In many practical situations, small inclusions are not the main subject of concern and it seemed to us desirable to devise a simple, general purpose and implementation friendly method that does not rely on any kind of mesh refinement technique while remaining robust as δ tends to zero.

In [155], we proposed an approach based on the following methodology. First, we computed an asymptotic expansion of u_δ with respect to δ . To state the result which is interesting for our study, we have to introduce a few objects. Let G , P denote respectively the Green function and logarithmic capacity potential, defined in normalized geometries, such that

$$\left\{ \begin{array}{l} -\Delta G = 0 \text{ in } \Omega \setminus \{0\} \\ G = 0 \text{ on } \partial\Omega \\ G(\mathbf{x}) = \frac{1}{2\pi} \ln(1/|\mathbf{x}|) + O(1) \text{ as } |\mathbf{x}| \rightarrow 0 \end{array} \right. \quad \left\{ \begin{array}{l} -\Delta P = 0 \text{ in } \Xi := \mathbb{R}^2 \setminus \overline{\omega} \\ P = 0 \text{ on } \partial\Xi \\ P(\boldsymbol{\xi}) = \frac{1}{2\pi} \ln(1/|\boldsymbol{\xi}|) + O(1) \text{ as } |\boldsymbol{\xi}| \rightarrow \infty \end{array} \right.$$

Classically, there exist constants G_0 , P_0 that depend only on the domains Ω , ω such that

$$\left\{ \begin{array}{l} G(\mathbf{x}) - (2\pi)^{-1} \ln |\mathbf{x}|^{-1} - G_0 = O(|\mathbf{x}|) \text{ for } |\mathbf{x}| \rightarrow 0, \\ P(\boldsymbol{\xi}) - (2\pi)^{-1} \ln |\boldsymbol{\xi}|^{-1} - P_0 = O(|\boldsymbol{\xi}|^{-1}) \text{ for } |\boldsymbol{\xi}| \rightarrow +\infty. \end{array} \right.$$

Next, define the gauge function

$$\lambda(\delta) := \frac{2\pi}{\ln \delta + 2\pi(P_0 - G_0)} \quad (4.2)$$

and the approximation

$$v_\delta(\mathbf{x}) := u_0(\mathbf{x}) + u_0(0) \lambda(\delta) G(\mathbf{x}).$$

We proved the following result.

Proposition 4.1.1 *For any disk $B_r \subset \Omega$, there exist constants C_r , $\delta_0 > 0$ independent of δ such that*

$$\|u_\delta - v_\delta\|_{H^1(\Omega \setminus \overline{B_r})} \leq C_r \delta |\ln \delta| \|f\|_{L^2(\Omega)}, \quad \forall \delta \in (0; \delta_0]. \quad (4.3)$$

This last result shows that $v_\delta = u_0 + u_0(0)\lambda(\delta)G$ provides a reasonable approximation (for example $\delta |\ln \delta| \approx 2.3 \cdot 10^{-9}$ for $\delta = 10^{-10}$) of u_δ at any fixed distance from the small hole. Thus, the far field of u_δ appears as the superposition of the limit field u_0 and a field “radiated” by a point source located at the center of the hole.

Numerically, u_0 , G can be approximated by functions u_0^h , G^h using a standard finite element method (here, h refers to some mesh size) and define $v_\delta^h = u_0^h + u_0^h(0)\lambda(\delta)G^h$. Then, (4.3) ensures that v_δ^h is a good approximation of the far field of u_δ . This procedure is rather simple to implement and it has been proven for slightly different problems that it gives good results. However, it requires solving two problems (plus the knowledge of the constant G_0) which we would like to avoid because it may be time consuming. Adapting this approach to the case of N inclusions would lead to $N + 1$ numerical solves. Similarly, looking for an approximation of u_δ as sharp as the first M terms of its asymptotic expansion would lead to M numerical solves when there is one inclusion. From this perspective, for practical computations, a method involving only one numerical solve would be much more interesting.

This is why we then derived an approximate model whose solution has the same far field asymptotics as u_δ , up to a remainder in $O(\delta^{1-\varepsilon})$, for all $\varepsilon > 0$. Let us decompose G under the form

$$G = s_{\log} + \tilde{G} \quad \text{with} \quad s_{\log}(\mathbf{x}) := \frac{1}{2\pi} \chi(\mathbf{x}) \ln(1/|\mathbf{x}|) \quad \text{and} \quad \tilde{G} \in H_0^1(\Omega) \cap \mathcal{C}^0(\Omega),$$

where $\chi \in \mathcal{C}_0^\infty(\Omega)$ is equal to one in a neighborhood of O . This allows us to write v_δ as

$$v_\delta = w_\delta + u_0(0)\lambda(\delta) s_{\log}$$

with $w_\delta := u_0 + u_0(0)\lambda(\delta)\tilde{G}$. Let us express the coefficient $u_0(0)\lambda(\delta)$ by means of w_δ . According to the definition of u_0 and G it is clear that w_δ belongs to $H_0^1(\Omega) \cap \mathcal{C}^0(\Omega)$. Moreover, observing that $\tilde{G} = G_0 + \hat{G}$ for some function $\hat{G} \in H^1(\Omega) \cap \mathcal{C}^0(\Omega)$ vanishing at 0, we find $w_\delta(0) = u_0(0)(1 + \lambda(\delta)G_0)$ and so $u_0(0)\lambda(\delta) = w_\delta(0)\lambda(\delta)/(1 + \lambda(\delta)G_0)$. Using Definition (4.2) of $\lambda(\delta)$, we deduce that

$$v_\delta = w_\delta + \frac{2\pi w_\delta(0)}{\ln \delta + 2\pi P_0} s_{\log}.$$

Let us emphasize that this expression for v_δ is interesting because it involves only one unknown function, namely w_δ . Now, we need to derive a problem characterizing w_δ . In the sense of distributions in Ω , there holds

$$-\Delta w_\delta = -\Delta(u_0 + u_0(0)\lambda(\delta)\tilde{G}) = f - \frac{2\pi w_\delta(0)}{\ln \delta + 2\pi P_0} \Delta \tilde{G}$$

with

$$\Delta \tilde{G}(\mathbf{x}) = (2\pi)^{-1} \left((\Delta \chi)(\mathbf{x}) \ln |\mathbf{x}| + 2\nabla \chi(\mathbf{x}) \cdot \nabla (\ln |\mathbf{x}|) \right).$$

Of course the term $w_\delta(0)$ is annoying to write a variational formulation in $H_0^1(\Omega)$ because the functional $w_\delta \mapsto w_\delta(0)$ is not defined in this space. So it cannot be exploited directly for discretization and then numerical computation. This is the motivation for considering a regularized counterpart where we replace $w_\delta(0)$ by $(2\pi\delta)^{-1} \int_{|\mathbf{x}|=\delta} w_\delta d\sigma$. Finally, this leads us to examine the following approximate model

$$\left| \begin{array}{l} \text{Find } \hat{w}_\delta \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega \nabla \hat{w}_\delta \cdot \nabla v d\mathbf{x} + \frac{2\pi}{\ln \delta + 2\pi P_0} \frac{1}{2\pi\delta} \int_{|\mathbf{x}|=\delta} \hat{w}_\delta d\sigma \int_\Omega \Delta \tilde{G} v d\mathbf{x} = \int_\Omega f v d\mathbf{x} \quad \forall v \in H_0^1(\Omega). \end{array} \right. \quad (4.4)$$

We emphasize that Problem (4.4) is set in Ω and not in Ω_δ . As a consequence, when discretizing it with a finite element method, we do not have to adapt the mesh to the small obstacle. Moreover (4.4) requires only one numerical resolution. After that, we established a result of consistency for the Galerkin discretization of (4.4) which is quasi-optimal and, this is important, uniform with respect to δ . This led us to the following result

Proposition 4.1.2 *Consider a finite dimensional space $V^h \subset H_0^1(\Omega)$ and denote by $\hat{w}_\delta^h$ the solution of the variational formulation (4.4) in V^h . For any disk $B_r \subset \Omega$ and for any $\varepsilon > 0$, there exists a constant $C > 0$ depending on r, ε but not on δ and h such that, for δ small enough,*

$$\begin{aligned} & \|u_\delta - (\hat{w}_\delta^h + \frac{2\pi}{\ln \delta + 2\pi P_0} \frac{1}{2\pi\delta} \int_{|x|=\delta} \hat{w}_\delta^h d\sigma s_{\log})\|_{H^1(\Omega \setminus \overline{B_r})} \\ & \leq C(\delta^{1-\varepsilon} + |\ln \delta|^{-1} \inf_{\varphi^h \in V^h} \|\tilde{G} - \varphi^h\|_{H^1(\Omega)}) \|f\|_{L^2(\Omega)} + C \inf_{\varphi^h \in V^h} \|u_0 - \varphi^h\|_{H^1(\Omega)}. \end{aligned}$$

In [155], we also implemented numerically the approach and indeed observed that it does not deteriorate as δ tends to zero.

In theory, it should be possible to adapt the method to obtain higher-order models that can be solved with only one numerical resolution. However this has to be written properly and tested numerically.

4.2 Small obstacle asymptotics for a 2D semi-linear problem

With X. Claeys and S.A. Nazarov, we continued our investigations concerning asymptotic analysis in presence of small objects by considering in [159] a 2D non-linear equation in a domain with a small Dirichlet obstacle of size δ . Let $\Omega_\delta \subset \mathbb{R}^2$ be as in the previous section. For $f \in L^2(\Omega)$ and some fixed integer $m \in \mathbb{N}$, we are interested in the following semi-linear problem

$$\begin{cases} \text{Find } u_\delta \in H_0^1(\Omega_\delta) \text{ such that} \\ -\Delta u_\delta + (u_\delta)^{2m+1} = f \quad \text{in } \Omega_\delta. \end{cases} \quad (4.5)$$

First, we remind the reader how to prove that (4.5) admits a unique solution. For any Lipschitz domain $\mathcal{O} \subset \mathbb{R}^2$, the Rellich-Kondrachov embedding theorem ensures that $H^1(\mathcal{O}) \subset L^p(\mathcal{O})$ for all $p \geq 1$. As a consequence, the form $(u, v) \mapsto \int_{\Omega_\delta} \nabla u \cdot \nabla v + (u)^{2m} uv \, d\mathbf{x}$ is well-defined and continuous in $H_0^1(\Omega_\delta) \times H_0^1(\Omega_\delta)$. Introduce the continuous operator $A_\delta : H_0^1(\Omega_\delta) \rightarrow H^{-1}(\Omega_\delta) := H_0^1(\Omega_\delta)^*$ such that

$$\langle A_\delta(u), v \rangle = \int_{\Omega_\delta} \nabla u \cdot \nabla v + (u)^{2m} uv \, d\mathbf{x} \quad \forall u, v \in H_0^1(\Omega_\delta).$$

Classically, one finds that $u_\delta \in H_0^1(\Omega_\delta)$ satisfies (4.5) if and only if it solves the problem

$$\begin{cases} \text{Find } u_\delta \in H_0^1(\Omega_\delta) \text{ such that} \\ \langle A_\delta(u_\delta), v \rangle = \int_{\Omega_\delta} f v \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega_\delta). \end{cases} \quad (4.6)$$

Simple differential calculus shows that $A_\delta(u) - f$ is the Fréchet differential evaluated at u of the cost functional $J_\delta : H_0^1(\Omega_\delta) \rightarrow \mathbb{R}$ defined by

$$J_\delta(\varphi) = \frac{1}{2} \|\nabla \varphi\|_{L^2(\Omega_\delta)}^2 + \frac{1}{2m+2} \|\varphi\|_{L^{2m+2}(\Omega_\delta)}^{2m+2} - \int_{\Omega_\delta} f \varphi \, d\mathbf{x} \quad \forall \varphi \in H_0^1(\Omega_\delta).$$

One can check that u_δ is a minimum of J_δ if and only if it satisfies Problem (4.6). Since J_δ is continuous, strictly convex and coercive, it admits a unique minimum (see *e.g.* [295, Chap. 3]). Therefore (4.6) (as well as (4.5)) has a unique solution. From the convexity of the map $t \mapsto t^{2m+2}$ on $\mathbb{R}$, we obtain $(s^{2m+1} - t^{2m+1})(s - t) \geq 0$ for all $s, t \in \mathbb{R}$. This implies $\langle A_\delta(u) - A_\delta(v), u - v \rangle \geq \|u - v\|_{H_0^1(\Omega_\delta)}^2$ for all $u, v \in H_0^1(\Omega_\delta)$ and leads to the stability estimate

$$\|u - v\|_{H_0^1(\Omega_\delta)} \leq \|A_\delta(u) - A_\delta(v)\|_{H^{-1}(\Omega_\delta)} \quad \forall u, v \in H_0^1(\Omega_\delta).$$

Here we use the notation $\|\cdot\|_{H_0^1(\Omega_\delta)} := \|\nabla \cdot\|_{L^2(\Omega_\delta)^2}$. To sum up, for all $\delta \in (0; 1]$, the operator $A_\delta : H_0^1(\Omega_\delta) \rightarrow H^{-1}(\Omega_\delta)$ is a uniformly continuous bijection, and its inverse $A_\delta^{-1} : H_0^1(\Omega_\delta) \rightarrow H^{-1}(\Omega_\delta)$ is also uniformly continuous. The same conclusion holds for the operator $A_0 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ defined by

$$\langle A_0(u), v \rangle = \int_{\Omega} \nabla u \cdot \nabla v + (u)^{2m} uv \, d\mathbf{x} \quad \forall u, v \in H_0^1(\Omega).$$

The goal of [159] was to construct an asymptotic expansion of the solution u_δ to Problem (4.5) as $\delta \rightarrow 0$. For the usual Dirichlet Laplacian, as mentioned in the previous section, neglecting the small obstacle, that is considering the solution of the problem without the obstacle, is a very crude approximation of order $|\ln \delta|^{-1}$. Such an estimate also holds for the above non linear problem. Therefore, it is essential to compute a corrective term.

While literature concerning asymptotic analysis for linear elliptic PDEs is very rich (see *e.g.* the classical monographs [462, 298, 283, 352]), for non linear PDEs, it seems that there are much less references, especially if one looks for error estimates. The problem we consider has been studied in [12] in the context of topological sensitivity analysis. In [12], the author provides an error estimate in $o(|\ln \delta|^{-1})$. The first outcome of [159] was an improvement of this result. More precisely, we showed an estimate in $o(\delta^{1-\varepsilon})$ with any $\varepsilon > 0$. In [282] (see also the related papers [439, 213]), the authors study the same equation as in (4.5), but in 3D for a Dirichlet obstacle and in 2D for a Neumann obstacle. We emphasize that the asymptotic expansion is very different for the 2D Dirichlet obstacle. Compared to the case of the Neumann condition in 2D, at least two remarkable differences in the analysis can be pointed out. For the Dirichlet case, the matching equation is non trivial and requires a solution procedure (based on a fixed point argument), whereas the matching condition is totally explicit in the Neumann case. Another remarkable difference lies in the magnitude of the first corrector term that is $O(|\ln \delta|^{-1})$ in the Dirichlet case, whereas it is only $O(\delta)$ for the Neumann case. Moreover, in [282], the compound method is used and error estimates are proved in Hölder spaces. In [159], we work with different tools, employing the method of matched asymptotic expansions and showing error estimates in Sobolev spaces (Hölder estimates are available in our case, too). In the literature, it seems that people prefer to use the compound method to derive asymptotic expansions for non linear PDEs (cf. *e.g.* [352, §5.7] where the equation $-\Delta u_\delta + |u_\delta|^2 = f_\delta$ with various boundary perturbations is discussed under the smallness assumption on the source term f_δ). The second objective of [159] was to investigate the method of matched asymptotic expansions for non linear PDEs.

For this technique, the general scheme is the following. First we compute an expansion of the solution with respect to δ far from the obstacle ω_δ . There, we search for an approximation of u_δ under the form $v_\delta := u_0 + \lambda(\delta)u_{1,\delta}$ where the functions $u_0, u_{1,\delta}$ have to be determined and where the gauge function $\lambda(\delta)$ (independent of $\mathbf{x}$) is supposed to vanish as $\delta \rightarrow 0$. Formally plugging this expansion into Equation (4.5) and retaining the first order terms as $\delta \rightarrow 0$, we find that u_0 must be a solution to the problem

$$\left| \begin{array}{l} \text{Find } u_0 \in H_0^1(\Omega) \text{ such that} \\ -\Delta u_0 + (u_0)^{2m+1} = f \quad \text{in } \Omega. \end{array} \right.$$

Using that $A_0 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is a continuous bijection, we can define u_0 as $u_0 := A_0^{-1}(f)$. In other words and quite naturally, we postulate that at first order, far from the origin, u_δ does not see the small obstacle.

To derive the correction term $u_{1,\delta}$ we impose that

$$\left| \begin{array}{ll} -\Delta(u_0 + \lambda(\delta)u_{1,\delta}) + (u_0 + \lambda(\delta)u_{1,\delta})^{2m+1} & = f \quad \text{in } \Omega \setminus \{O\} \\ u_0 + \lambda(\delta)u_{1,\delta} & = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (4.7)$$

Since $A_0 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is a bijection, we have to look for $u_{1,\delta}$ in a larger space than $H_0^1(\Omega)$ (otherwise, we would conclude $u_{1,\delta} = 0$ which is not interesting). Denote by G the Green function for the Laplace operator with homogeneous Dirichlet boundary condition such that $-\Delta G = \delta_O$ in Ω , $G = 0$ on $\partial\Omega$ (δ_O is the Dirac distribution centered at O). To simplify the notation, we write $G(\cdot)$ instead of $G(\cdot, O)$. By analogy with the more classical asymptotics for the 2D Dirichlet Laplace problem with a small obstacle (see for example [352, Chap. 2]), we impose the condition $u_{1,\delta} - G \in H^1(\Omega)$ (as a consequence, $u_{1,\delta}$ admits a logarithmic singularity at the origin). Expanding (4.7), we find that $u_{1,\delta}$ must be solution to the problem

$$\left| \begin{array}{l} \text{Find } u_{1,\delta} \text{ such that } u_{1,\delta} - G \in H^1(\Omega) \text{ and} \\ \qquad \qquad \qquad -\Delta u_{1,\delta} + (2m+1)(u_0)^{2m}u_{1,\delta} \\ \qquad \qquad \qquad + \lambda(\delta) \sum_{k=2}^{2m+1} \binom{2m+1}{k} (u_0)^{2m+1-k} \lambda(\delta)^{k-2} (u_{1,\delta})^k = 0 \quad \text{in } \Omega \setminus \{O\} \\ \qquad \qquad \qquad u_{1,\delta} = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (4.8)$$

In (4.8), $\binom{2m+1}{k}$ denotes the binomial coefficient. Here, we already observe one consequence of the fact that we are dealing with a non-linear problem. We proved that (4.8) admits a unique solution for $\lambda(\delta) \in \mathbb{R}$ such that $|\lambda(\delta)|$ is small enough.

Then, and we will not detail the rest of the analysis, we construct an expansion close to the obstacle where we expect a rapid variation of the field. Finally, we match the two expansions in an intermediate region to define completely the unknown terms of the far and near field expansions. The main novelty appearing in the study of the non linear PDE occurs during the matching step which leads us to solve a non linear (and non explicit) equation to define the gauge functions of the expansion.

In [159], we justified this formal asymptotic procedure by proving an error estimate between u_δ and our expansion as δ tends to zero. Moreover, we constructed an approximate model which admits a solution corresponding to the first two terms of the far field expansion of u_δ . From a practical point of view, it is much simpler to discretize this approximate model than to compute separately each of the terms appearing in the expansion of u_δ . The derivation of this problem was the third main outcome of the paper.

Admittedly, in this work we considered a rather academic problem. A possible direction to continue this study would be to consider other types of non linear PDEs that could be more relevant for applications. Note that in [58, 13], stronger non linearities are dealt with in the field of topological sensitivity analysis. Besides, considering higher order expansions seems a difficult problem with this approach.

4.3 Quasi-reversibility methods

In the article [99], with L. Bourgeois, we were interested in an ill-posed Cauchy problem for the Laplace equation for which uniqueness holds. Let us start by presenting this problem. Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d > 1$, the boundary $\partial\Omega$ of which is partitioned into two sets Γ and $\tilde{\Gamma}$. More precisely, Γ and $\tilde{\Gamma}$ are non empty open sets for the topology induced on $\partial\Omega$ from the topology on $\mathbb{R}^d$, $\partial\Omega = \bar{\Gamma} \cup \bar{\tilde{\Gamma}}$ and $\Gamma \cap \tilde{\Gamma} = \emptyset$ (see Figure 4.2). The Cauchy problem we are interested in consists, for some data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, in finding $u \in H^1(\Omega)$ such that

$$\left| \begin{array}{lcl} \Delta u & = & 0 \quad \text{in } \Omega \\ u & = & g_0 \quad \text{on } \Gamma \\ \partial_\nu u & = & g_1 \quad \text{on } \Gamma, \end{array} \right. \quad (4.9)$$

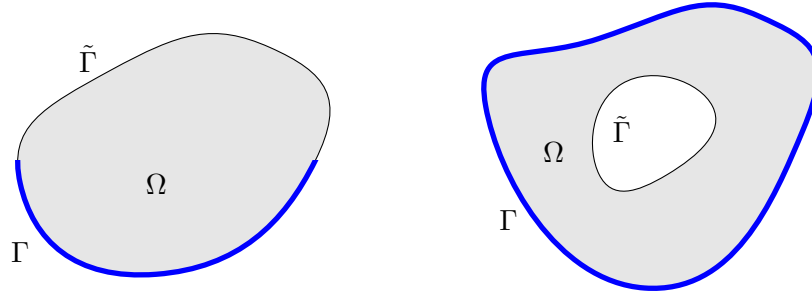


Figure 4.2: Examples of domains Ω . The thick blue lines represent the support of measurements.

where ν is the outward unit normal to $\partial\Omega$. This kind of problem arises when some part $\tilde{\Gamma}$ of the boundary of a structure is not accessible, while the complementary part Γ is the support of measurements which provide the Cauchy data (g_0, g_1) . It is important to note that in practice, those measurements are contaminated by some noise. Due to Holmgren's theorem, the Cauchy problem (4.9) has at most one solution. However it is ill-posed in the sense of Hadamard: existence may not hold for some data (g_0, g_1) , as for example shown in [41]. A possibility to regularize Problem (4.9) is to use the quasi-reversibility method, which goes back to [325] and was revisited in [310]. The original idea was to replace an ill-posed Boundary Value Problem such as (4.9) by a family, depending on a small parameter ε , of well-posed fourth-order BVPs. Much later, L. Bourgeois introduced the notion of mixed formulation of quasi-reversibility for the Cauchy problem of the Laplace equation [95]. This notion was extended to general abstract linear ill-posed problems in [104]. The idea is to replace the ill-posed second-order BVP by a family, again depending on a small parameter ε , of second-order systems of two coupled BVPs: the advantage is that the order of the regularized problem is the same as the original one, which is interesting when it comes to the numerical resolution. The price to pay is the introduction of a second unknown function λ_ε in addition to the principal unknown u_ε . Such mixed formulation of quasi-reversibility is the following: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_{g_0} \times \tilde{V}_0$ such that for all $(v, \mu) \in V_0 \times \tilde{V}_0$,

$$\left\{ \begin{array}{l} \varepsilon \int_{\Omega} \nabla u_\varepsilon \cdot \nabla v \, d\mathbf{x} + \int_{\Omega} \nabla v \cdot \nabla \lambda_\varepsilon \, d\mathbf{x} = 0 \\ \int_{\Omega} \nabla u_\varepsilon \cdot \nabla \mu \, d\mathbf{x} - \int_{\Omega} \nabla \lambda_\varepsilon \cdot \nabla \mu \, d\mathbf{x} = \langle g_1, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}, \end{array} \right. \quad (4.10)$$

where $V_{g_0} := \{u \in H^1(\Omega), u|_{\Gamma} = g_0\}$, $V_0 := \{u \in H^1(\Omega), u|_{\Gamma} = 0\}$ and $\tilde{V}_0 = \{\lambda \in H^1(\Omega), \lambda|_{\tilde{\Gamma}} = 0\}$. In (4.10), the brackets stand for duality pairing between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$. We observe that in view of Poincaré inequality, the standard norm of $H^1(\Omega)$ in the spaces V_0 and $\tilde{V}_0$ is equivalent to the semi-norm $\|\cdot\|_{H_0^1(\Omega)}$ defined by $\|\cdot\|_{H_0^1(\Omega)}^2 = \int_{\Omega} |\nabla \cdot|^2 \, d\mathbf{x}$. Let us denote by $(\cdot, \cdot)_{H_0^1(\Omega)}$ the corresponding scalar product. We remark that the weak formulation (4.10) is equivalent to the strong problem

$$\left\{ \begin{array}{ll} \Delta u_\varepsilon = 0 & \text{in } \Omega \\ \Delta \lambda_\varepsilon = 0 & \text{in } \Omega \\ u_\varepsilon = g_0 & \text{on } \Gamma \\ \partial_\nu u_\varepsilon - \partial_\nu \lambda_\varepsilon = g_1 & \text{on } \Gamma \\ \lambda_\varepsilon = 0 & \text{on } \tilde{\Gamma} \\ \varepsilon \partial_\nu u_\varepsilon + \partial_\nu \lambda_\varepsilon = 0 & \text{on } \tilde{\Gamma}, \end{array} \right.$$

where we observe that the two unknowns u_ε and λ_ε are harmonic functions which are coupled by the boundary $\partial\Omega$. We have the following theorem.

Theorem 4.3.1 *For all $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, Problem (4.10) has a unique solution $(u_\varepsilon, \lambda_\varepsilon) \in V_{g_0} \times \tilde{V}_0$. Moreover, there exists a constant C which depends only on the geometry*

such that

$$\forall \varepsilon \in (0; 1], \quad \sqrt{\varepsilon} \|u_\varepsilon\|_{H^1(\Omega)} + \|\lambda_\varepsilon\|_{H^1(\Omega)} \leq C (\|g_0\|_{H^{1/2}(\Gamma)} + \|g_1\|_{H^{-1/2}(\Gamma)}).$$

If in addition we assume that (g_0, g_1) is such that problem (4.9) has a (unique) solution u (the data are said to be compatible), then there exists a constant C which depends only on the geometry such that

$$\forall \varepsilon > 0, \quad \|u_\varepsilon\|_{H^1(\Omega)} + \frac{\|\lambda_\varepsilon\|_{H^1(\Omega)}}{\sqrt{\varepsilon}} \leq C \|u\|_{H^1(\Omega)}$$

and

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{H^1(\Omega)} = 0.$$

To prove such theorem, we need the following lemma, which establishes an equivalent weak formulation to Problem (4.9) and which is proved in [104].

Lemma 4.3.2 For $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, the function $u \in H^1(\Omega)$ is a solution to Problem (4.9) if and only if $u|_\Gamma = g_0$ and for all $\mu \in H^1(\Omega)$ with $\mu|_{\tilde{\Gamma}} = 0$, we have

$$\int_{\Omega} \nabla u \cdot \nabla \mu \, d\mathbf{x} = \langle g_1, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}. \quad (4.11)$$

Proof of Theorem 4.3.1. Let us begin with the first part of the theorem. There exists a continuous lifting operator $g_0 \mapsto U$ from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$ such that $U|_\Gamma = g_0$. Let us define $\hat{u}_\varepsilon = u_\varepsilon - U \in V_0$. By replacing in (4.10), we obtain that $(\hat{u}_\varepsilon, \lambda_\varepsilon) \in V_0 \times \tilde{V}_0$ satisfies, for all $(v, \mu) \in V_0 \times \tilde{V}_0$, the system

$$\begin{cases} \varepsilon \int_{\Omega} \nabla \hat{u}_\varepsilon \cdot \nabla v \, d\mathbf{x} + \int_{\Omega} \nabla v \cdot \nabla \lambda_\varepsilon \, d\mathbf{x} &= -\varepsilon \int_{\Omega} \nabla U \cdot \nabla v \, d\mathbf{x} \\ \int_{\Omega} \nabla \hat{u}_\varepsilon \cdot \nabla \mu \, d\mathbf{x} - \int_{\Omega} \nabla \lambda_\varepsilon \cdot \nabla \mu \, d\mathbf{x} &= \langle g_1, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)} - \int_{\Omega} \nabla U \cdot \nabla \mu \, d\mathbf{x}. \end{cases}$$

Well-posedness then relies on the Lax-Milgram Lemma applied to the coercive bilinear form

$$a_\varepsilon((u, \lambda), (v, \mu)) = \varepsilon \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x} + \int_{\Omega} \nabla v \cdot \nabla \lambda \, d\mathbf{x} - \int_{\Omega} \nabla u \cdot \nabla \mu \, d\mathbf{x} + \int_{\Omega} \nabla \lambda \cdot \nabla \mu \, d\mathbf{x}$$

in V_0 . Choosing $v = \hat{u}_\varepsilon$ and $\mu = \lambda_\varepsilon$ and subtracting the two above equations, we obtain

$$\varepsilon \int_{\Omega} |\nabla \hat{u}_\varepsilon|^2 \, d\mathbf{x} + \int_{\Omega} |\nabla \lambda_\varepsilon|^2 \, d\mathbf{x} = -\varepsilon \int_{\Omega} \nabla U \cdot \nabla \hat{u}_\varepsilon \, d\mathbf{x} - \langle g_1, \lambda_\varepsilon \rangle + \int_{\Omega} \nabla U \cdot \nabla \lambda_\varepsilon \, d\mathbf{x}.$$

The Cauchy-Schwarz inequality implies

$$\varepsilon \|\hat{u}_\varepsilon\|_{H_0^1(\Omega)}^2 + \|\lambda_\varepsilon\|_{H_0^1(\Omega)}^2 \leq \varepsilon \|U\|_{H_0^1(\Omega)} \|\hat{u}_\varepsilon\|_{H_0^1(\Omega)} + \|g_1\|_{H^{-1/2}(\Gamma)} \|\lambda_\varepsilon\|_{H^{1/2}(\Gamma)} + \|U\|_{H_0^1(\Omega)} \|\lambda_\varepsilon\|_{H_0^1(\Omega)}.$$

The equivalence of the norm $\|\cdot\|_{H_0^1(\Omega)}$ and the standard $H^1(\Omega)$ norm in the spaces V_0 and $\tilde{V}_0$, the continuity of the trace operator and the continuity of the lifting operator $g_0 \mapsto U$ yield

$$\varepsilon \|\hat{u}_\varepsilon\|_{H^1(\Omega)}^2 + \|\lambda_\varepsilon\|_{H^1(\Omega)}^2 \leq C\varepsilon \|g_0\|_{H^{1/2}(\Gamma)} \|\hat{u}_\varepsilon\|_{H^1(\Omega)} + (c \|g_1\|_{H^{-1/2}(\Gamma)} + C \|g_0\|_{H^{1/2}(\Gamma)}) \|\lambda_\varepsilon\|_{H^1(\Omega)}.$$

Using Young's inequality to deal with the right-hand side of the above inequality, the result follows. Let us prove the second part of the theorem. In the case when the Cauchy data (g_0, g_1) is associated with the solution u , then u satisfies the weak formulation (4.11). By subtracting (4.11) from the second equation of (4.10), we obtain that for all $\mu \in \tilde{V}_0$,

$$\int_{\Omega} \nabla (u_\varepsilon - u) \cdot \nabla \mu \, d\mathbf{x} - \int_{\Omega} \nabla \lambda_\varepsilon \cdot \nabla \mu \, d\mathbf{x} = 0. \quad (4.12)$$

Now setting $v = u_\varepsilon - u \in V_0$ in the first line of (4.10), $\mu = \lambda_\varepsilon \in \tilde{V}_0$ in (4.12) and subtracting the two obtained equations, we get

$$\varepsilon \int_{\Omega} \nabla u_\varepsilon \cdot \nabla (u_\varepsilon - u) \, d\mathbf{x} + \int_{\Omega} |\nabla \lambda_\varepsilon|^2 \, d\mathbf{x} = 0.$$

We deduce that the term $(u_\varepsilon, u_\varepsilon - u)_{H_0^1(\Omega)}$ in the above sum is nonpositive, which from the Cauchy-Schwarz inequality implies $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)}$ and then $\|\lambda_\varepsilon\|_{H_0^1(\Omega)} \leq \sqrt{\varepsilon} \|u\|_{H_0^1(\Omega)}$. Hence there exists a constant C such that

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)} \quad \text{and} \quad \|\lambda_\varepsilon\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon} \|u\|_{H^1(\Omega)}.$$

It remains to prove that $u_\varepsilon \rightarrow u$ in $H^1(\Omega)$ when $\varepsilon \rightarrow 0$. The sequence (u_ε) is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence, still denoted (u_ε) , such that $u_\varepsilon \rightharpoonup w$ in $H^1(\Omega)$ when $\varepsilon \rightarrow 0$, with $w \in H^1(\Omega)$. Since the affine space V_{g_0} is convex and closed, it is weakly closed. This guarantees that $w \in V_{g_0}$. Besides, by passing to the limit in the second equation of (4.10), we obtain that w satisfies the weak formulation (4.11). Uniqueness in Problem (4.9) then implies that $w = u$, so that (u_ε) weakly converges to u in $H^1(\Omega)$. But

$$\|u_\varepsilon - u\|_{H_0^1(\Omega)}^2 = (u_\varepsilon, u_\varepsilon - u)_{H_0^1(\Omega)} - (u, u_\varepsilon - u)_{H_0^1(\Omega)} \leq -(u, u_\varepsilon - u)_{H_0^1(\Omega)},$$

so that weak convergence implies strong convergence. Lastly, a standard contradiction argument enables us to conclude that the whole sequence (u_ε) strongly converges to u in $H^1(\Omega)$. ■

Remark 4.3.3 *Another type of mixed formulation of quasi-reversibility was introduced in [205], in which the additional unknown lies in $H_{\text{div}}(\Omega)$ instead of $H^1(\Omega)$. In addition, a notion of iterative formulation of quasi-reversibility was introduced and analyzed in [204]. However we believe that the quasi-reversibility formulation (4.10) is the easiest one to handle to establish regularity results of the weak solutions.*

The estimates of Theorem 4.3.1 involve $H^1(\Omega)$ norms of the regularized solution $(u_\varepsilon, \lambda_\varepsilon)$ in the case of a Lipschitz domain Ω and for the natural regularity of the Cauchy data (g_0, g_1) , that is $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. These estimates were derived in two different cases: the data (g_0, g_1) are compatible or not.

The main concern of [99] was to analyze, when the domain Ω and the Cauchy data (g_0, g_1) are more regular than Lipschitz and $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, respectively, the additional regularity of the solution $(u_\varepsilon, \lambda_\varepsilon)$, whether the data (g_0, g_1) are compatible or not. We also wanted to obtain estimates in the corresponding norms. In order to simplify the analysis, the additional regularity of the data (g_0, g_1) is formulated in the following way: we assume that (g_0, g_1) is such that there exists a function U in $H^2(\Omega)$ with $(U|_\Gamma, \partial_\nu U|_\Gamma) = (g_0, g_1)$ and that we can define a continuous lifting operator $(g_0, g_1) \mapsto U$. Denoting $f = \Delta U \in L^2(\Omega)$ and considering the new translated unknown $u - U \rightarrow u$, the initial Cauchy problem (4.9) can be transformed into a homogeneous one (however still ill-posed): for $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma \\ \partial_\nu u &= 0 & \text{on } \Gamma. \end{cases} \quad (4.13)$$

We emphasize that this regularity assumption made on the data is not an assumption of regularity of the solution u . It is simple to construct smooth data in the sense above such that the corresponding

u is only in $H^1(\Omega)$ and not in $H^2(\Omega)$. The mixed formulation of quasi-reversibility for Problem (4.13) takes the following form: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_0 \times \tilde{V}_0$ such that for all $(v, \mu) \in V_0 \times \tilde{V}_0$,

$$\begin{cases} \varepsilon \int_{\Omega} \nabla u_\varepsilon \cdot \nabla v \, d\mathbf{x} + \int_{\Omega} \nabla v \cdot \nabla \lambda_\varepsilon \, d\mathbf{x} &= 0 \\ \int_{\Omega} \nabla u_\varepsilon \cdot \nabla \mu \, d\mathbf{x} - \int_{\Omega} \nabla \lambda_\varepsilon \cdot \nabla \mu \, d\mathbf{x} &= \int_{\Omega} f \mu \, d\mathbf{x}. \end{cases} \quad (4.14)$$

Note that the strong equations corresponding to Problem (4.14) are

$$\begin{cases} -(1 + \varepsilon)\Delta u_\varepsilon &= f & \text{in } \Omega \\ -(1 + \varepsilon)\Delta \lambda_\varepsilon &= -\varepsilon f & \text{in } \Omega \\ u_\varepsilon &= 0 & \text{on } \Gamma \\ \partial_\nu u_\varepsilon - \partial_\nu \lambda_\varepsilon &= 0 & \text{on } \Gamma \\ \lambda_\varepsilon &= 0 & \text{on } \tilde{\Gamma} \\ \varepsilon \partial_\nu u_\varepsilon + \partial_\nu \lambda_\varepsilon &= 0 & \text{on } \tilde{\Gamma}. \end{cases}$$

The analog of Theorem 4.3.1 is the following.

Theorem 4.3.4 *For all $f \in L^2(\Omega)$ and $\varepsilon > 0$, Problem (4.14) has a unique solution $(u_\varepsilon, \lambda_\varepsilon) \in V_0 \times \tilde{V}_0$. There exists a constant C which depends only on the geometry such that*

$$\forall \varepsilon \in (0; 1], \quad \sqrt{\varepsilon} \|u_\varepsilon\|_{H^1(\Omega)} + \|\lambda_\varepsilon\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \quad (4.15)$$

If in addition we assume that f is such that Problem (4.13) has a (unique) solution u , then there exists a constant C which depends only on the geometry such that

$$\forall \varepsilon > 0, \quad \|u_\varepsilon\|_{H^1(\Omega)} + \frac{\|\lambda_\varepsilon\|_{H^1(\Omega)}}{\sqrt{\varepsilon}} \leq C \|u\|_{H^1(\Omega)} \quad (4.16)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{H^1(\Omega)} = 0.$$

The goal of [99] then was to study the regularity of the solution $(u_\varepsilon, \lambda_\varepsilon)$ to Problem (4.14) and to complete the statements (4.15) and (4.16) of Theorem 4.3.4 by giving estimates in stronger norms. One objective was the following. In practice, one has to solve Problem (4.14) with two additional approximations. Firstly, the data f is altered by some noise of amplitude δ . Secondly, Problem (4.14) is discretized, for instance with the help of a Finite Element Method based on a mesh of size h . It is then desirable to estimate the error between the approximated solution and the exact solution as a function of ε , δ and h . Such error estimate for the $H^1(\Omega)$ norm needs the solution to be in a Sobolev space $H^s(\Omega)$, with $s > 1$. Let us mention that in [115] (see also [110, 111, 112, 114]), a discretized method was proposed in order to regularize the Cauchy problem (4.9) in the presence of noisy data without introducing a regularized problem such as (4.14) at the continuous level. In some sense, the method of [115] relies on a single asymptotic parameter, that is h , instead of two in our method, that is ε and h . However, we believe that from the theoretical point of view, the study of regularity of quasi-reversibility solutions is an interesting problem in itself, which has never been investigated. The difficulty stems from the fact that it leads to analyze the regularity of the solution to a problem involving a small parameter ε which degenerates when ε tends to 0. There are other contributions (see *e.g.* [302, 197, 156, 159, 384, 385]) where regularity results or asymptotic expansions are obtained in situations where the limit problem has a different nature from the regularized one. For example in [197], the authors study a mixed Neumann-Robin problem where the small parameter ε is the inverse of the Robin coefficient. But while both the perturbed problem and the limit one are well-posed in [197], only the perturbed problem is well-posed in our case, the limit (Cauchy) problem being ill-posed (in any framework). Our contribution is original

in this sense.

In our work, we proved regularity results of the solutions to these regularized problems which hold uniformly in ε when ε tends to zero. We considered two situations: smooth domains and 2D polygonal geometries. In presence of corners, due to the particular structure of the regularized problems, classical techniques *à la* Grisvard do not work and instead, we applied the Kondratiev approach (already used in §1.3, §2.10, §3.1.2). The main point was to write the procedure in detail to keep track of the dependence in ε in all the estimates. With these results, we derived some error estimates between the exact solution and the quasi-reversibility solution in the presence of two perturbations: noisy data and discretization with the help of a Finite Element Method. Let us give an idea of what they look like.

Consider $V_{0,h}$ and $\tilde{V}_{0,h}$ finite dimensional subspaces approximating respectively V_0 and $\tilde{V}_0$. For a noise level $\delta > 0$, let $f^\delta \in L^2(\Omega)$ (the available data in practice) be a noisy approximation of f such that

$$\|f^\delta - f\|_{L^2(\Omega)} \leq \delta.$$

The discretized version of the mixed formulation of quasi-reversibility (4.14) writes: for $\varepsilon > 0$, find $(u_{\varepsilon,h}^\delta, \lambda_{\varepsilon,h}^\delta) \in V_{0,h} \times \tilde{V}_{0,h}$ such that for all $(v_h, \mu_h) \in V_{0,h} \times \tilde{V}_{0,h}$,

$$\begin{cases} \varepsilon \int_{\Omega} \nabla u_{\varepsilon,h}^\delta \cdot \nabla v_h \, d\mathbf{x} + \int_{\Omega} \nabla v_h \cdot \nabla \lambda_{\varepsilon,h}^\delta \, d\mathbf{x} &= 0 \\ \int_{\Omega} \nabla u_{\varepsilon,h}^\delta \cdot \nabla \mu_h \, d\mathbf{x} - \int_{\Omega} \nabla \lambda_{\varepsilon,h}^\delta \cdot \nabla \mu_h \, d\mathbf{x} &= \int_{\Omega} f^\delta \mu_h \, d\mathbf{x}. \end{cases} \quad (4.17)$$

We also denote by $(u_{\varepsilon,h}, \lambda_{\varepsilon,h})$ the solution to problem (4.17) associated with f instead of f^δ . In practice, $u_{\varepsilon,h}^\delta$ is the only approximate function of the exact solution u which is accessible. Therefore, our goal is to assess the norm of the discrepancy $u_{\varepsilon,h}^\delta - u$ in the domain Ω . In this view, we write

$$\|u_{\varepsilon,h}^\delta - u\|_{H^1(\Omega)} \leq \|u_{\varepsilon,h}^\delta - u_{\varepsilon,h}\|_{H^1(\Omega)} + \|u_{\varepsilon,h} - u_\varepsilon\|_{H^1(\Omega)} + \|u_\varepsilon - u\|_{H^1(\Omega)}, \quad (4.18)$$

and estimate each term of this decomposition. The first one corresponds to the error due to the noisy data. We proved the following lemma.

Lemma 4.3.5 *There exists a constant $C > 0$ which depends only on the geometry such that*

$$\|u_{\varepsilon,h}^\delta - u_{\varepsilon,h}\|_{H^1(\Omega)} \leq C \frac{\delta}{\sqrt{\varepsilon}}. \quad (4.19)$$

The second term of (4.18) represents the error due to discretization. By exploiting our regularity analysis, the core of [99], we deduced the following estimate.

Lemma 4.3.6 *There is a constant $C > 0$ which depends only on the geometry and on u such that*

$$\|u_{\varepsilon,h} - u_\varepsilon\|_{H^1(\Omega)} \leq C \frac{h^{s-1}}{\varepsilon}, \quad (4.20)$$

for a certain $s > 1$ that we can characterize.

Estimating the third term in (4.18) is strongly related to the stability of the Cauchy problem for the Laplace equation, a topic which has a long history. It is well-known that since such problem is exponentially ill-posed, the corresponding stability estimate is at best of logarithmic type (see for example [96]). To our best knowledge, an estimate of $\eta(\varepsilon) := \|u_\varepsilon - u\|_{H^1(\Omega)}$, which tends to 0 when ε tends to 0, is unknown. However, a logarithmic stability estimate for $\|u_\varepsilon - u\|_{L^2(\Omega)}$ can be derived from Theorem 1.9 in [8].

Lemma 4.3.7 *There exists a constant $C > 0$ which depends only on the geometry and on u and a constant $\mu \in (0; 1)$ which depends only on the geometry such that*

$$\|u_\varepsilon - u\|_{L^2(\Omega)} \leq C \frac{1}{(\log(1/\varepsilon))^\mu}.$$

In conclusion, by gathering (4.18), (4.19) and (4.20), we end up with the final estimate

$$\|u_{\varepsilon,h}^\delta - u\|_{H^1(\Omega)} \leq C \frac{\delta}{\sqrt{\varepsilon}} + C \frac{h^{s-1}}{\varepsilon} + \eta(\varepsilon), \quad (4.21)$$

where η converges to 0 when ε tends to 0 at best with a logarithmic convergence rate in view of Lemma 4.3.7. An important application of the estimate (4.21) is that when $\delta \rightarrow 0$, we have to choose $\varepsilon = \varepsilon(\delta)$ and $h = h(\varepsilon)$ such that

$$\lim_{\delta \rightarrow 0} \frac{\delta}{\sqrt{\varepsilon(\delta)}} = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{h^{s-1}(\varepsilon)}{\varepsilon} = 0$$

in order to obtain a good approximation of the exact solution from noisy data and by using our Finite Element Method.

An intriguing question around this topic concerns the derivation of an asymptotic expansion, and not only a regularity study, of the solution u_ε with respect to ε as ε tends to zero. The main difficulty for that lies again in the fact that the limit problems are Cauchy problems that are ill-posed in any framework. Is it possible to propose an asymptotic expansion that can be justified by error estimates? This is not clear at all and deserves to be studied.

4.4 Generalized impedance boundary conditions with vanishing or sign-changing impedance

In the article [100], with L. Bourgeois, we considered a Laplace type problem with a Generalized Impedance Boundary Condition (GIBC) involving a vanishing or sign-changing coefficient. GIBCs are often used in the context of asymptotic analysis for partial differential equations to obtain simplified models. Imagine for example that one is interested in the scattering of an electromagnetic wave by an inclusion of perfectly conducting material coated with a thin dielectric layer of variable thickness. One can show that the solution of the corresponding problem is well approximated by the solution of a scattering problem for the inclusion alone supplemented with an *ad hoc* second-order GIBC. In this model, the complexity of the initial geometry is incorporated in the boundary condition. This can be useful in particular to reduce computational costs because it allows one to avoid meshing the thin layer, see *e.g.* [19]. Note that this approach can also be exploited to solve the inverse problem consisting in finding information on the obstacle from the measurement of scattered fields [97, 98]. For modeling aspects and derivation of GIBCs in electromagnetism, we refer the reader to [255]. Similarly, in aeroacoustics, the so-called Ingard-Myers boundary conditions are used to model the presence of a liner located on the surface of a duct [348]. They also consist of a second-order GIBC.

In order to describe the setting of our study, let us present in more detail another situation where GIBCs arise. Let $\Omega \subset \mathbb{R}^2$ be an open, connected, bounded set with a Lipschitz continuous boundary $\partial\Omega$. Additionally, assume that there holds $\Gamma \subset \partial\Omega$ with $\Gamma := (-1; 1) \times \{0\}$ and set $\Gamma_0 := \partial\Omega \setminus \bar{\Gamma}$ (see Figure 4.3 left). Now let us perturb slightly, in a smooth way, the boundary of Ω . To proceed, introduce some smooth profile function g supported in $[-1; 1]$ satisfying $g(-1) = g(1) = 0$ and for $\varepsilon > 0$ small, define the domain Ω^ε such that

$$\partial\Omega^\varepsilon = \bar{\Gamma}_0 \cup \Gamma^\varepsilon \quad \text{with} \quad \Gamma^\varepsilon := \{(x, \varepsilon g(x)) \mid x \in (-1; 1)\}.$$

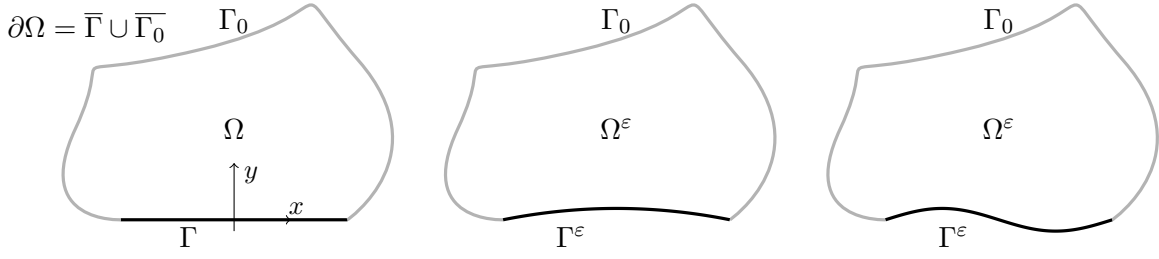


Figure 4.3: Left: domain Ω . Center and right: examples of perturbed domains Ω^ε .

Examples of such geometries are given in Figure 4.3 center and right. In Ω^ε , we study the model problem

$$\begin{cases} -\Delta u^\varepsilon + u^\varepsilon = f & \text{in } \Omega^\varepsilon \\ \partial_{\nu^\varepsilon} u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \quad (4.22)$$

where f is a given source term which vanishes in a neighborhood of Γ and ν^ε stands for the outward unit normal vector to $\partial\Omega^\varepsilon$. Let us recall how to obtain formally an asymptotic expansion of u^ε with respect to ε small. Consider the ansatz

$$u^\varepsilon = u_0 + \varepsilon u_1 + \dots \quad (4.23)$$

where u_0, u_1 are functions to determine and where the dots correspond to higher order terms. On Γ^ε , we have the expansions

$$\nu^\varepsilon = \frac{1}{\sqrt{1 + \varepsilon^2 (g'(x))^2}} \begin{pmatrix} \varepsilon g'(x) \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} g'(x) \\ 0 \end{pmatrix} + \dots \quad (4.24)$$

$$\begin{aligned} \nabla u^\varepsilon(x, \varepsilon g(x)) &= \nabla u^\varepsilon(x, 0) + \varepsilon g(x) \begin{pmatrix} \partial_{xy}^2 u^\varepsilon(x, 0) \\ \partial_{yy}^2 u^\varepsilon(x, 0) \end{pmatrix} + \dots \\ &= \nabla u^\varepsilon(x, 0) + \varepsilon g(x) \begin{pmatrix} \partial_{xy}^2 u^\varepsilon(x, 0) \\ -\partial_{xx}^2 u^\varepsilon(x, 0) + u^\varepsilon(x, 0) \end{pmatrix} + \dots \end{aligned} \quad (4.25)$$

Now we insert (4.23) in (4.22) and exploit (4.24), (4.25). Collecting the terms of orders $\varepsilon^0, \varepsilon^1$, we find that u_0, u_1 satisfy respectively the problems

$$\begin{cases} -\Delta u_0 + u_0 = f & \text{in } \Omega \\ \partial_\nu u_0 = 0 & \text{on } \partial\Omega \end{cases} \quad \begin{cases} -\Delta u_1 + u_1 = 0 & \text{in } \Omega \\ \partial_\nu u_1 = 0 & \text{on } \Gamma_0 \\ \partial_\nu u_1 = -\partial_x(g(x)\partial_x u_0) + g(x)u_0 & \text{on } \Gamma. \end{cases}$$

Then by rectifying the boundary of Ω^ε using “almost identical” diffeomorphisms to transform the perturbed domain into the original geometry Ω (see *e.g.* [294, Chap. 7, §6.5]), under additional assumptions of regularity for f , one can prove the estimate, for ε small enough,

$$\|u^\varepsilon - (u_0 + \varepsilon u_1)\|_{H^1(\Omega \setminus \bar{\omega})} \leq C\varepsilon^2,$$

where $C > 0$ is a constant independent of ε and ω is a neighborhood of Γ . In practice, as mentioned above, instead of computing successively $u_0, u_1, \dots$, very often one prefers to work with a model problem (see also §4.1), whose dependence with respect to ε is rather explicit, which provides via a simple calculation an approximation of u^ε up to a given order. In our case, to approach $u_0 + \varepsilon u_1$ in one shot, one can consider the problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = -\varepsilon \partial_x(g(x)\partial_x u) + \varepsilon g(x)u & \text{on } \Gamma. \end{cases} \quad (4.26)$$

The condition on Γ appearing in (4.26), which makes the analysis of this problem not straightforward, is a particular instance of the so-called Ventcel boundary conditions, which are themselves a subclass of GIBCs. Ventcel boundary conditions are second order differential conditions which have been named after the pioneering works of Feller and Ventcel [231, 465, 232, 466]. Since then, there have been many studies concerning the Laplace operator with Ventcel boundary conditions [230, 16, 201, 141, 202, 200, 59, 142]. For investigations in non smooth domains, one can consult [334, 425, 396]. The value of $\varepsilon > 0$ as well as the term $\varepsilon g u$ play no major role in the well-posedness in the Fredholm sense of (4.26) and to simplify, we work with the condition

$$\partial_\nu u = -\partial_x(g(x)\partial_x u) \quad \text{on } \Gamma, \quad (4.27)$$

where ν is the outward unit normal vector to $\partial\Omega$. Up to now, in most of the articles mentioned above only the cases $g = -1$ or $g = 1$ on the whole boundary have been considered. However, for certain problems, as the one which led us to (4.26), it may be relevant to consider some g which vanish or whose sign is not constant on Γ . For instance, such situation arises in a water wave problem where the small variation of the ocean bottom shape with respect to a reference flat depth is approached by the boundary condition (4.27), like in [103]. In this case the function g coincides with this shape variation and is likely to vanish or change sign. Then it is natural to wonder what can be said concerning the well-posedness of the corresponding problem in that situation.

This is precisely the question we studied with L. Bourgeois in [100]. Let us mention that a variable g is allowed in [241] but it does not vanish. Besides, note that degenerate elliptic problems are studied in [399] in the context of modeling of resonant waves in 2D plasma. These problems share similarities with ours but are nonetheless different.

To simplify, we focus our attention on two model problems. Let us divide Γ into the two segments

$$\Gamma_- := (-1; 0) \times \{0\} \quad \text{and} \quad \Gamma_+ := (0; 1) \times \{0\}.$$

To study the case of a g which vanishes at a point (the origin), we work on the variational problem whose strong formulation is

$$\left\{ \begin{array}{lll} -\Delta u + u & = & f \quad \text{in } \Omega \\ \partial_\nu u & = & 0 \quad \text{on } \partial\Omega \setminus \overline{\Gamma_+} \\ \partial_\nu u & = & s \partial_x(x^\alpha \partial_x u) \quad \text{on } \Gamma_+, \end{array} \right. \quad (4.28)$$

where $s = 1$ or $s = -1$ and $\alpha \geq 0$ is a real number. Moreover here and below, f is a given element of $L^2(\Omega)$.

To address the situation where g changes sign in (4.27), we study the variational problem whose strong formulation is

$$\left\{ \begin{array}{lll} -\Delta u + u & = & f \quad \text{in } \Omega \\ \partial_\nu u & = & 0 \quad \text{on } \Gamma_0 \\ \partial_\nu u & = & -\partial_x(|x|^\alpha \partial_x u) \quad \text{on } \Gamma_- \\ \partial_\nu u & = & \partial_x(x^\alpha \partial_x u) \quad \text{on } \Gamma_+, \end{array} \right. \quad (4.29)$$

with again $\alpha \geq 0$. In (4.28) and (4.29), the coefficient α can be viewed as a simple polynomial rate which characterizes how fast the impedance vanishes at the origin. The most relevant case is probably $\alpha = 1$, which in (4.29) corresponds to a smooth change of sign. The situation $\alpha = 0$, *i.e.* a jump of the impedance function which may seem less realistic because one needs smoothness in the asymptotic expansion leading to (4.26), is also considered in our paper. We will see that the value of α plays a crucial role in the results of well-posedness.

Definition of the problems

To consider the case of an impedance vanishing at the origin, we introduce the space

$$V_\alpha(\Omega) := \{v \in H^1(\Omega) \mid x^{\alpha/2} \partial_x v \in L^2(\Gamma_+)\}$$

and study the weak formulation

$$\left| \begin{array}{l} \text{Find } u \in V_\alpha(\Omega) \text{ such that for all } v \in V_\alpha(\Omega) \\ \int_\Omega \nabla u \cdot \nabla \bar{v} + u \bar{v} \, d\mathbf{x} + s \int_{\Gamma_+} x^\alpha \partial_x u \partial_x \bar{v} \, dx = \int_\Omega f \bar{v} \, d\mathbf{x}. \end{array} \right. \quad (4.30)$$

Here the term x^α is our impedance function. We denote by $a(\cdot, \cdot)$ the sesquilinear form appearing in the left-hand side of (4.30) and by $\ell(\cdot)$ the antilinear form of the right-hand side. For the space $V_\alpha(\Omega)$, we have the following result.

Proposition 4.4.1 *For all $\alpha \geq 0$, $V_\alpha(\Omega)$ is a Hilbert space when equipped with the inner product*

$$(u, v)_{V_\alpha(\Omega)} = \int_\Omega \nabla u \cdot \nabla \bar{v} + u \bar{v} \, d\mathbf{x} + \int_{\Gamma_+} x^\alpha \partial_x u \partial_x \bar{v} \, dx.$$

It is readily seen that the forms $a(\cdot, \cdot)$ and $\ell(\cdot)$ are respectively continuous in $V_\alpha(\Omega) \times V_\alpha(\Omega)$ and $V_\alpha(\Omega)$. With the help of the Riesz representation theorem, hence we can define the bounded operator $A : V_\alpha(\Omega) \rightarrow V_\alpha(\Omega)$ and the element $L \in V_\alpha(\Omega)$ such that

$$(Au, v)_{V_\alpha(\Omega)} = a(u, v) \quad \text{and} \quad (L, v)_{V_\alpha(\Omega)} = \ell(v), \quad \forall u, v \in V_\alpha(\Omega).$$

With this notation, the weak formulation (4.30) is equivalent to find $u \in V_\alpha(\Omega)$ such that $Au = L$.

To investigate the case of an impedance which is both vanishing at the origin and whose sign is non constant, we introduce the space, for $\alpha \geq 0$,

$$W_\alpha(\Omega) := \{v \in H^1(\Omega) \mid |x|^{\alpha/2} \partial_x v \in L^2(\Gamma_-), x^{\alpha/2} \partial_x v \in L^2(\Gamma_+)\},$$

and consider the weak formulation

$$\left| \begin{array}{l} \text{Find } u \in W_\alpha(\Omega) \text{ such that for all } v \in W_\alpha(\Omega) \\ \int_\Omega \nabla u \cdot \nabla \bar{v} + u \bar{v} \, d\mathbf{x} - \int_{\Gamma_-} |x|^\alpha \partial_x u \partial_x \bar{v} \, dx + \int_{\Gamma_+} x^\alpha \partial_x u \partial_x \bar{v} \, dx = \int_\Omega f \bar{v} \, d\mathbf{x}. \end{array} \right. \quad (4.31)$$

We denote by $b(\cdot, \cdot)$ the sesquilinear form appearing in the left-hand side of (4.31). One shows that $W_\alpha(\Omega)$ is a Hilbert space for all $\alpha \geq 0$ when equipped with the inner product

$$(u, v)_{W_\alpha(\Omega)} = \int_\Omega \nabla u \cdot \nabla \bar{v} + u \bar{v} \, d\mathbf{x} + \int_{\Gamma_-} |x|^\alpha \partial_x u \partial_x \bar{v} \, dx + \int_{\Gamma_+} x^\alpha \partial_x u \partial_x \bar{v} \, dx.$$

With the Riesz representation theorem, we define the bounded operator $B : W_\alpha(\Omega) \rightarrow W_\alpha(\Omega)$ and the element $\tilde{L} \in W_\alpha(\Omega)$ such that

$$(Bu, v)_{W_\alpha(\Omega)} = b(u, v) \quad \text{and} \quad (\tilde{L}, v)_{W_\alpha(\Omega)} = \ell(v), \quad \forall u, v \in W_\alpha(\Omega).$$

With this notation, the weak formulation (4.31) is equivalent to find $u \in W_\alpha(\Omega)$ such that $Bu = \tilde{L}$.

Statement of the results

We start with the variational formulation (4.30). Concerning the relation with the strong problem (4.28), we have the following result:

Theorem 4.4.2 *For $s = \pm 1$, for any $\alpha \geq 0$, if u satisfies (4.30), then it solves (4.28) where the GIBC on Γ_+ holds in $\mathcal{D}'(\Gamma_+)$ (distributional sense).*

Next we consider the well-posedness of (4.30). In the good sign case $s = 1$, we have the following theorem, which is a direct application of the Lax-Milgram theorem since the bilinear form $a(\cdot, \cdot)$ is coercive.

Theorem 4.4.3 *For $s = 1$, the weak formulation (4.30) has a unique solution for all $\alpha \geq 0$.*

The bad sign case $s = -1$ is more delicate to address because then the coercivity of $a(\cdot, \cdot)$ in $V_\alpha(\Omega)$ is not clear. The main achievement of [100] was to emphasize that the well-posedness (in the Fredholm sense) of (4.30) depends on the parameter α . We proved a positive result for $\alpha \in [0; 1)$ and a negative result for $\alpha = 1$.

Theorem 4.4.4 *For $s = -1$ and $\alpha \in [0; 1)$, the operator $A : V_\alpha(\Omega) \rightarrow V_\alpha(\Omega)$ is Fredholm of index zero. As a consequence, Problem (4.30) admits a solution when A is injective.*

Theorem 4.4.5 *For $s = -1$ and $\alpha = 1$, the operator $A : V_\alpha(\Omega) \rightarrow V_\alpha(\Omega)$ is not of Fredholm type.*

Now we present the results for the variational formulation (4.31) with an impedance which is both vanishing and sign-changing. They are very similar to the ones of Theorems 4.4.2, 4.4.4 and 4.4.5.

Theorem 4.4.6 *For any $\alpha \geq 0$, if u satisfies (4.31), then it solves (4.29) where the GIBC on $\Gamma_\pm$ holds in $\mathcal{D}'(\Gamma_\pm)$.*

Theorem 4.4.7 *For $\alpha \in [0; 1)$, the operator $B : W_\alpha(\Omega) \rightarrow W_\alpha(\Omega)$ is Fredholm of index zero. As a consequence, Problem (4.31) admits a solution when B is injective.*

Theorem 4.4.8 *For $\alpha = 1$, the operator $B : W_\alpha(\Omega) \rightarrow W_\alpha(\Omega)$ is not of Fredholm type.*

Remark 4.4.9 *Comparing all these results, we can say that the well-posedness of Problem (4.31) is more affected by the fact that the impedance is vanishing than by the fact that its sign is non constant.*

Let us make a few comments. The proof of the positive results (Theorems 4.4.4 and 4.4.7) is based on the reformulation as 1D problems by working with Dirichlet-to-Neumann operators. We proved compact embedding results for the weighted Sobolev spaces involved in the analysis to show that, roughly speaking, the volume terms in (4.30), (4.31) are only compact perturbations of the ones on $\Gamma_\pm$ when $\alpha \in [0; 1)$. In the study of B , i.e. in the case of a sign-changing impedance, we also used a T-coercivity argument a bit as in §1.2.1. The proof of the negative results (Theorems 4.4.5 and 4.4.8) relies on the computation of singularities and the construction of Weyl's sequences similar to what we did in the study of the plasmonic problem (1.1) with a corner in the interface. In [100], we also worked on the equivalence between the strong and weak formulations, which is not straightforward.

For $s = -1$ and $\alpha > 1$, we expect that A and B are not Fredholm but unfortunately, we have been unable to establish this result. This is due to the fact that then there are no singularities with separate variables in polar coordinates at the origin. In our study, for $s = -1$, we did not consider the question of injectivity of A and B . For $\alpha \in [0; 1)$, due to the Fredholm property, we know that these operators have a kernel of finite dimension. However showing that this kernel

reduces to the null function is an open question. We do not expect that this occurs in all geometries but even finding a simple Ω where we can prove that this is true reveals complications because we can not use separation of variables due to the form of the operators. Finally, in [100] we presented simple numerical experiments corroborating the Fredholm/non Fredholm property of the operators A, B according to the value of α . However there is no justification for this numerics. It would be interesting to establish results of convergence for our numerical methods when the mesh is refined and when A, B are isomorphisms.

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